1. Let \( V \) be a finite-dimensional \( F \)-vector space of dimension \( n \) and \( T : V \to V \) a linear transformation. Show that \( T \) is an isomorphism if and only if there exist ordered bases \( \mathcal{A} \) and \( \mathcal{B} \) such that \( [T]_\mathcal{A}^\mathcal{B} = I_n \) (here \( I_n \) is the \( n \times n \)-identity matrix). (5 pts)

**Solution:** Suppose there are such ordered bases \( \mathcal{A} = \{v_1, \ldots, v_n\} \) and \( \mathcal{B} = \{w_1, \ldots, w_n\} \). Then \( T \) has a matrix representation that is an invertible matrix, therefore it is an isomorphism. More precisely, \( T \) is the unique linear transformation such that \( T(v_i) = w_i \) for \( 1 \leq i \leq n \), and we can define the inverse \( T^{-1} \) to be the unique linear transformation such that \( T^{-1}(W - I) = v_i \) for \( 1 \leq i \leq n \).

Conversely, if \( T \) is an isomorphism, let \( \mathcal{A} = \{v_1, \ldots, v_n\} \) be any ordered basis of \( V \) and let \( w_i = T(v_i) \). Since \( T \) is one-to-one and onto, the set \( \mathcal{B} = \{w_1, \ldots, w_n\} \) is a basis of \( W \), and it is obvious from the definition of the matrix representation hat \( [T]_\mathcal{A}^\mathcal{B} = I_n \).

2. Let \( V \) and \( W \) be finite-dimensional \( F \)-vector spaces and suppose that \( T : V \to W \) is an isomorphism with inverse \( T^{-1} \). Show that the map \( \Phi_T : \mathcal{L}(V) \to \mathcal{L}(W) \) given by \( \Phi_T(R) = T \circ R \circ T^{-1} \) for \( R \in \mathcal{L}(V) \) is an isomorphism. (5 pts)

**Solution:** First, we check that \( \Phi_T \) is linear. Indeed, if \( R_1 \) and \( R_2 \) are in \( \mathcal{L}(V) \) and \( c \in F \) then \( \Phi_T(cR_1 + R_2) = T \circ (cR_1 + R_2) \circ T^{-1} = cT \circ R_1 \circ T^{-1} + T \circ R_2 \circ T^{-1} \) since \( T \) and \( T^{-1} \) are linear. Moreover, the map \( \Phi_{T^{-1}} : \mathcal{L}(W) \to \mathcal{L}(V) \) defined by \( \Phi_{T^{-1}}(S) = T^{-1} \circ S \circ T \) is plainly an inverse of \( \Phi_T \), so \( \Phi_T \) is an invertible linear transformation, that is, and isomorphism.

3. Let \( W \subseteq V \) be a subspace of a finite-dimensional \( F \)-vector space. Show that there is a linear transformation \( P_W : V \to V \) such that \( P_W \circ P_W = P_W \) and \( R(P_W(w)) = W \) and for all \( w \in W \), \( P_W(w) = w \). (This is called a projector onto \( W \).) What is the range of \( P_W \)? (5 pts)

**Solution:** Choose a basis \( \beta' = \{v_1, \ldots, v_k\} \) of \( W \) and complete it to a basis \( \beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\} \) of \( V \). Now define the linear transformation \( P_W : V \to V \) on the basis by \( P_W(v_i) = v_i \) if \( 1 \leq i \leq k \) and \( P_W(v_i) = 0 \) if \( k < i \leq n \). Since \( P_W \) is the identity on the basis \( \beta' \) of \( W \), we have \( P_W(w) = w \) for all \( w \in W \); we immediately check that \( P_W^2 = P_W \) on the basis \( \beta \), so these are equal as linear transformations; and the range is spanned by \( P_W(\beta) = \beta' \cup \{0\} \), so \( R(P_W) = W \). The rank of \( P_W \) is \( k = \dim(W) \).

4. Let \( V \) be a finite-dimensional \( F \)-vector space. The vector space \( V^* = \mathcal{L}(V,F) \) is called the dual vector space of \( V \). Show that the map \( i : V \to V^{**} = \mathcal{L}(V^*,F) \) to the dual of the dual vector space such that \( i(v)(f) = f(v) \) for \( v \in V \) and \( f : V \to F \) is an isomorphism. (5 pts)

**Solution:** First we show that \( i \) is linear. Indeed, let \( v \) and \( w \) be in \( V \) and \( c \in F \). Then for any \( f \in V^* \), we have \( i(cv + w)(f) = f(cv + w) = cf(v) + f(w) = ci(v)(f) + i(w)(f) \) since \( f \) is a linear transformation, and by the definition of \( i \). Moreover, we know that \( V \) and \( V^{**} \) have the same finite dimension, so to prove that \( i \) is an isomorphism it suffices to show it is one-to-one. Suppose \( v \neq 0 \) is in \( V \). Then we can find a linear transformation \( f : V \to F \) such that \( f(v) = 1 \neq 0 \) (by completing \( v \) to a basis of \( V \) and then simply defining \( f \) on that basis such that \( f(v) = 1 \)); but this means \( i(v)(f) = 1 \neq 0 \), so \( i(v) \neq 0 \in V^{**} \). That is, the null space \( N(i) = \{0\} \), so \( i \) is one-to-one as needed.