Math 115A **Homework 2** **Due October 24th, 2008**

1. For each of the following maps $T : V \to W$, check if T is an F-linear transformation. If yes, write out a proof. If not, prove that. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^3$, $W = \mathbb{R}^2$ and $T(a_1, a_2, a_3) = (a_1a_2, a_3 + 2a_1)$.

Let $\lambda = 2 \in \mathbb{R}$ and $\mathbf{a} = (a_1, a_2, a_3) = (1, 1, 1)$. Then $T(\lambda\mathbf{a}) = T(2, 2, 2) = (4, 6)$ but $\lambda T(\mathbf{a}) = (2, 6)$. That is, T does not commute with scalar multiplication and is therefore not a linear transformation.

b) $F = \mathbb{C}$, $V = \mathbb{C}^2$, $W = \mathbb{C}$ and $T(z, w) = \overline{z}$.

Let $\lambda = i \in \mathbb{C}$ and $\mathbf{v} = (1, 0) \in V$. Then $T(\lambda\mathbf{v}) = -i$ but $\lambda T(\mathbf{v}) = i$. That is, T does not commute with scalar multiplication and is therefore not a linear transformation.

c) $F = \mathbb{R}$, $V = \mathbf{C}(\mathbb{R})$ the set of continuous real-valued functions on the real numbers, $W = \mathbb{R}$ and $T(f) = \int_0^1 x^2 f(x)dx$.

The map $U : V \to V$ that is defined by $U(f)(x) = x^2 f(x)$ is a linear transformation because multiplication of real numbers is commutative, and multiplication and addition are distributive. Moreover, the map $S : V \to W = \mathbb{R}$ defined by $S(f) = \int_0^1 f(x)dx$ is linear (as shown in calculus). Therefore $T = S \circ U$ is a composite of linear transformations and as such, is a linear transformation.

2. Let $\mathbb{R}^+$ be the set of positive real numbers. (**Warning:** Positive means just that - zero is not in that set.) Show that there is a structure of an $\mathbb{R}$-vector space on $\mathbb{R}^+$ such that the map $\mathbf{exp} : \mathbb{R} \to \mathbb{R}^+$ defined by $\mathbf{exp}(x) = e^x$ is a linear transformation. That is, exhibit an "addition" and "scalar multiplication" on $\mathbb{R}^+$ satisfying the vector space axioms and show that with these definitions, **exp** is $\mathbb{R}$-linear. (3 pts)

We define an addition on $\mathbb{R}^+$ by setting $x \boxplus y := xy$, and a scalar multiplication by $\lambda \boxdot x := x^\lambda$. Since multiplication of real numbers is comutative and associative, so is the addition $\boxplus$. The associativity for scalar multiplication holds because $(x^a)^b = x^{(ab)}$. Distributivity follows from the fact that $(xy)^a = x^a y^a$ and $x^{(a+b)} = x^a x^b$. The zero element is 1, and the negative $\boxminus x$ of $x \in \mathbb{R}^+$ is x^{-1}.
Finally, it follows from the exponential laws that **exp** is a linear transformation with respect to the usual $\mathbb{R}$-vector space structure on $\mathbb{R}$ and the vector space structure on $\mathbb{R}^+$ just defined.

3. Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are F-linear transfomations. (3 pts)

a) Prove: if the Null space of T_1 and the null space of T_2 are both zero, then the Null space of $T_2 \circ T_1$ is zero.

Suppose $x \in N(T_2 \circ T_1)$. Then $T_1(x) \in N(T_2) = \{0\}$, so $T_1(x) = 0$. This means that $x \in N(T_1) = \{0\}$. Since x was an arbitrary element of $N(T_2 \circ T_1)$, this proves that $N(T_2 \circ T_1) = \{0\}$, as asserted.

b) Show by example that the converse statement to a) does not hold.

The converse of the statement in a) is: "If the Null space of $T_2 \circ T_1$ is zero, then the Null space of T_1 and the Null space of T_2 are both zero".

Let $U = \{0\}$, $V = F$ (or any non-zero F-vector space), $W = \{0\}$ and T_1 and T_2 both the zero linear transformation. Then the Null space of $T_2 \circ T_1$ is $\{0\}$, simply because it is a subspace of V. On the other hand, the Null space of T_2 is equal to V, which is not the zero vector space.

4. Let $T : V \to W$ be an F-linear transformation of finite-dimensional vector spaces with Null space $N(T)$, and let $n = \dim(V)$. Prove that there is a basis $\{v_1, \ldots, v_n\}$ of V and an $r \in \{0, \ldots, n\}$ such that $T(v_i) = 0$ for $1 \leq i \leq r$ and $\{T(v_{r+1}), T(v_{r+2}), \ldots, T(v_n)\} \subseteq W$ is linearly independent. (4 pts)

Let $r = \dim_F(N(T))$. Choose a basis $\{v_1, \ldots, v_r\}$ of $N(T)$. Using the replacement theorem, we can complete this to a basis $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of V.
I claim that the set $\{T(v_{r+1}), \ldots, T(v_n)\} \subseteq W$ is linearly independent. In fact, suppose we have $a_{r+1}, \ldots a_n \in F$ such that the linear combination $a_{r+1}T(v_{r+1}) + \cdots + a_nT(v_n) = 0 \in W$. Since T is linear, this implies that $T(a_{r+1}v_{r+1} + \cdots + a_nv_n) = 0 \in W$, so that $a_{r+1}v_{r+1} + \cdots + a_nv_n \in N(T)$. Because $\{v_1, \ldots, v_r\}$ spans $N(T)$, this means there are $a_1, \ldots, a_r \in F$ such that

$$a_{r+1}v_{r+1} + \cdots + a_nv_n = a_1v_1 + \cdots + a_rv_r,$$

or equivalently, such that

$$(-a_1)v_1 + \cdots + (-a_r)v_r + a_{r+1}v_{r+1} + \cdots + a_nv_n = 0 \in V.$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent, this implies that $a_1 = \cdots = a_n = 0$; in particular, $a_{r+1} = \cdots = a_n = 0$. That is, the set $\{T(v_{r+1}), \ldots, T(v_n)\} \subseteq W$ is linearly independent, as asserted.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $T(e_1) = e_2$, $T(e_2) = e_3$ and $T(e_3) = e_1$. Write down the matrix for this transformation with respect to the basis $\{e_1 + e_2, e_2, e_3 - e_1\}$. (4 pts)

I'll leave this as an exercise.