Math 115A

Homework 5

Due February 12th, 2010

1. For each of the following $F$-vector spaces $V$ and pairs of ordered bases $\mathcal{A}$ and $\mathcal{B}$ of $V$, compute the change-of-coordinate matrix $Q$ for changing coordinates from $\mathcal{A}$ to $\mathcal{B}$. (2 pts each)

   a) $F = \mathbb{R}$, $V = W = \mathbb{R}^3$, and $\mathcal{A} = \{(2, 1, 0), (1, 2, 1), (0, 0, 1)\}$ and $\mathcal{B} = \{(0, 0, 1), (1, 2, 1), (2, 1, 0)\}$.

   b) $F = \mathbb{R}$, $V = \mathbb{C}$, and $\mathcal{A} = \{1 + i, 1 - i\}$ and $\mathcal{B} = \{2i, 1 - 2i\}$.

   c) $F = \mathbb{R}$, $V = \mathcal{P}_2(\mathbb{R})$, and $\mathcal{A} = \{1, 1 + x, 1 + x^2\}$ and $\mathcal{B} = \{x, 1 + x + x^2, 1 - x\}$.

2. Find all the eigenvalues for each of the following linear transformations $T : V \to V$ of the $F$-vector space $V$. (2 pts each)

   a) $F = \mathbb{R}$, $V = \mathbb{R}^2$ and $T$ the linear transformation such that $T(1, 0) = (1/2, 3/2)$ and $T(0, 1) = (3/2, 1/2)$.

   b) $F = \mathbb{R}$, $V = C^\infty(\mathbb{R})$ the vector space of infinitely differentiable functions and $T$ the linear transformation defined as $T(f) = f'$ (that is, the derivative).

   c) $F = \mathbb{R}$, $V = \mathbb{R}^2$ and $T$ the linear transformation given by the matrix

   \[
   \begin{pmatrix}
   0 & 1 \\
   -1 & 0
   \end{pmatrix}.
   \]

   d) $F = \mathbb{C}$, $V = \mathbb{C}^2$ and $T$ the linear transformation given by the matrix

   \[
   \begin{pmatrix}
   0 & 1 \\
   -1 & 0
   \end{pmatrix}.
   \]

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that for all linear transformations $R \in \mathcal{L}(\mathbb{R}^2)$, $R \circ T = T \circ R$. Prove that there exists a scalar $a \in \mathbb{R}$ such that $T = aI_2$. (3 pts)

**Solution**: we will prove this for any two-dimensional vector space $V$ over a general field $F$. Note that a similar proof applies to any finite-dimensional vector space. Let $\{v, w\}$ be some basis of $V$. Let $R_1$ be the unique linear transformation such that $R_1(v) = v$ and $R_1(w) = 0$, and $R_2$ the unique linear transformation such that $R_2(v) = w$ and $R_2(w) = v$. Since $T$ commutes with $R_1$, both $v$ and $w$ have to be eigenvectors of $T$, with eigenvalues $a_v$ and $a_w$. On the other hand, since $T$ commutes with $R_2$, the eigenvalues $a_v$ and $a_w$ have to be equal; call this scalar $a$. Then $T(v) = av$ and $T(w) = aw$, and since $\{v, w\}$ is a basis, $T = aI$.

4. Let $V$ be an $F$-vector space of dimension $n$, and let $a_1, \ldots, a_n \in F$ be scalars. Show that there exists a linear transformation $T \in \mathcal{L}(V)$ with eigenvalues $a_1, \ldots, a_n$. (3 pts)

**Solution**: Choose any basis $\{v_1, \ldots, v_n\}$ and let $T$ be the (unique) linear transformation defined on the basis by $T(v_i) = a_iv_i$. 