Math 115A **Homework 4** **Due November 21st, 2008**

1. Given the field F, the F-vector space V and the linear transformation $T : V \to V$, compute all the eigenvalues and eigenvectors of T. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbf{C}^\infty(\mathbb{R})$ the space of infinitely differentiable functions, and $T : V \to V$ the second derivative, that is, $T(f)(x) = f''(x)$.

All real numbers occur as eigenvalues. If $\lambda > 0$, then the eigenvectors with eigenvalue λ are just functions of the form $c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ for scalars c_1 and c_2. If $\lambda = 0$, then the eigenvectors are linear polynomials. Finally, if $\lambda < 0$, then the eigenvectors are functions of the form $c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$, for scalars c_1 and c_2. (Note the eigenspaces are all of dimension two, but they look very differently depending on whether λ is positive, 0, or negative.)

b) $F = \mathbb{C}$, $V = \mathcal{P}_3$ the space of complex polynomials of degree at most 3, and $T : V \to V$ given by $T(f)(z) = f(z) + f'(z)$.

The only eigenvalue is 1. The eigenspace E_1 consists of the constant polynomials.

2. Let F be a field, V an F-vector space and $T : V \to V$ a linear transformation such that there exists a natural number k with $T^k = 0$. Determine all the eigenvalues of T. (*Remark:* Such a linear transformation is called nilpotent.) (4 pts)

If $\lambda \in F$ is an eigenvalue of T, then λ^k is an eigenvalue of T^k. Since $T^k = 0$, we have that $\lambda^k = 0$; because F is a field, this implies $\lambda = 0$. Therefore the only possible eigenvalue of T is 0.
I claim that 0 actually occurs as an eigenvalue, that is, that $N(T) \neq \{0\}$. Indeed, let n be the minimal positive integer such that $T^n = 0$. Then there exists $v \in V$ such that $T^{n-1}(v) \neq 0$ (otherwise n wasn't minimal). On the other hand, $T(T^{n-1}(v)) = T^n(v) = 0$, so that $T^{n-1}(v) \in N(T)$ is an eigenvector with eigenvalue 0.

3. Let F be a field, V an F-vector space, and $S : V \to V$ and $T : V \to V$ two linear transformations such that $S \circ T = T \circ S$. Suppose $\lambda \in F$ is an eigenvalue of S and $v \in V_\lambda = \{v \in V | S(v) = \lambda v\}$. Show that $T(v) \in V_\lambda$. Assuming in addition that $\dim_F(V_\lambda) = 1$ and $v \neq 0$, show that v is an eigenvector of T. (4 pts)

First, let $v \in V_\lambda$. then $S(T(v)) = T(S(v)) = T(\lambda v) = \lambda T(v)$, so $T(v) \in V_\lambda$.
Now suppose V_λ has dimension 1. If $v \in V_\lambda$ is a non-zero element, then $\{v\}$ is a basis of V_λ. We just proved that $T(v) \in V_\lambda$. Since $\{v\}$ is a basis of that space, there exists a scalar $\mu \in F$ such that $T(v) = \mu v$. (That is, we can express $T(v)$ as a linear combination of the single vector v, which means, it is a multiple of that vector.) Since $v \neq 0$, it is an eigenvector of T with eigenvalue μ.

4. Let $A \in M(n \times n, F)$. Using the properties of the determinant from section 4.4. of the textbook, prove that the characteristic polynomial $p_A(\lambda)$ is a polynomial of degree n in the variable λ with constant term $\det(A)$. (4 pts)

First we prove the following claim.

Claim: Let B be a square matrix whose entries are linear functions $b_{i,j} - c_{i,j}\lambda$ with $b_{i,j}$ and $c_{i,j}$ in F. Let m be the numbers of pairs (i,j) such that $c_{i,j} \neq 0$ (that is, the numbers of λ's appearing in the matrix), and call this number the *degree* of B. Then $\det(B)$ is a polynomial in λ over F of degree at most m.
Proof: If $m = 0$, then $\det(B)$ is itself a scalar, that is, a polynomial of degree 0. We proceed by induction on m. Suppose now that $m > 0$ and we have proved the claim for all square matrices B' as above of degree less than m. Choose a pair (i,j) such that $c_{i,j} \neq 0$. Expand the determinant of B using the i-th row; the result is an alternating sum of products $(b_{i,k} - c_{i,k}\lambda)\det(B_{i,k})$ where the matrices $B_{i,k}$ are obtained from B by deleting the i-th row and k-th column. Clearly, each of the matrices $B_{i,k}$ has degree at most $m-1$ (since one of the entries deleted was the non-constant entry $b_{i,j} - c_{i,j}\lambda$); by induction, then, the term $(b_{i,k} - c_{i,k}\lambda)\det(B_{i,k})$ is a polynomial of degree at most $1 + (m-1) = m$. Since a sum of polynomials of degree at most m is a polynomial of degree at most m, this proves the claim.

