1. Given the field $F$, the $F$-vector space $V$ and the linear transformation $T : V \to V$, compute all the eigenvalues and eigenvectors of $T$. (2 pts each)

a) $F = \mathbb{R}$, $V = C^\infty(\mathbb{R})$ the space of infinitely differentiable functions, and $T : V \to V$ the second derivative, that is, $T(f)(x) = f''(x)$.

All real numbers occur as eigenvalues. If $\lambda > 0$, then the eigenvectors with eigenvalue $\lambda$ are just functions of the form $c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ for scalars $c_1$ and $c_2$. If $\lambda = 0$, then the eigenvectors are linear polynomials. Finally, if $\lambda < 0$, then the eigenvectors are functions of the form $c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$, for scalars $c_1$ and $c_2$. (Note the eigenspaces are all of dimension two, but they look very differently depending on whether $\lambda$ is positive, 0, or negative.)

b) $F = \mathbb{C}$, $V = P_3$ the space of complex polynomials of degree at most 3, and $T : V \to V$ given by $T(f)(z) = f(z) + f'(z)$.

The only eigenvalue is 1. The eigenspace $E_1$ consists of the constant polynomials.

2. Let $F$ be a field, $V$ an $F$-vector space and $T : V \to V$ a linear transformation such that there exists a natural number $k$ with $T^k = 0$. Determine all the eigenvalues of $T$. (Remark: Such a linear transformation is called nilpotent.) (4 pts)

If $\lambda \in F$ is an eigenvalue of $T$, then $\lambda^k$ is an eigenvalue of $T^k$. Since $T^k = 0$, we have that $\lambda^k = 0$; because $F$ is a field, this implies $\lambda = 0$. Therefore the only possible eigenvalue of $T$ is 0.

I claim that 0 actually occurs as an eigenvalue, that is, that $N(T) \neq \{0\}$. Indeed, let $n$ be the minimal positive integer such that $T^n = 0$. Then there exists $v \in V$ such that $T^{n-1}(v) \neq 0$ (otherwise $n$ wasn’t minimal). On the other hand, $T(T^{n-1}(v)) = T^n(v) = 0$, so that $T^{n-1}(v) \in N(T)$ is an eigenvector with eigenvalue 0.

3. Let $F$ be a field, $V$ an $F$-vector space, and $S : V \to V$ and $T : V \to V$ two linear transformations such that $S \circ T = T \circ S$. Suppose $\lambda \in F$ is an eigenvalue of $S$ and $v \in V_\lambda = \{v \in V | S(v) = \lambda v\}$. Show that $T(v) \in V_\lambda$. Assuming in addition that $\dim_F(V_\lambda) = 1$ and $v \neq 0$, show that $v$ is an eigenvector of $T$. (4 pts)

First, let $v \in V_\lambda$. then $S(T(v)) = T(S(v)) = T(\lambda v) = \lambda T(v)$, so $T(v) \in V_\lambda$.

Now suppose $V_\lambda$ has dimension 1. If $v \in V_\lambda$ is a non-zero element, then $\{v\}$ is a basis of $V_\lambda$. We just proved that $T(v) \in V_\lambda$. Since $\{v\}$ is a basis of that space, there exists a scalar $\mu \in F$ such that $T(v) = \mu v$. (That is, we can express $T(v)$ as a linear combination of the single vector $v$, which means, it is a multiple of that vector.) Since $v \neq 0$, it is an eigenvector of $T$ with eigenvalue $\mu$.

4. Let $A \in M(n \times n, F)$. Using the properties of the determinant from section 4.4. of the textbook, prove that the characteristic polynomial $p_A(\lambda)$ is a polynomial of degree $n$ in the variable $\lambda$ with constant term $\det(A)$. (4 pts)

First we prove the following claim.
Claim: Let $B$ be a square matrix whose entries are linear functions $b_{i,j} - c_{i,j}\lambda$ with $b_{i,j}$ and $c_{i,j}$ in $F$. Let $m$ be the numbers of pairs $(i, j)$ such that $c_{i,j} \neq 0$ (that is, the numbers of $\lambda$'s appearing in the matrix), and call this number the degree of $B$. Then $\det(B)$ is a polynomial in $\lambda$ over $F$ of degree at most $m$.

Proof: If $m = 0$, then $\det(B)$ is itself a scalar, that is, a polynomial of degree 0. We proceed by induction on $m$. Suppose now that $m > 0$ and we have proved the claim for all square matrices $B'$ as above of degree less than $m$. Choose a pair $(i, j)$ such that $c_{i,j} \neq 0$. Expand the determinant of $B$ using the $i$-th row; the result is an alternating sum of products $(b_{i,k} - c_{i,k}\lambda)\det(B_{i,k})$ where the matrices $B_{i,k}$ are obtained from $B$ by deleting the $i$-th row and $k$-th column. Clearly, each of the matrices $B_{i,k}$ has degree at most $m - 1$ (since one of the entries deleted was the non-constant entry $b_{i,j} - c_{i,j}\lambda$); by induction, then, the term $(b_{i,k} - c_{i,k}\lambda)\det(B_{i,k})$ is a polynomial of degree at most $1 + (m - 1) = m$. Since a sum of polynomials of degree at most $m$ is a polynomial of degree at most $m$, this proves the claim.

