Math 115A **Homework 4** **Due November 21st, 2008**

1. Given the field F, the F-vector space V and the linear transformation $T : V \to V$, compute all the eigenvalues and eigenvectors of T. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbf{C}^\infty(\mathbb{R})$ the space of infinitely differentiable functions, and $T : V \to V$ the second derivative, that is, $T(f)(x) = f''(x)$.

b) $F = \mathbb{C}$, $V = \mathcal{P}_3$ the space of complex polynomials of degree at most 3, and $T : V \to V$ given by $T(f)(z) = f(z) + f'(z)$.

2. Let F be a field, V an F-vector space and $T : V \to V$ a linear transformation such that there exists a natural number k with $T^k = 0$. Determine all the eigenvalues of T. (*Remark:* Such a linear transformation is called nilpotent.) (4 pts)

3. Let F be a field, V an F-vector space, and $S : V \to V$ and $T : V \to V$ two linear transformations such that $S \circ T = T \circ S$. Suppose $\lambda \in F$ is an eigenvalue of S and $v \in V_\lambda = \{v \in V | S(v) = \lambda v\}$. Show that $T(v) \in V_\lambda$. Assuming in addition that $\dim_F(V_\lambda) = 1$ and $v \neq 0$, show that v is an eigenvector of T. (4 pts)

4. Let $A \in M(n \times n, F)$. Using the properties of the determinant from section 4.4. of the textbook, prove that the characteristic polynomial $p_A(\lambda)$ is a polynomial of degree n in the variable λ with constant term $\det(A)$. (4 pts)

5. Let V be a finite-dimensional $\mathbb{R}$-vector space and let $T : V \to V$ be a linear involution, that is, a linear transformation such that $T^2 = \mathbf{id}_V$. What are the possible eigenvalues of V? Show that there is a basis of V consisting of eigenvectors of T. (*Hint:* Build, inductively, a basis S of V such that for any $v \in S$ $T(v) \in S$ or $-T(v) \in S$. Then show that such a basis can be modified to obtain a basis consisting of eigenvectors.) (*Remark:* The same argument works over any field F with the property that $1 + 1 \neq 0$.) (4 pts)