Math 115A **Homework 1** **Due October 10th, 2008**

1. For each of the following, check if $W \subseteq V$ is an F-subspace of the F-vector space V. If yes, write out a proof. If not, prove that. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^3$ and $W = \{v = (v_1, v_2, v_3) \in V | 2v_1 - v_2 = 1\}$.

Since $2 \cdot 0 - 0 \neq 1$, $(0, 0, 0) \notin W$, so W is not a subspace.

b) $F = \mathbb{Q}$, $V = \mathbb{R}$ and $W = \{x \in \mathbb{R} | x\sqrt{2} \in \mathbb{Q}\}$.

Take $x \in W$ and $y \in W$, and let $\lambda \in \mathbb{Q}$. Then $\sqrt{2}(x + y) = \sqrt{2}x + \sqrt{2}y \in \mathbb{Q}$, so $x + y \in W$. Moreover, $\sqrt{2}(\lambda x) = \lambda(\sqrt{2}x) \in \mathbb{Q}$, so that $\lambda x \in W$. Finally, $\sqrt{2} \cdot 0 = 0 \in \mathbb{Q}$, whence $0 \in W$. That is, W is a subspace.

c) $F = \mathbb{R}$, $V = \mathbf{C}(\mathbb{R})$ the set of continuous real-valued functions on the real numbers and $W = \{f \in V | \int_0^1 f(x)dx = 0\}$.

Let $f \in W$ and $g \in W$, and let $\lambda \in \mathbb{R}$. Then $\int_0^1 (f + g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0 + 0 = 0$, so that $f + g \in W$. Also, $\int_0^1 (\lambda f)(x)dx = \lambda \int_0^1 f(x)dx = \lambda \cdot 0 = 0$, so $\lambda f \in W$. Finally, 0 is obviously in W. Therefore, W is a subspace.

2. For each of the following subsets $S \subseteq V$ of the F-vector space V, check if S is linearly independent. Prove you assertions. (2 pts each)

a) $F = \mathbb{Q}$, $V = \mathbb{R}$ and $S = \{1, \sqrt{2}\}$.

The general linear combination of $\{1, \sqrt{2}\}$ looks like $a + b\sqrt{2}$ for a and b rational numbers. Suppose $a + b\sqrt{2} = 0$. If $b \neq 0$, then $\sqrt{2} = -(a/b)$ is rational; since this is not the case, we conclude that $b = 0$. But then it immediately follows that $a = 0$. That is, the only linear combination expressing 0 is the trivial one, so the set is linearly independent.

b) $F = \mathbb{R}$, $V = \mathbb{C}$ and $S = \{1, i\}$.

An $\mathbb{R}$-linear combination of $\{1, i\}$ is of the form $a + bi$ - that is, it's simply a complex number. Because a complex number is 0 if and only if its real and imaginary part are both 0, the set $\{1, i\}$ is linearly independent over $\mathbb{R}$.

c) $F = \mathbb{C}$, $V = \mathbb{C}$ and $S = \{1, i\}$.

Let $a = -i \in \mathbb{C}$ and $b = 1 \in \mathbb{C}$. Then $a + bi = -i + i = 0$ is a non-trivial linear combination expressing 0; thus, $\{1, i\}$ is linearly dependent over $\mathbb{C}$.

3. Let S be a set and write $\mathbb{R}^S$ for the set of all real-valued functions on S. (4 pts)

a) Explain how $\mathbb{R}^S$ has a natural structure as an $\mathbb{R}$-vector space, giving the addition and scalar multiplication and checking the axioms.

Let f and g be two elements of $\mathbb{R}^S$, that is, functions from S to $\mathbb{R}$. Define $f + g$ via $(f + g)(s) = f(s) + g(s)$, and if $\lambda \in \mathbb{R}$, define $\lambda f \in \mathbb{R}^S$ via $(\lambda f)(s) = \lambda f(s)$. The addition is associative and commutative because the addition of real numbers is; for $f \in \mathbb{R}^S$, the function $(-1)f$ is a negative; scalar multiplication is associative because multiplication of real numbers is; the function f_0 defined by $f_0(s) = 0$ for all $s \in S$ is a zero element; and, finally, the distributive laws (right and left) hold, once more, because they hold true for real numbers.

b) For $s \in S$, write $\chi_s : S \to \mathbb{R}$ for the function given by $\chi_s(s) = 1$ and $\chi_s(t) = 0$ for $t \neq s$. Show that the set $\{\chi_s | s \in S\} \subseteq \mathbb{R}^S$ is linearly independent.

Let $\lambda_1\chi_{s_1} + \lambda_2\chi_{s_2} + \cdots + \lambda_r\chi_{s_r}$ be a general linear combination of elements in the set $\{\chi_s | s \in S\}$. That is, $\lambda_i \in \mathbb{R}$ and $s_i \in S$, with $s_i \neq s_j$ if $i \neq j$ (recall that, in a linear combination, the same vector can only appear once). Suppose that

$$\lambda_1\chi_{s_1} + \lambda_2\chi_{s_2} + \cdots + \lambda_r\chi_{s_r} = 0.$$

(Recall that 0 here stands for the zero element of the vector space $\mathbb{R}^S$, that is, the zero function f_0.)
We need to prove that this implies $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. Remember that we are dealing with functions on S here; that means we are allowed to evaluate everything at an element of S. So, evaluate the linear combination at $s_i \in S$.
Observe that $\chi_{s_j}(s_i) = 0$ for $j \neq i$ since $s_j \neq s_i$ in that case, and that $\chi_{s_i}(s_i) = 1$. In other words,

$$(\lambda_1\chi_{s_1} + \lambda_2\chi_{s_2} + \cdots + \lambda_r\chi_{s_r})(s_i) = \lambda_i.$$

By assumption, we also have that

$$(\lambda_1\chi_{s_1} + \lambda_2\chi_{s_2} + \cdots + \lambda_r\chi_{s_r})(s_i) = f_0(s_i) = 0.$$

That is, $\lambda_i = 0$, for any $i \in \{1, \ldots, r\}$, as required.

c) Find an example of a set S such that the set $\{\chi_s | s \in S\} \subseteq \mathbb{R}^S$ is not a basis of the vector space $\mathbb{R}^S$. Prove your assertion.

We already know that this set is linearly independent. That means we have to find a set S such that the functions χ_s do *not* form a generating set of $\mathbb{R}^S$.
Observe that if $f : S \to \mathbb{R}$ can be expressed by a linear combination of functions of the form χ_s, then there are only finitely many $t \in S$ such that $f(s) \neq 0$ (this is because a linear combination is always a finite sum, and each of the χ_s is non-zero only at one point - at s). Now let S be an infinite set, and $f : S \to \mathbb{R}$ be defined by $f(s) = 1$ for all $s \in S$. Then f cannot be expressed as a linear combination of functions in $\{\chi_s | s \in S\}$, since it has non-zero value at infinitely many points of S. Therefore, $\{\chi_s | s \in S\} \subseteq \mathbb{R}^S$ is not a generating system, a fortiori, it is not a basis.

4. Let V be an F-vector space, and $S \subseteq T \subseteq V$ be subsets. Prove: if S is linearly dependent, then so is T. (4 pts)

Since S is linearly dependent, there is a non-trivial linear combination of elements of S expressing zero. But elements of S are also elements of T, so the very same linear combination shows that T is linearly dependent.