Math 115A  Homework 4  Due November 21st, 2008

1. Given the field \( F \), the \( F \)-vector space \( V \) and the linear transformation \( T : V \to V \), compute all the eigenvalues and eigenvectors of \( T \). (2 pts each)

   a) \( F = \mathbb{R} \), \( V = C^\infty(\mathbb{R}) \) the space of infinitely differentiable functions, and \( T : V \to V \) the second derivative, that is, \( T(f)(x) = f''(x) \).

   b) \( F = \mathbb{C} \), \( V = P_3 \) the space of complex polynomials of degree at most 3, and \( T : V \to V \) given by \( T(f)(z) = f(z) + f'(z) \).

2. Let \( F \) be a field, \( V \) an \( F \)-vector space and \( T : V \to V \) a linear transformation such that there exists a natural number \( k \) with \( T^k = 0 \). Determine all the eigenvalues of \( T \). (Remark: Such a linear transformation is called nilpotent.) (4 pts)

3. Let \( F \) be a field, \( V \) an \( F \)-vector space, and \( S : V \to V \) and \( T : V \to V \) two linear transformations such that \( S \circ T = T \circ S \). Suppose \( \lambda \in F \) is an eigenvalue of \( S \) and \( v \in V_\lambda = \{v \in V | S(v) = \lambda v \} \). Show that \( T(v) \in V_\lambda \). Assuming in addition that \( \dim_F(V_\lambda) = 1 \) and \( v \neq 0 \), show that \( v \) is an eigenvector of \( T \). (4 pts)

4. Let \( A \in M(n \times n, F) \). Using the properties of the determinant from section 4.4. of the textbook, prove that the characteristic polynomial \( p_A(\lambda) \) is a polynomial of degree \( n \) in the variable \( \lambda \) with constant term \( \det(A) \). (4 pts)

5. Let \( V \) be a finite-dimensional \( \mathbb{R} \)-vector space and let \( T : V \to V \) be a linear involution, that is, a linear transformation such that \( T^2 = \text{id}_V \). What are the possible eigenvalues of \( V \)? Show that there is a basis of \( V \) consisting of eigenvectors of \( T \). (Remark: The same argument works over any field \( F \) with the property that \( 1 + 1 \neq 0 \).) (4 pts)