MATH 115A - Lecture 4 - Fall 2008
Midterm 2 - November 14, 2008

NAME:

STUDENT ID #:

This is a closed-book and closed-note examination.
Calculators are not allowed.
Please show all your work.
Use only the paper provided. You may write on the back if you need more space, but clearly indicate this on the front.
There are 5 problems for a total of 100 points.

POINTS:

1.

2.

3.

4.

5.

1. (20 points) Let $\mathcal{P}_3$ be the $\mathbb{R}$-vector space of polynomials of degree at most 3. Let $\mathcal{A} = \{1, x, x^2, x^3\}$ be the standard ordered basis of $\mathcal{P}^3$ and let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the linear transformation such that $T(f)(x) = f(x) + 2f'(x) - f''(x)$. Compute the matrix representation $[T]_{\mathcal{A}}$.

We calculate:

$$\begin{aligned} T(1) &= 1 \\ T(x) &= x + 2 \\ T(x^2) &= x^2 + 4x - 2 \\ T(x^3) &= x^3 + 6x^2 - 6x \end{aligned}$$

to obtain the matrix representation

$$[T]_{\mathcal{A}} = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. (20 points) Suppose V is an F-vector space and $T : V \to V$ is a linear transformation. Show: if $\lambda \in F$ is an eigenvalue of T, then λ^2 is an eigenvalue of T^2.

Let $v \in V$ be an eigenvector with eigenvalue λ. That is, $v \neq 0$ and $T(v) = \lambda v$. Then $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$, so v is also an eigenvector of T^2, with eigenvalue λ^2. In particular, λ^2 is an eigenvalue of T^2.

3. (20 points) Let $\mathcal{P}_2$ be the $\mathbb{R}$-vector space of polynomials of degree at most two. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by $T(f)(x) = f(x) + f'(x)$. Find all the eigenvalues of T and give a basis for each of the eigenspaces. Is T diagonalizable?

The matrix representation of T with respect to the basis $\mathcal{A} = \{1, x, x^2\}$ is

$$[T]_{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore the characteristic polynomial is $p_T(\lambda) = (1-\lambda)^3$, so the only eigenvalue is $\lambda_1 = 1$. To compute a basis for the eigenspace E_1, we have to solve the equation $(T - id)(v) = 0$, which writing

$$[v]_{\mathcal{A}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

translates to the system

$$\begin{aligned} y &= 0 \\ 2z &= 0 \\ 0 &= 0. \end{aligned}$$

Clearly, the solutions are x arbitrary, and $y = z = 0$. That is, a basis for the eigenspace is (for example) given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

That means the eigenspace E_1 is of dimension 1. As there are no other eigenvalues, and the dimension of $\mathcal{P}_2$ is equal to $3 > 1$, we conclude that T is not diagonalizable.

4. (20 points) Suppose a linear transformation $T : V \to V$ is diagonalizable. Prove that the transformation $-T$ is also diagonalizable.

By assumption, T is diagonalizable. Therefore, there is a basis $\mathcal{A}$ of V consisting of eigenvectors of T. Let $v \in \mathcal{A}$, and let λ be the corresponding eigenvalue, so that $T(v) = \lambda v$. Then $(-T)(v) = -(T(v)) = (-\lambda)v$. Therefore, v is also an eigenvector of$-T$, with eigenvalue $-\lambda$. Since $v \in \mathcal{A}$ was arbitrary, we conclude that $\mathcal{A}$ is a basis of V consisting of eigenvectors of $-T$, whence $-T$ is diagonalizable.

5. (20 points) Suppose V and W are finite-dimensional F-vector spaces of the same dimension. Show that there is an isomorphism $T : V \to W$.

Let $n = \dim_F(V) = \dim_F(W)$. Choose a basis $\{v_1, \ldots, v_n\}$ of V and a basis $\{w_1, \ldots, w_n\}$ of W. There is a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $1 \leq i \leq n$, and a unique linear transformation $S : W \to V$ such that $S(w_i) = v_i$ for $1 \leq i \leq n$. Obviously,

$$T(S(w_i)) = w_i = id_W(w_i)$$

and

$$S(T(v_i)) = v_i = id_V(v_i)$$

for $1 \leq i \leq n$, so that $T \circ S = id_W$ and $S \circ T = id_V$. That is, T is an isomorphism with inverse S. (Alternatively, it is clear that T is onto since its image contains a basis. By the rank-nullity theorem, T is also onoe-to-one, hence it is an isomorphism.)