Math 115A **Homework 4** **Due February 5th, 2010**

1. Let V be a finite-dimensional F-vector space of dimension n and $T : V \to V$ a linear transformation. Show that T is an isomorphism if and only if there exist ordered bases $\mathcal{A}$ and $\mathcal{B}$ such that $[T]_{\mathcal{A}}^{\mathcal{B}} = I_n$ (here I_n is the $n \times n$-identity matrix). (5 pts)

Solution: Suppose there are such ordered bases $\mathcal{A} = \{v_1, \ldots, v_n\}$ and $\mathcal{B} = \{w_1, \ldots, w_n\}$. Then T has a matrix representation that is an invertible matrix, therefore it is an isomorphism. More precisely, T is the unique linear transformation such that $T(v_i) = w_i$ for $1 \leq i \leq n$, and we can define the inverse T^{-1} to be the unique linear transformation such that $T^{-1}(W - I) = v_i$ for $1 \leq i \leq n$.
Conversely, if T is an isomorphism, let $\mathcal{A} = \{v_1, \ldots, v_n\}$ be any ordered basis of V and let $w_i = T(v_i)$. Since T is one-to-one and onto, the set $\mathcal{B} = \{w_1, \ldots, w_n\}$ is a basis of W, and it is obvious from the definition of the matrix representation hat $[T]_{\mathcal{A}}^{\mathcal{B}} = I_n$.

2. Let V and W be finite-dimensional F-vector spaces and suppose that $T : V \to W$ is an isomorphism with inverse T^{-1}. Show that the map $\Phi_T : \mathcal{L}(V) \to \mathcal{L}(W)$ given by $\Phi_T(R) = T \circ R \circ T^{-1}$ for $R \in \mathcal{L}(V)$ is an isomorphism. (5 pts)

Solution: First, we check that Φ_T is linear. Indeed, if R_1 and R_2 are in $\mathcal{L}(V)$ and $c \in F$ then $\Phi_T(cR_1 + R_2) = T \circ (cR_1 + R_2) \circ T^{-1} = cT \circ R_1 \circ T^{-1} + T \circ R_2 \circ T^{-1}$ since T and T^{-1} are linear. Moreover, the map $\Phi_{T^{-1}} : \mathcal{L}(W) \to \mathcal{L}(V)$ defined by $\Phi_{T^{-1}}(S) = T^{-1} \circ S \circ T$ is plainly an inverse of Φ_T, so Φ_T is an invertible linear transformation, that is, and isomorphism.

3. Let $W \subseteq V$ be a subspace of a finite-dimensional F-vector space. Show that there is a linear transformation $P_W : V \to V$ such that $P_W \circ P_W = P_W$, $R(P_W) = W$ and for all $w \in W$, $P_W(w) = w$. (This is called a projector onto W.) What is the rank of P_W? (5 pts)

Solution: Choose a basis $\beta' = \{v_1, \ldots, v_k\}$ of W and complete it to a basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Now define the linear transformation $P_W : V \to V$ on the basis by $P_W(v_i) = v_i$ if $1 \leq i \leq k$ and $P_W(v_i) = 0$ if $k < i \leq n$. Since P_W is the identity on the basis β' of W, we have $P_W(w) = w$ for all $w \in W$; we immediately check that $P_W^2 = P_W$ on the basis β, so these are equal as linear transformations; and the range is spanned by $P_W(\beta) = \beta' \cup \{0\}$, so $R(P_W) = W$. The rank of P_W is $k = dim(W)$.

4. Let V be a finite-dimensional F-vector space. The vector space $V^* = \mathcal{L}(V, F)$ is called the dual vector space of V. Show that the map $i : V \to V^{**} = \mathcal{L}(V^*, F)$ to the dual of the dual vector space such that $i(v)(f) = f(v)$ for $v \in V$ and $f : V \to F$ is an isomorphism. (5 pts)

Solution: First we show that i is linear. Indeed, let v and w be in V and $c \in F$. Then for any $f \in V^*$, we have $i(cv + w)(f) = f(cv + w) = cf(v) + f(w) = ci(v)(f) + i(w)(f)$ since f is a linear transformation, and by the definition of i. Moreover, we know that V and V^{**} have the same finite dimension, so to prove that i is an isomorphism it suffices to show it is one-to-one. Suppose $v \neq 0$ is in V. Then we can find a linear transformation $f : V \to F$ such that $f(v) = 1 \neq 0$ (by completing v to a basis of V and then simply defining f on that basis such that $f(v) = 1$); but this means $i(v)(f) = 1 \neq 0$, so $i(v) \neq 0 \in V^{**}$. That is, the null space $N(i) = \{0\}$, so i is one-to-one as needed.