Math 115A **Homework 5** **Due December 5, 2008**

1. Given the field F, the F-vector space V prove or disprove that the given function

$$< , >: V \times V \to F$$

is an inner product. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^\infty$ the space of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers such that $a_n = 0$ for all but finitely many n, and

$$< (a_n), (b_n) >= \sum_{n=0}^{\infty} a_n b_n.$$

This is easily seen to be an inner product.

b) $F = \mathbb{C}$, $V = \mathcal{P}_3$ the space of complex polynomials of degree at most 3, and

$$< f, g >= f(0)\overline{g(0)} + f(1)\overline{g(1)} + f(2)\overline{g(2)} + f(3)\overline{g(3)}.$$

The linearity properties and conjugate symmetry are easy. It is positive definite because $< f, f >$ is a sum of norm squares and a polynomial of degree at most 3 has at most 3 zeroes.

2. Let V be a finite-dimensional inner product space and let $S : V \to V$ and $T : V \to V$ be two self-adjoint linear transformations. Assume $S \circ T = T \circ S$. Show that S and T are simultaneously diagonalizable, that is, there exists a basis $\mathcal{B}$ of V such that each $v \in \mathcal{B}$ is an eigenvector for both S and T. (You may use the theorem that every self-adjoint transformation is diagonalizable.) (4 pts)

Let λ be an eigenvalue of S and E_λ the corresponding eigenspace. As shown on the previous homework, $T(E_\lambda) \subseteq E_\lambda$ because $ST = TS$. That is, $T|_{E_\lambda} : E_\lambda \to E_\lambda$ is a linear transformation; moreover, it is clearly self-adjoint with respect to the restriction of the inner product to E_λ. Therefore, E_λ has a basis $\mathcal{B}_\lambda$ consisting of eigenvectors of T; obviously, these are also eigenvectors of S (with eigenvalue λ).
Now let $\lambda_1, \dots, \lambda_r$ be the collection of all the eigenvalues of S and apply the above argument to each of the eigenspaces E_{λ_i} obtaining bases $\mathcal{B}_i$. Since S is self-adjoint, hence diagonalizable, the union $\mathcal{B} = \bigcup_{i=1}^r \mathcal{B}_i$ is a basis of V, consisting of eigenvectors for both S and T.

3. Let $V = \{f \in \mathbf{C}^\infty(\mathbb{R}) | f(x) = f(x + 2\pi)\}$ be the real inner product space of 2π-periodic infinitely differentiable real-valued functions with inner product $< f, g >= \int_0^{2\pi} f(x)g(x)dx$. Let $\Delta : V \to V$ be the linear transformation given by $\Delta(f)(x) = f''(x)$. Show that Δ is self-adjoint. (*Remark:* This is the one-dimensional version of the Laplace operator, a very important self-adjoint differential operator.) (4 pts)

Note: As stated, this is not quite correct. One also needs to impose the condition of elements of V that $f'(0) = f'(2\pi)$.
In that case, applying integration by parts twice shows that $\int_0^{2\pi} f''(x)g(x)dx = \int_0^{2\pi} f(x)g''(x)dx$, as needed.

4. Let $A \in M(n \times n, \mathbb{R})$ be a symmetric matrix such that all eigenvalues of A are positive. Show that $< v, w >_A := v^t A w$ defines an inner product on $\mathbb{R}^n$. Conversely, given an inner product $< \, , \, >$ on $\mathbb{R}^n$, show that there exists a symmetric matrix A with all eigenvalues positive such that $< v, w > = v^t A w$ for all v and w in $\mathbb{R}^n$. (4 pts)

It is obvious that $< -, - >_A$ is bilinear and symmetric. Choose an orthonormal (with respect to the standard inner product) basis $\{v_1, \dots, v_n\}$ consisting of eigenvectors of A. This is possible because A is self-adjoint. Let $\lambda_i > 0$ be the eigenvalue for v_i; it is positive by assumption. Now let $W = \sum_{i=1}^n a_i v_i \in \mathbb{R}^n$ be an arbitrary column vector. (Note that all w can be expressed in terms of the v_i because the latter form a basis.)
Then

$$< w, w >_A = \sum_{i=1}^n a_i^2 \lambda_i \geq 0$$

and this is equal to zero if and only if $a_i = 0$ for all i, that is, if and only if $W = 0$. Hence, $< -, - >_A$ is positive definite, therefore, and inner product.

Now assume conversely that $< -, - >$ is some inner product on $\mathbb{R}^n$. Let $a_{ij} =< e_j, e_i >$ and let A be the matrix with entries a_{ij}. Since $< -, - >$ is a real inner product, it is symmetric, therefore $a_{ij} = a_{ji}$ for all i and j, so that A is a symmetric matrix.
Obviously, $e_j^t A e_i = a_{ij} =< e_j, e_i >$, so $< -, - > = < -, - >_A$ on the basis vectors. Because both are bilinear, they are equal. Finally we saw above that $< -, - >_A$ is positive definite if and only if all eigenvalues of A are positive; since $< -, - > = < -, - >_A$ is an inner product by assumption and hence positive definite, we conclude that A is a symmetric matrix with positive eigenvalues, as asserted.

5. Let V be an inner product space, and let $U \subseteq V$ and $W \subseteq V$ be two subspaces. Suppose for all $u \in U$ and $w \in W$, we have $< u, w > = 0$. Show that $U \cap W = \{0\}$. (4 pts)

Let $x \in U \cap W$. Then $< x, x > = 0$, therefore $x = 0$.