

MATH 115A - Lecture 4 - Fall 2008
Midterm 1 - October 20, 2008

NAME:

STUDENT ID #:

This is a closed-book and closed-note examination.

Calculators are not allowed.

Please show all your work.

Use only the paper provided. You may write on the back if you need more space, but clearly indicate this on the front.

There are 5 problems for a total of 100 points.

POINTS:

1.

2.

3.

4.

5.

1. (10 points each) For the given subset W of the F -vector space V , prove or disprove that W is a subspace of V .

(a) $F = \mathbb{R}$, $V = \mathbb{R}^3$ and $W = \{(a_1, a_2, a_3) \in V \mid a_1 - a_2 = a_3\}$.

Clearly, the zero vector $(0, 0, 0)$ is in W . Now suppose $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are in W . Then $(a_1 + b_1) - (a_2 + b_2) = a_1 - a_2 + b_1 - b_2 = a_3 + b_3$, so $\mathbf{a} + \mathbf{b} \in W$. If $\lambda \in \mathbb{R}$ is a scalar, then $\lambda a_1 - \lambda a_2 = \lambda(a_1 - a_2) = \lambda a_3$, so $\lambda \mathbf{a} \in W$. That is, W is closed under addition and scalar multiplication and contains 0, hence, it is a subspace.

(b) $F = \mathbb{Q}$, $V = \mathbb{C}$ and $W = \{a + bi \in V \mid ab \in \mathbb{Q}\}$.

The two complex numbers $x = 1$ and $y = \sqrt{2} + (1/\sqrt{2})i$ are both in W . However, $(1 + \sqrt{2})/\sqrt{2}$ is not rational, so that $x + y$ is not in W . That is, W is not closed under addition and is therefore not a subspace.

2. (20 points) Let $S = \{(1, 2, 3), (1, 3, 2), (1, 1, 1)\} \subseteq \mathbb{R}^3$. Is S a basis of \mathbb{R}^3 ? If so, prove it. If not, disprove it.

Because the dimension of \mathbb{R}^3 is 3, it suffices to show either that S is linearly independent, or that S is a generating set. Either is easily accomplished by solving linear equations.

3. (20 points) The set of complex numbers \mathbb{C} can be viewed as both an \mathbb{R} -vector space and a \mathbb{C} -vector space. Give an example of a map $T : \mathbb{C} \rightarrow \mathbb{C}$ that is \mathbb{R} -linear, but not \mathbb{C} -linear.

For example, the map $T : \mathbb{C} \rightarrow \mathbb{C}$ defined by $T(a + bi) = a$ is easily checked to be \mathbb{R} -linear, but not \mathbb{C} -linear. First, we check that T is additive. Let $x = a + bi$ and $y = c + di$ be two complex numbers. Then $T(x + y) = T((a + c) + (b + d)i) = (a + c) = T(x) + T(y)$.

Next, let $\lambda \in \mathbb{R}$ be a real scalar, and $x = a + bi$ a complex number. Then $T(\lambda x) = T((\lambda a) + (\lambda b)i) = \lambda a = \lambda T(x)$. Thus, T is \mathbb{R} -linear.

Finally, we show T is not \mathbb{C} -linear. Let $x = i \in \mathbb{C}$ (element of the vector space) and $\lambda = i \in \mathbb{C}$ (complex scalar). then $T(\lambda x) = -1$, but $\lambda T(x) = 0$.

4. (20 points) Let $D : \mathbf{C}^1(\mathbb{R}) \rightarrow \mathbf{C}(\mathbb{R})$ be the map on continuously differentiable functions given by the derivative, that is, $D(f)(x) = f'(x)$. Show that D is linear (you may assume what you know from calculus) and give a basis for the Null space (that is, the kernel) of D .

Let f and g be elements of $\mathbf{C}^1(\mathbb{R})$, and let $c \in \mathbb{R}$ be a scalar. Then $D(cf + g) = D(cf) + D(g) = cD(f) + D(g)$, where the first equality holds because the derivative of a sum is the sum of derivatives, and the second because the derivative of a multiple is the multiple of the derivative. This shows that D is a linear transformation.

The Null space of D is the set of all continuously differentiable functions f such that $f'(x) = 0$ for all $x \in \mathbb{R}$. Such functions are the constant ones, that is, $N(D) = \{c \mid c \in \mathbb{R}\}$. A basis for this space is any one constant non-zero function, for example, $\{1\}$.

5. (20 points) Let V be the \mathbb{R} -vector space of functions in one variable x of the form $f(x) = a + bx + cx^2$ where a , b and c are arbitrary real numbers. Define a map $T : V \rightarrow \mathbb{R}^3$ by $T(f) = (f(0), f(1), f(2))$. Show that T is a linear transformation and determine the dimension of its Null space $N(T)$ and its range $R(T)$.

Let f and g be in V , and let $c \in \mathbb{R}$ be a scalar. Then $T(f + g) = ((f + g)(0), (f + g)(1), (f + g)(2)) = (f(0) + g(0), f(1) + g(1), f(2) + g(2)) = (f(0), f(1), f(2)) + (g(0), g(1), g(2)) = T(f) + T(g)$ and $T(cf) = ((cf)(0), (cf)(1), (cf)(2)) = (cf(0), cf(1), cf(2)) = c(f(0), f(1), f(2)) = cT(f)$. That is, T is a linear transformation.

We compute the range of T . I claim T is onto. To show that, it suffices to give elements f , g and h in V such that $\{T(f), T(g), T(h)\} \subseteq \mathbb{R}^3$ is a basis. Let $f(x) = x(x - 1)/2$, $g(x) = -x(x - 2)$ and $h(x) = (x - 1)(x - 2)/2$. Then $T(f) = e_1$, $T(g) = e_2$ and $T(h) = e_3$. Thus, T is onto and the dimension of the range is 3.

Since the dimension of V is equal to the dimension of \mathbb{R}^3 , the dimension theorem implies that the dimension of the Null space is 0.