1. For each of the following, check if \( W \subseteq V \) is an \( F \)-subspace of the \( F \)-vector space \( V \). If yes, write out a proof. If not, prove that. (2 pts each)

a) \( F = \mathbb{R}, V = \mathbb{R}^3 \) and \( W = \{v = (v_1, v_2, v_3) \in V | 2v_1 - v_2 = 1\} \).

Since \( 2 \cdot 0 - 0 \neq 1 \), \( (0, 0, 0) \notin W \), so \( W \) is not a subspace.

b) \( F = \mathbb{Q}, V = \mathbb{R} \) and \( W = \{x \in \mathbb{R} | x \sqrt{2} \in \mathbb{Q}\} \).

Take \( x \in W \) and \( y \in W \), and let \( \lambda \in \mathbb{Q} \). Then \( \sqrt{2}(x + y) = \sqrt{2}x + \sqrt{2}y \in \mathbb{Q} \), so \( x + y \in W \). Moreover, \( \sqrt{2}(\lambda x) = \lambda(\sqrt{2}x) \in \mathbb{Q} \), so that \( \lambda x \in W \). Finally, \( \sqrt{2} \cdot 0 = 0 \in \mathbb{Q} \), whence \( 0 \in W \). That is, \( W \) is a subspace.

c) \( F = \mathbb{R}, V = C(\mathbb{R}) \) the set of continuous real-valued functions on the real numbers and \( W = \{f \in V | \int_0^1 f(x)dx = 0\} \).

Let \( f \in W \) and \( g \in W \), and let \( \lambda \in \mathbb{R} \). Then \( \int_0^1 (f + g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0 + 0 = 0 \), so that \( f + g \in W \). Also, \( \int_0^1 (\lambda f)(x)dx = \lambda \int_0^1 f(x)dx = \lambda \cdot 0 = 0 \), so \( \lambda f \in W \). Finally, \( 0 \) is obviously in \( W \). Therefore, \( W \) is a subspace.

2. For each of the following subsets \( S \subseteq V \) of the \( F \)-vector space \( V \), check if \( S \) is linearly independent. Prove your assertions. (2 pts each)

a) \( F = \mathbb{Q}, V = \mathbb{R} \) and \( S = \{1, \sqrt{2}\} \).

The general linear combination of \( \{1, \sqrt{2}\} \) looks like \( a + b\sqrt{2} \) for \( a \) and \( b \) rational numbers. Suppose \( a + b\sqrt{2} = 0 \). If \( b \neq 0 \), then \( \sqrt{2} = -(a/b) \) is rational; since this is not the case, we conclude that \( b = 0 \). But then it immediately follows that \( a = 0 \). That is, the only linear combination expressing \( 0 \) is the trivial one, so the set is linearly independent.

b) \( F = \mathbb{R}, V = C \) and \( S = \{1, i\} \).

An \( \mathbb{R} \)-linear combination of \( \{1, i\} \) is of the form \( a + bi \) - that is, it’s simply a complex number. Because a complex number is \( 0 \) if and only if its real and imaginary part are both \( 0 \), the set \( \{1, i\} \) is linearly independent over \( \mathbb{R} \).

c) \( F = \mathbb{C}, V = \mathbb{C} \) and \( S = \{1, i\} \).

Let \( a = -i \in \mathbb{C} \) and \( b = 1 \in \mathbb{C} \). Then \( a + bi = -i + i = 0 \) is a non-trivial linear combination expressing \( 0 \); thus, \( \{1, i\} \) is linearly dependent over \( \mathbb{C} \).

3. Let \( S \) be a set and write \( \mathbb{R}^S \) for the set of all real-valued functions on \( S \). (4 pts)

a) Explain how \( \mathbb{R}^S \) has a natural structure as an \( \mathbb{R} \)-vector space, giving the addition and scalar multiplication and checking the axioms.

