Math 115A **Homework 5** **Due December 5, 2008**

1. Given the field F, the F-vector space V prove or disprove that the given function

$$< \, , \, >: V \times V \to F$$

is an inner product. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^\infty$ the space of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers such that $a_n = 0$ for all but finitely many n, and

$$< (a_n), (b_n) >= \sum_{n=0}^{\infty} a_n b_n.$$

b) $F = \mathbb{C}$, $V = \mathcal{P}_3$ the space of complex polynomials of degree at most 3, and

$$< f, g >= f(0)\overline{g(0)} + f(1)\overline{g(1)} + f(2)\overline{g(2)} + f(3)\overline{g(3)}.$$

2. Let V be a finite-dimensional inner product space and let $S : V \to V$ and $T : V \to V$ be two self-adjoint linear transformations. Assume $S \circ T = T \circ S$. Show that S and T are simultaneously diagonalizable, that is, there exists a basis $\mathcal{B}$ of V such that each $v \in \mathcal{B}$ is an eigenvector for both S and T. (You may use the theorem that every self-adjoint transformation is diagonalizable.) (4 pts)

3. Let $V = \{f \in \mathbf{C}^\infty(\mathbb{R}) | f(x) = f(x + 2\pi)\}$ be the real inner product space of 2π-periodic infinitely differentiable real-valued functions with inner product $< f, g >= \int_0^{2\pi} f(x)g(x)dx$. Let $\Delta : V \to V$ be the linear transformation given by $\Delta(f)(x) = f''(x)$. Show that Δ is self-adjoint. (*Remark:* This is the one-dimensional version of the Laplace operator, a very important self-adjoint differential operator.) (4 pts)

4. Let $A \in M(n \times n, \mathbb{R})$ be a symmetric matrix such that all eigenvalues of A are positive. Show that $< v, w >_A := v^t A w$ defines an inner product on $\mathbb{R}^n$. Conversely, given an inner product $< \, , \, >$ on $\mathbb{R}^n$, show that there exists a symmetric matrix A with all eigenvalues positive such that $< v, w >= v^t A w$ for all v and w in $\mathbb{R}^n$. (4 pts)

5. Let V be an inner product space, and let $U \subseteq V$ and $W \subseteq V$ be two subspaces. Suppose for all $u \in U$ and $w \in W$, we have $< u, w >= 0$. Show that $U \cap W = \{0\}$. (4 pts)