Now we are going to show that $p_A(\lambda)$ is a polynomial of degree precisely $n$. This is certainly the case if $n = 1$. We again proceed by induction on $n$. Assume $n > 1$ and we have proved the claim for all matrices $A' \in M((n-1)\times(n-1), F)$. Expanding the determinant of $A - \lambda I_n$ using the first row, we see that the function $p_A(\lambda)$ is equal to $f_1(\lambda) = (a_{1,1} - \lambda)\det(A_{1,1} - \lambda I_{n-1})$ plus an alternating sum $\sum_{k=2}^{n}(\lambda - a_{1,k})\det(B_{1,k})$ where the matrix $B_{1,k}$ is obtained from $A - \lambda I_n$ by deleting the first row and $k$-th column. Obviously, $B_{1,k}$ has degree at least $n - 1$; therefore the functions $f_k(\lambda)$ are polynomials of degree at most $n - 1$, by the Claim. On the other hand, $f_1(\lambda) = (a_{1,1} - \lambda)p_{A_{1,1}}(\lambda)$, a product of a linear term with function that is, by induction, a polynomial of degree equal to $n - 1$; that is, $f_1$ is a polynomial of degree $n$.

We conclude that $p_A(\lambda)$ is a sum of a polynomial of degree $n$ and a polynomial of degree at most $n - 1$, and is therefore a polynomial of degree equal to $n$, as asserted. Finally, we compute the constant term by simply setting $\lambda = 0$.

5. Let $V$ be a finite-dimensional $\mathbb{R}$-vector space and let $T : V \to V$ be a linear involution, that is, a linear transformation such that $T^2 = \text{id}_V$. What are the possible eigenvalues of $V$?
Show that there is a basis of $V$ consisting of eigenvectors of $T$. (Hint: Build, inductively, a basis $S$ of $V$ such that for any $v \in S$ $T(v) \in S$ or $-T(v) \in S$. Then show that such a basis can be modified to obtain a basis consisting of eigenvectors.) (Remark: The same argument works over any field $F$ with the property that $1 + 1 \neq 0$.) (4 pts)

Note that clearly the only possible eigenvalues of $T$ are 1 and $-1$. Proceeding from here, I will present two proofs of the assertion that $T$ is diagonalizable.

First proof: Let $n = \dim_F(V)$. We prove by induction on $k$ that for each $0 \leq k \leq n$, there is a linearly independent set $S_k = \{v_1, \ldots, v_k\}$ in $V$ consisting of eigenvectors of $T$. Letting $k = n$, we then get a basis consisting of eigenvectors, showing that $T$ is diagonalizable. Let $k = 0$. Obviously, the empty set is linearly independent and consists of eigenvectors. This is the base step for our induction.

Now suppose $k < n$ and we have constructed $S_k = \{v_1, \ldots, v_k\}$. Since $k < n$, we can find a vector $v \in V$ such that $v \notin \text{Span}(S_k)$. Let $w = T(v)$. Since $T^2 = \text{id}$, we have that $v = T(w)$. Hence, we have the following two obvious identities:

$$T(v + w) = v + w$$

and

$$T(v - w) = -(v - w).$$
Suppose both $v + w$ and $v - w$ are in the span of $S_k$. Then so is $v = (1/2)((v + w) + (v - w))$, contradicting our choice of $v$. Now set $v_{k+1} = v + w$ if $v + w$ is not in the span of $S_k$, and $v_{k+1} = v - w$ otherwise. In either case $v_{k+1}$ is not in the span of $S_k$ and in particular, is non-zero. By the above identities, $v_{k+1}$ is therefore an eigenvector of $T$, and $S_{k+1} = \{v_1, \ldots, v_k, v_{k+1}\}$ is a linearly independent set, consisting of eigenvectors. This finishes the inductive step and therefore the proof.

**Second proof:** Write $E_1$ and $E_{-1}$ for the respective eigenspaces with eigenvalues 1 and $-1$, and let $S_1$ and $S_{-1}$ be bases of the respective eigenspaces. Let $v \in V$, and set $w = T(v)$. Then $v + w \in E_1$, $v - w \in E_{-1}$ (by the identities from the first proof) and $v = (1/2)(v + w) + (1/2)(v - w)$. That is, every vector in $V$ is a sum of vectors in $E_1$ and $E_{-1}$. Therefore $S = S_1 \cup S_{-1}$ spans $V$. Since $S_1$ and $S_{-1}$ are bases of eigenspaces for distinct eigenvalues, the set $S$ is also linearly independent. Hence, $S$ is a basis of $V$ consisting of eigenvectors of $T$, so that $T$ is diagonalizable.