Let \( f \) and \( g \) be two elements of \( \mathbb{R}^S \), that is, functions from \( S \) to \( \mathbb{R} \). Define \( f + g \) via \( (f + g)(s) = f(s) + g(s) \), and if \( \lambda \in \mathbb{R} \), define \( \lambda f \in \mathbb{R}^S \) via \( (\lambda f)(s) = \lambda f(s) \). The addition is associative and commutative because the addition of real numbers is; for \( f \in \mathbb{R}^S \), the function \( (-1)f \) is a negative; scalar multiplication is associative because multiplication of
Real numbers is; the function \( f_0 \) defined by \( f_0(s) = 0 \) for all \( s \in S \) is a zero element; and, finally, the distributive laws (right and left) hold, once more, because they hold true for real numbers.

b) For \( s \in S \), write \( \chi_s : S \to \mathbb{R} \) for the function given by \( \chi_s(s) = 1 \) and \( \chi_s(t) = 0 \) for \( t \neq s \). Show that the set \( \{ \chi_s|s \in S \} \subseteq \mathbb{R}^S \) is linearly independent.

Let \( \lambda_1 \chi_{s_1} + \lambda_2 \chi_{s_2} + \cdots + \lambda_r \chi_{s_r} \) be a general linear combination of elements in the set \( \{ \chi_s|s \in S \} \). That is, \( \lambda_i \in \mathbb{R} \) and \( s_i \in S \), with \( s_i \neq s_j \) if \( i \neq j \) (recall that, in a linear combination, the same vector can only appear once). Suppose that 
\[
\lambda_1 \chi_{s_1} + \lambda_2 \chi_{s_2} + \cdots + \lambda_r \chi_{s_r} = 0.
\]
(Recall that 0 here stands for the zero element of the vector space \( \mathbb{R}^S \), that is, the zero function \( f_0 \).)

We need to prove that this implies \( \lambda_1 = \lambda_2 = \cdots = \lambda_r = 0 \). Remember that we are dealing with functions on \( S \) here; that means we are allowed to evaluate everything at an element of \( S \). So, evaluate the linear combination at \( s_i \in S \).

Observe that \( \chi_{s_j}(s_i) = 0 \) for \( j \neq i \) since \( s_j \neq s_i \) in that case, and that \( \chi_{s_i}(s_i) = 1 \). In other words,
\[
(\lambda_1 \chi_{s_1} + \lambda_2 \chi_{s_2} + \cdots + \lambda_r \chi_{s_r})(s_i) = \lambda_i.
\]
By assumption, we also have that
\[
(\lambda_1 \chi_{s_1} + \lambda_2 \chi_{s_2} + \cdots + \lambda_r \chi_{s_r})(s_i) = f_0(s_i) = 0.
\]
That is, \( \lambda_i = 0 \), for any \( i \in \{1, \ldots, r\} \), as required.

c) Find an example of a set \( S \) such that the set \( \{ \chi_s|s \in S \} \subseteq \mathbb{R}^S \) is not a basis of the vector space \( \mathbb{R}^S \). Prove your assertion.

We already know that this set is linearly independent. That means we have to find a set \( S \) such that the functions \( \chi_s \) do not form a generating set of \( \mathbb{R}^S \).

Observe that if \( f : S \to \mathbb{R} \) can be expressed by a linear combination of functions of the form \( \chi_s \), then there are only finitely many \( t \in S \) such that \( f(s) \neq 0 \) (this is because a linear combination is always a finite sum, and each of the \( \chi_s \) is non-zero only at one point - at \( s \)).

Now let \( S \) be an infinite set, and \( f : S \to \mathbb{R} \) be defined by \( f(s) = 1 \) for all \( s \in S \). Then \( f \) cannot be expressed as a linear combination of functions in \( \{ \chi_s|s \in S \} \), since it has non-zero value at infinitely many points of \( S \). Therefore, \( \{ \chi_s|s \in S \} \subseteq \mathbb{R}^S \) is not a generating system, a fortiori, it is not a basis.

4. Let \( V \) be an \( F \)-vector space, and \( S \subseteq T \subseteq V \) be subsets. Prove: if \( S \) is linearly dependent, then so is \( T \). (4 pts)

Since \( S \) is linearly dependent, there is a non-trivial linear combination of elements of \( S \) expressing zero. But elements of \( S \) are also elements of \( T \), so the very same linear combination shows that \( T \) is linearly dependent.