Math 115A **Homework 5** **Due February 12th, 2010**

1. For each of the following F-vector spaces V and pairs of ordered bases $\mathcal{A}$ and $\mathcal{B}$ of V, compute the change-of-coordinate matrix Q for changing coordinates from $\mathcal{A}$ to $\mathcal{B}$. (2 pts each)

a) $F = \mathbb{R}$, $V = W = \mathbb{R}^3$, and $\mathcal{A} = \{(2,1,0),(1,2,1),(0,0,1)\}$ and $\mathcal{B} = \{(0,0,1),(1,2,1),(2,1,0)\}$.

b) $F = \mathbb{R}$, $V = \mathbb{C}$, and $\mathcal{A} = \{1+i, 1-i\}$ and $\mathcal{B} = \{2i, 1-2i\}$.

c) $F = \mathbb{R}$, $V = \mathcal{P}_2(\mathbb{R})$, and $\mathcal{A} = \{1, 1+x, 1+x^2\}$ and $\mathcal{B} = \{x, 1+x+x^2, 1-x\}$.

2. Find all the eigenvalues for each of the following linear transformations $T : V \to V$ of the F-vector space V. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^2$ and T the linear transformation such that $T(1,0) = (1/2, 3/2)$ and $T(0,1) = (3/2, 1/2)$.

b) $F = \mathbb{R}$, $V = C^\infty(\mathbb{R})$ the vector space of infinitely differentiable functions and T the linear transformation defined as $T(f) = f'$ (that is, the derivative).

c) $F = \mathbb{R}$, $V = \mathbb{R}^2$ and T the linear transformation given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

d) $F = \mathbb{C}$, $V = \mathbb{C}^2$ and T the linear transformation given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that for all linear transformations $R \in \mathcal{L}(\mathbb{R}^2)$, $R \circ T = T \circ R$. Prove that there exists a scalar $a \in \mathbb{R}$ such that $T = aI_2$. (3 pts)

Solution: we will prove this for any two-dimensional vector space V over a general field F. Note that a similar proof applies to any finite-dimensional vector space. Let $\{v, w\}$ be some basis of V. Let R_1 be the unique linear transformation such that $R_1(v) = v$ and $R_1(w) = 0$, and R_2 the unique linear transformation such that $R_2(v) = w$ and $R_2(w) = v$. Since T commutes with R_1, both v and w have to be eigenvectors of T, with eigenvalues a_v and a_w. On the other hand, since T commutes with R_2, the eigenvalues a_v and a_w have to be equal; call this scalar a. Then $T(v) = av$ and $T(w) = aw$, and since $\{v, w\}$ is a basis, $T = aI$.

4. Let V be an F-vector space of dimension n, and let $a_1, \ldots, a_n \in F$ be scalars. Show that there exists a linear transformation $T \in \mathcal{L}(V)$ with eigenvalues $a_1, \ldots, a_n$. (3 pts)

Solution: Choose any basis $\{v_1, \ldots, v_n\}$ and let T be the (unique) linear transformation defined on the basis by $T(v_i) = a_i v_i$.