1. For each of the following functions $T$, determine if $T$ is $F$-linear. Prove your answer. (2 pts each)

a) $F = \mathbb{R}$, $V = W = \mathbb{R}^3$ and $T : V \to W$ given by $T(v) = 2v - (1,0,0)$.

**Solution:** If $T$ was linear, then $T(0,0,0) = (0,0,0)$. Since $T(0,0,0) = (-1,0,0) \neq (0,0,0)$, $T$ is, in fact, not linear.

b) $F = \mathbb{C}$, $V = W = \mathbb{R}$ and $T : V \to W$ given by $T(a + bi) = a - b$.

**Solution:** Note that $W = \mathbb{R}$ is not an $F$-vector space! Therefore $T$ can’t be $F$-linear.

c) $F = \mathbb{R}$, $V = C^1(\mathbb{R})$, $W = C(\mathbb{R})$ and $T : V \to W$ given by $T(f) = f' - f$.

**Solution:** We need to check that $T$ is additive and commutes with scalar multiplication. So let $F$ and $g$ be continuously differentiable functions (elements of $V$) and let $c \in \mathbb{R}$ be a scalar. Then $T(f + g) = (f + g)' - (f + g) = f' - f + g' - g = T(f) + T(g)$ so $T$ is additive. And $T(cf) = (cf)' - cf = c(f' - f) = cT(f)$, so $T$ commutes with scalar multiplication. Thus, $T$ is linear.

2. Find a basis for each of the following vector spaces. Prove your answer. (2 pts each)

a) $F = \mathbb{R}$, $V = \{a + bi \in \mathbb{C} | a + b = 0 \}$.

**Solution:** A basis of $V$ is for example $\{1 - i\}$. Indeed, this set has one non-zero element, therefore it is linearly independent. Moreover, all elements of $V$ are of the form $x - xi = x(1 - i)$ for some $x \in \mathbb{R}$, so $\{1 - i\}$ is a generating set for $V$.

b) $F = \mathbb{R}$, $V = \{f \in C^1(\mathbb{R}) | f' = 0 \}$.

**Solution:** $V$ is the vector space of constant functions; any non-zero constant function can serve as a basis.

c) $F = \mathbb{R}$, $V = \mathbb{C}^2$.

**Solution:** The set $S = \{(1,0), (0,1), (i,0), (0,i)\}$ is a basis. Indeed, if $a$, $b$, $c$, $d$ are real numbers such that $a(1,0) + b(0,1) + c(i,0) + d(0,i) = (0,0)$ then clearly $a = b = c = d = 0$, so $S$ is linearly independent. On the other hand, a general element in $V$ is of the form

$$ (a + ci, b + di) = a(1,0) + b(0,1) + c(i,0) + d(0,i) $$

for some $a$, $b$, $c$, $d$ in $\mathbb{R}$, so $S$ is also a generating set.

3. Let $T_1 : V \to W$ and $T_2 : W \to U$ be $F$-linear transformations. Suppose the Null space $N(T_2 \circ T_1) = \{0\}$. Prove that $N(T_1) = \{0\}$. Show by example that $N(T_2)$ need not be zero. (3 pts)
**Solution:** Suppose that \( v \in N(T_1) \), so \( T_1(v) = 0 \). Then \((T_2 \circ T_1)(v) = T_2(T_1(v)) = 0\) since \( T_2 \) is linear; therefore \( v \in N(T_2 \circ T_1) = \{0\} \), whence \( v = 0 \). Since \( v \) was arbitrary in \( N(T_1) \), this proves that \( N(T_1) = \{0\} \).

As an example, let \( F = \mathbb{R} \), \( V = U = \{0\} \) and \( W = \mathbb{R} \). Let \( T_1 \) and \( T_2 \) be the zero linear transformations. Then obviously \( N(T_2 \circ T_1) = \{0\} \) because it is a subspace of \( V = \{0\} \); but \( N(T_2) = \mathbb{R} \neq \{0\} \).

4. Let \( T : V \to W \) be an \( F \)-linear transformation and assume that \( N(T) = 0 \). Show that there exists an \( F \)-linear transformation \( S : W \to V \) such that \( S \circ T = id_V \). (5 pts)

**Solution:** Choose a basis \( B \) of \( V \). Then \( T(B) \subseteq W \) is linearly independent since \( N(T) = \{0\} \). Indeed, if \( v_1, \ldots, v_r \in B \) and \( a_1, \ldots, a_r \in F \) such that \( \sum_{i=1}^r a_iT(v_i) = 0 \), then \( \sum_{i=1}^r a_i v_i \in N(T) \), so \( \sum_{i=1}^r a_i v_i = 0 \), and since \( B \) is linearly independent, \( a_i = 0 \) for all \( i \).

Next we complete \( T(B) \) to a basis \( C = T(B) \cup A \) of \( W \) (any linearly independent set is contained in a basis), and there is a unique linear transformation \( S : W \to V \) such that \( S(T(v)) = v \) for \( v \in B \) and \( S(w) = 0 \) for \( w \in A \). Now \( S \circ T : V \to V \) is a linear transformation such that for all \( v \in B \), \( S \circ T(v) = v \). Since the identity transformation \( id_V \) has the same property, and since \( B \) is a basis for \( V \), we conclude that \( S \circ T = id_V \), as needed.