1. For each of the following linear transformations $T$, compute a basis for the Null space $N(T)$ and determine the dimension of the range $R(T)$. (2 pts each)

a) $F = \mathbb{R}$, $V = W = \mathbb{R}^3$ and $T : V \to W$ given by $T(x, y, z) = (x - y, 0, z)$.

**Solution:** A vector $(x, y, z)$ is in the Null space if and only if $x = y$ and $z = 0$. That is, $N(T) = \{(a, a, 0) | a \in \mathbb{R}\}$. A basis for this space is $\{(1, 1, 0)\}$. Since this is a set containing only one non-zero vector, it is linearly independent; and if $(a, a, 0) \in N(T)$, then $(a, a, 0) = a(1, 1, 0)$ so it is a generating set for $N(T)$. By the dimension theorem, the dimension of the range of $T$ is 2.

b) $F = \mathbb{R}$, $V = \mathbb{R}^4$, $W = C(\mathbb{R})$ and $T : V \to W$ where $T(v_1, v_2, v_3, v_4)$ is the function $f(x) = v_1x - v_2e^x + (v_3 - v_4)\sin(x)$.

**Solution:** Let $f(x) = T(v_1, v_2, v_3, v_4)$. Note that $f(0) = v_2$. Therefore, if $v = (v_1, v_2, v_3, v_4) \in N(T)$, then $v_2 = 0$. Now if $v_2 = 0$, then $f(2\pi) = 2\pi v_1$, and hence $v \in N(T)$ implies $v_1 = 0$. Finally, if $v_1 = v_2 = 0$, then $f(\pi/2) = v_3 - v_4$, so $v \in N(T)$ also implies that $v_3 = v_4$. Conversely, if $v = (0, 0, a, a)$ then $T(v)$ is the zero function. Thus we have proved that $N(T) = \{(0, 0, a, a) | a \in \mathbb{R}\}$. A basis for this space is given by $\{(0, 0, 1, 1)\}$. By the dimension theorem, the dimension of the range of $T$ is 3.

c) $F = \mathbb{C}$, $V = \mathbb{C}^2$, $W = \mathbb{C}^3$ and $T : V \to W$ given by $T(z, w) = (2z - w, w - z, w + z)$.

**Solution:** If $T(z, w) = (0, 0, 0)$ then $w = z$ and also $w = -z$. But this implies that $w = z = 0$, so the Null space is $\{0\}$. A basis for this space is given by the empty set. By the dimension theorem, the range has dimension 2.

2. Let $V = \mathcal{P}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of polynomials over $\mathbb{R}$ of degree at most 2, and let $W = \mathbb{R}^3$. Let $\mathcal{A}$ be the ordered basis (ordering from left to right) $\{1, 1 + x, 1 + x + x^2\}$ of $V$, and let $\mathcal{B}$ be the standard ordered basis. For each of the following linear transformations $T$ compute its matrix representation with respect to these bases. (2 pts each)

a) $T : V \to W$ given by $T(f) = (f(0), f'(0), f''(0))$.

b) $T : V \to W$ given by $T(f) = (f(1), f(2), f(3))$.

c) $T : W \to V$ given by $T(a, b, c) = a + bx + cx^2$.

3. Let $V$ and $W$ be finite-dimensional $F$-vector spaces and $T : V \to W$ a linear transformation. Suppose the dimension of the Null space of $T$ is $n$. Prove that there are ordered bases $\mathcal{A}$ of $V$ and $\mathcal{B}$ of $W$, respectively, such that the first $n$ columns of the matrix representation $[T]_{\mathcal{A}}^{\mathcal{B}}$ are zero. (4 pts)

**Solution:** Choose an ordered basis $\mathcal{A}' = \{v_1, \ldots, v_n\}$ of the Null space $N(T)$. Complete this to an ordered basis $\mathcal{A}$ of $V$ such that the first $n$ vectors in $\mathcal{A}$ are $\{v_1, \ldots, v_n\}$. Now choose any ordered basis $\mathcal{B}$ of $W$. Since $T(v_1) = \cdots = T(v_n) = 0$, the first $n$ columns of the matrix representation $[T]_{\mathcal{A}}^{\mathcal{B}}$ are zero, as required.
4. Let $T : V \to W$ be an $F$-linear transformation of finite-dimensional vector spaces and assume that $N(T) = 0$. Let $\mathcal{A}$ be an ordered basis of $V$ and $\mathcal{B}$ an ordered basis of $W$. Prove that the columns of $[T]_{\mathcal{B}}^\mathcal{A}$ are linearly independent. (4 pts)

Solution: Let $\mathcal{A} = \{v_1, \ldots, v_n\}$. Since the null space $N(T) = \{0\}$, the set $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent. The columns of the matrix representation $[T]_{\mathcal{B}}^\mathcal{A}$ are the coordinate vectors $[T(v_i)]_{\mathcal{B}}$, for $1 \leq i \leq n$. Since "taking the coordinate vector" is an isomorphism and in particular one-to-one, these columns are linearly independent.