

1. For each of the following linear transformations  $T$ , compute a basis for the Null space  $N(T)$  and determine the dimension of the range  $R(T)$ . (2 pts each)

a)  $F = \mathbb{R}$ ,  $V = W = \mathbb{R}^3$  and  $T : V \rightarrow W$  given by  $T(x, y, z) = (x - y, 0, z)$ .

**Solution:** A vector  $(x, y, z)$  is in the Null space if and only if  $x = y$  and  $z = 0$ . That is,  $N(T) = \{(a, a, 0) | a \in \mathbb{R}\}$ . A basis for this space is  $\{(1, 1, 0)\}$ . Since this is a set containing only one non-zero vector, it is linearly independent; and if  $(a, a, 0) \in N(T)$ , then  $(a, a, 0) = a(1, 1, 0)$  so it is a generating set for  $N(T)$ . By the dimension theorem, the dimension of the range of  $T$  is 2.

b)  $F = \mathbb{R}$ ,  $V = \mathbb{R}^4$ ,  $W = C(\mathbb{R})$  and  $T : V \rightarrow W$  where  $T(v_1, v_2, v_3, v_4)$  is the function  $f(x) = v_1x - v_2e^x + (v_3 - v_4)\sin(x)$ .

**Solution:** Let  $f(x) = T(v_1, v_2, v_3, v_4)$ . Note that  $f(0) = v_2$ . Therefore, if  $v = (v_1, v_2, v_3, v_4) \in N(T)$ , then  $v_2 = 0$ . Now if  $v_2 = 0$ , then  $f(2\pi) = 2\pi v_1$ , and hence  $v \in N(T)$  implies  $v_1 = 0$ . Finally, if  $v_1 = v_2 = 0$ , then  $f(\pi/2) = v_3 - v_4$ , so  $v \in N(T)$  also implies that  $v_3 = v_4$ . Conversely, if  $v = (0, 0, a, a)$  then  $T(v)$  is the zero function. Thus we have proved that  $N(T) = \{(0, 0, a, a) | a \in \mathbb{R}\}$ . A basis for this space is given by  $\{(0, 0, 1, 1)\}$ . By the dimension theorem, the dimension of the range of  $T$  is 3.

c)  $F = \mathbb{C}$ ,  $V = \mathbb{C}^2$ ,  $W = \mathbb{C}^3$  and  $T : V \rightarrow W$  given by  $T(z, w) = (2z - w, w - z, w + z)$ .

**Solution:** If  $T(z, w) = (0, 0, 0)$  then  $w = z$  and also  $w = -z$ . But this implies that  $w = z = 0$ , so the Null space is  $\{0\}$ . A basis for this space is given by the empty set. By the dimension theorem, the range has dimension 2.

2. Let  $V = \mathcal{P}_2(\mathbb{R})$  be the  $\mathbb{R}$ -vector space of polynomials over  $\mathbb{R}$  of degree at most 2, and let  $W = \mathbb{R}^3$ . Let  $\mathcal{A}$  be the ordered basis (ordering from left to right)  $\{1, 1 + x, 1 + x + x^2\}$  of  $V$ , and let  $\mathcal{B}$  be the standard ordered basis. For each of the following linear transformations  $T$  compute its matrix representation with respect to these bases. (2 pts each)

a)  $T : V \rightarrow W$  given by  $T(f) = (f(0), f'(0), f''(0))$ .

b)  $T : V \rightarrow W$  given by  $T(f) = (f(1), f(2), f(3))$ .

c)  $T : W \rightarrow V$  given by  $T(a, b, c) = a + bx + cx^2$ .

3. Let  $V$  and  $W$  be finite-dimensional  $F$ -vector spaces and  $T : V \rightarrow W$  a linear transformation. Suppose the dimension of the Null space of  $T$  is  $n$ . Prove that there are ordered bases  $\mathcal{A}$  of  $V$  and  $\mathcal{B}$  of  $W$ , respectively, such that the first  $n$  columns of the matrix representation  $[T]_{\mathcal{B}}^{\mathcal{A}}$  are zero. (4 pts)

**Solution:** Choose an ordered basis  $\mathcal{A}' = \{v_1, \dots, v_n\}$  of the Null space  $N(T)$ . Complete this to an ordered basis  $\mathcal{A}$  of  $V$  such that the first  $n$  vectors in  $\mathcal{A}$  are  $\{v_1, \dots, v_n\}$ . Now choose any ordered basis  $\mathcal{B}$  of  $W$ . Since  $T(v_1) = \dots = T(v_n) = 0$ , the first  $n$  columns of the matrix representation  $[T]_{\mathcal{B}}^{\mathcal{A}}$  are zero, as required.

4. Let  $T : V \rightarrow W$  be an  $F$ -linear transformation of finite-dimensional vector spaces and assume that  $N(T) = 0$ . Let  $\mathcal{A}$  be an ordered basis of  $V$  and  $\mathcal{B}$  an ordered basis of  $W$ . Prove that the columns of  $[T]_{\mathcal{A}}^{\mathcal{B}}$  are linearly independent. (4 pts)

**Solution:** Let  $\mathcal{A} = \{v_1, \dots, v_n\}$ . Since the Null space  $N(T) = \{0\}$ , the set  $\{T(v_1), \dots, T(v_n)\}$  is linearly independent. The columns of the matrix representation  $[T]_{\mathcal{A}}^{\mathcal{B}}$  are the coordinate vectors  $[T(v_i)]_{\mathcal{B}}$ , for  $1 \leq i \leq n$ . Since "taking the coordinate vector" is an isomorphism and in particular one-to-one, these columns are linearly independent.