Now we are going to show that $p_A(\lambda)$ is a polynomial of degree precisely n. This is certainly the case if $n = 1$. We again proceed by induction on n. Assume $n > 1$ and we have proved the claim for all matrices $A' \in M((n-1)\times(n-1), F)$. Expanding the determinant of $A - \lambda I_n$ using the first row, we see that the function $p_A(\lambda)$ is equal to $f_1(\lambda) = (a_{1,1} - \lambda)\det(A_{1,1} - \lambda I_{n-1})$ plus an alternating sum $\sum_{k=2}^{n}(-1)^{k-1} f_k(\lambda)$ of terms of the form $f_k(\lambda) = a_{1,k}\det(B_{1,k})$, where the matrix $B_{1,k}$ is obtained from $A - \lambda I_n$ by deleting the first row and k-th column.
Obviously, $B_{1,k}$ has degree at most $(n-1)$; therefore the functions $f_k(\lambda)$ are polynomials of degree at most $n-1$, by the **Claim**. On the other hand, $f_1(\lambda) = (a_{1,1} - \lambda)p_{A_{1,1}}(\lambda)$, a product of a linear term with function that is, by induction, a polynomial of degree equal to $n-1$; thay is, f_1 is a polynomial of degree n.
We conclude that $p_A(\lambda)$ is a sum of a polynomial of degree n and a polynomial of degree at most $n-1$, and is therefore a polynomial of degree equal to n, as asserted.
Finally, we compute the constant term by simply setting $\lambda = 0$.

5. Let V be a finite-dimensional $\mathbb{R}$-vector space and let $T : V \to V$ be a linear involution, that is, a linear transformation such that $T^2 = \mathbf{id}_V$. What are the possible eigenvalues of V? Show that there is a basis of V consisting of eigenvectors of T. (*Hint:* Build, inductively, a basis S of V such that for any $v \in S$ $T(v) \in S$ or $-T(v) \in S$. Then show that such a basis can be modified to obtain a basis consisting of eigenvectors.) (*Remark:* The same argument works over any field F with the property that $1 + 1 \neq 0$.) (4 pts)

Note that clearly the only possible eigenvalues of T are 1 and -1. Proceeding from here, I will present two proofs of the assertion that T is diagonalizable.

First proof: Let $n = \dim_F(V)$. We prove by induction on k that for each $0 \leq k \leq n$, there is a linearly independent set $S_k = \{v_1, \ldots, v_k\}$ in V consisting of eigenvectors of T. Letting $k = n$, we then get a basis consisting of eigenvectors, showing that T is diagonalizable.
Let $k = 0$. Obviously, the empty set is linearly independent and consists of eigenvectors. This is the base step for our induction.
Now suppose $k < n$ and we have constructed $S_k = \{v_1, \ldots, v_k\}$. Since $k < n$, we can find a vector $v \in V$ such that $v \notin Span(S_k)$. Let $w = T(v)$. Since $T^2 = id$, we have that $v = T(w)$. Hence, we have the following two obvious identities:

$$T(v + w) = v + w$$

and

$$T(v - w) = -(v - w).$$

Suppose both $v+w$ and $v-w$ are in the span of S_k.
Then so ist $v=(1/2)((v+w)+(v-w))$, contradicting our choice of v. Now set $v_{k+1}=v+w$ if $v+w$ is not in the span of S_k, and $v_{k+1}=v-w$ otherwise. In either case v_{k+1} is not in the span of S_k and in particular, is non-zero. By the above identities, v_{k+1} is therefore an eigenvector of T, and $S_{k+1}=\{v_1,\ldots,v_k,v_{k+1}\}$ is a linearly independent set, consisting of eigenvectors. This finishes the inductive step and therefore the proof.

Second proof: Write E_1 and E_{-1} for the respective eigenspaces with eigenvalues 1 and -1, and let S_1 and S_{-1} be bases of the respective eigenspaces. Let $v\in V$, and set $w=T(v)$. Then $v+w\in E_1$, $v-w\in E_{-1}$ (by the identities from the first proof) and $v=(1/2)(v+w)+(1/2)(v-w)$. That is, every vector in V is a sum of vectors in E_1 and E_{-1}. Therefore $S=S_1\cup S_{-1}$ spans V. Since S_1 and S_{-1} are bases of eigenspaces for distinct eigenvalues, the set S is also linearly independent. Hence, S is a basis of V consisting of eigenvectors of T, so that T is diagonalizable.