

1. For each of the following F -vector spaces V and linear transformations $T \in \mathcal{L}(V)$, find all eigenvalues of T and their algebraic and geometric multiplicities. (2 pts each)

a) $F = \mathbb{R}$, $V = M(2 \times 2, \mathbb{R})$, and T the linear transformation that sends a matrix $A \in M(2 \times 2, \mathbb{R})$ to its transpose $T(A) = A^t$.

b) $F = \mathbb{C}$, $V = \mathbb{C}^2$, and T the linear transformation given by $T(z, w) = (z + w, z - w)$.

c) $F = \mathbb{R}$, $V = \mathcal{P}_2(\mathbb{R})$, and T given by $T(f)(x) = xf'(x) + f(x)$.

2. Let V be a finite-dimensional F -vector space and $T \in \mathcal{L}(V)$ a linear transformation such that there exists a natural number n with $(T - id)^n = 0$. (Here, for a linear transformation S , S^n means the n -fold composition of S with itself.) Find all the eigenvalues of T . (4 pts)

Solution: If v is an eigenvector of T with eigenvalue λ then $(T - id)^n(v) = (\lambda - 1)^n(v) = 0$ so $(\lambda - 1)^n = 0$. Therefore the only possible eigenvalue is $\lambda = 1$. On the other hand, because $(T - id)^n = 0$, the Null space of $(T - id)$ can't just be $\{0\}$ (unless $V = \{0\}$), which implies that 1 is in fact an eigenvalue of T .

3. Let V be an F -vector space of dimension n and let $T \in \mathcal{L}(V)$ be a linear transformation with n distinct eigenvalues. (Recall that this implies that T is diagonalizable.) Further, let $S \in \mathcal{L}(V)$ such that $S \circ T = T \circ S$. Show that S is diagonalizable. (5 pts)

Solution: Choose a basis $\{v_1, \dots, v_n\}$ consisting of eigenvectors of T , with the (distinct) eigenvalues $\lambda_1, \dots, \lambda_n$. For any i , $T(S(v_i)) = S(T(v_i)) = S(\lambda_i v_i) = \lambda_i S(v_i)$. Therefore, $S(v_i) \in E_{\lambda_i}$ but the eigenspace is one-dimensional since T has n distinct eigenvalues. This implies that $S(v_i) = a_i v_i$ for some $a_i \in F$; that is, v_i is an eigenvector of S as well. That is, $\{v_1, \dots, v_n\}$ is a basis consisting of eigenvectors of S , so S is diagonalizable.

4. Let V be a finite-dimensional F -vector space and $T \in \mathcal{L}(V)$ a linear transformation such that $T^2 = T$ and $T \neq 0$ and $T \neq id$. Find all eigenvalues of T . (5 pts)

Solution: Since $T^2 = T$ any eigenvalue λ of T has to have the property that $\lambda^2 = \lambda$. Thus $\lambda = 0$ or $\lambda = 1$. Next note that for any vector $v \in V$, we can write $v = T(v) + (1 - T)(v)$ and $T(v)$ is an eigenvector with eigenvalue 1, while $(1 - T)(v)$ is an eigenvector with eigenvalue 0. This means that V is spanned by eigenvectors, so that T is diagonalizable. Now the hypothesis that $T \neq 0$ and $T \neq id$ implies that both 0 and 1 do in fact occur as eigenvalues of T .