Math 115A **Homework 3** **Due November 7th, 2008**

1. Let $Id : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity linear transformation. For the ordered bases $\mathcal{A}$, $\mathcal{B}$, compute the matrix representation $[Id]^{\mathcal{B}}_{\mathcal{A}}$. (To be clear, the bases are ordered by reading from left to right.) (2 pts each)

a) $\mathcal{A} = \{2e_2, e_3 - e_1, e_1\}$ and $\mathcal{B} = \{e_1, 2e_2, e_3 - e_1\}$.

b) $\mathcal{A} = \{e_1, e_1 + e_2, e_1 + e_2 + e_3\}$ and $\mathcal{B} = \{2e_1 - 3e_2, e_3 + e_2, 4e_2\}$.

2. Prove: If F is a field, V is an F-vector space of dimension n, and $\mathcal{A}$ is any ordered basis of V, then the matrix representation $[Id]^{\mathcal{A}}_{\mathcal{A}}$ of the identity linear transformation is the $n \times n$-identity matrix I_n. (4 pts)

This is essentially obvious. Let $\mathcal{A} = \{v_1, \ldots, v_n\}$. The i-th column of the matrix $[Id]^{\mathcal{A}}_{\mathcal{A}}$ is the column vector $[Id(v_i)]_{\mathcal{A}} = e_i$. That is precisely the i-th column of the identity matrix.

3. Prove: Let F be a field and let V be an F-vector space of dimension n. Suppose $T : V \to V$ is a linear transformation of rank n. Then there are ordered bases $\mathcal{A}$ and $\mathcal{B}$ of V such that $[T]^{\mathcal{B}}_{\mathcal{A}} = I_n$ is the $n \times n$-identity matrix. (4 pts)

Choose any ordered basis $\mathcal{A} = \{v_1, \ldots, v_n\}$. The set $\{T(v_1), \ldots, T(v_n)\}$ spans the range of T, which is V; since V has dimension n, $\mathcal{B} = \{T(v_1), \ldots, T(v_n)\}$ is an ordered basis of V. The i-th column of $[T]^{\mathcal{B}}_{\mathcal{A}}$ is $[T(v_i)]_{\mathcal{B}} = e_i$, hence $[T]^{\mathcal{B}}_{\mathcal{A}} = I_n$.

4. Let F be a field and let V and W be F-vector spaces. Suppose $v \in V$. Prove that the map $E_v : \mathcal{L}(V, W) \to W$ defined by $E_v(T) = T(v)$ is a linear transformation. Moreover, show that E_v is onto provided $v \neq 0$. (You may assume that V has finite dimension.) (4 pts)

Let S and T be linear transformations from V to W and let $\lambda \in F$ be a scalar. Then

$$E_v(S + \lambda T) = (S + \lambda T)(v) = S(v) + \lambda T(v) = E_v(S) + \lambda E_v(T)$$

where the first equality is by definition of the addition and scalar multiplication on $\mathcal{L}(V, W)$. That is, E_v is linear.
Now suppose $v \neq 0$. Suppose $w \in W$. Let $S \subseteq V$ be any basis of V. By the replacement theorem, we can find a subset $T \subseteq S$ such that $B := T \cup \{v\}$ is a basis of V. Now there is a unique linear transformation $T : V \to W$ such that $T(v) = w$ and $T(v') = 0$ for $v' \in T$. But for such a T, we have $E_v(T) = T(v) = w$. Since $w \in W$ was arbitrary, this means that E_v is onto.

5. Let F be a field and V an F-vector space. The vector space $\mathcal{L}(V, F)$ is called the *dual vector space* of V and also written V^*. Prove that $\dim_F(V) = \dim_F(V^*)$ if V is finite dimensional. (4 pts)

Choose an ordered basis $\mathcal{A}$ of V and an ordered basis (i.e., a non-zero element) $\mathcal{B}$ of F. Then the matrix representation $[\ ]^{\mathcal{B}}_{\mathcal{A}} : V^* \to M(1 \times n, F)$ is an isomorphism. The latter space is simply the space of row vectors of length n, which has dimension n. Therefore V^* also has dimension n. (*Remark:* If V is not finite-dimensional, then there is no isomorphism between V and V^* - the dual vector space is "larger" than the vector space.)