Math 115A **Homework 2** **Due January 22nd, 2010**

1. For each of the following functions T, determine if T is F-linear. Prove your answer. (2 pts each)

a) $F = \mathbb{R}$, $V = W = \mathbb{R}^3$ and $T : V \to W$ given by $T(v) = 2v - (1, 0, 0)$.

Solution: If T was linear, then $T(0, 0, 0) = (0, 0, 0)$. Since $T(0, 0, 0) = (-1, 0, 0) \neq (0, 0, 0)$, T is, in fact, not linear.

b) $F = \mathbb{C}$, $V = \mathbb{C}$, $W = \mathbb{R}$ and $T : V \to W$ given by $T(a + bi) = a - b$.

Solution: Note that $W = \mathbb{R}$ is not an F-vector space! Therefore T can't be F-linear.

c) $F = \mathbb{R}$, $V = C^1(\mathbb{R})$, $W = C(\mathbb{R})$ and $T : V \to W$ given by $T(f) = f' - f$.

Solution: We need to check that T is additive and commutes with scalar multiplication. So let F and g be continuously differentiable functions (elements of V) and let $c \in \mathbb{R}$ be a scalar. Then $T(f+g) = (f+g)' - (f+g) = f' - f + g' - g = T(f) + T(g)$ so T is additive. And $T(cf) = (cf)' - (cf) = c(f' - f) = cT(f)$, so T commutes with scalar multiplication. Thus, T is linear.

2. Find a basis for each of the following vector spaces. Prove your answer. (2 pts each)

a) $F = \mathbb{R}$, $V = \{a + bi \in \mathbb{C} | a + b = 0\}$.

Solution: A basis of V is for example $\{1 - i\}$. Indeed, this set has one non-zero element, therefore it is linearly independent. Moreover, all elements of V are of the form $x - xi = x(1 - i)$ for some $x \in \mathbb{R}$, so $\{1 - i\}$ is a generating set for V.

b) $F = \mathbb{R}$, $V = \{f \in C^1(\mathbb{R}) | f' = 0\}$.

Solution: V is the vector space of constant functions; any non-zero constant function can serve as a basis.

c) $F = \mathbb{R}$, $V = \mathbb{C}^2$.

Solution: The set $S = \{(1, 0), (0, 1), (i, 0), (0, i)\}$ is a basis. Indeed, if a, b, c, d are real numbers such that

$$a(1, 0) + b(0, 1) + c(i, 0) + d(0, i) = (0, 0)$$

then clearly $a = b = c = d = 0$, so S is linearly independent. On the other hand, a general element in V is of the form

$$(a + ci, b + di) = a(1, 0) + b(0, 1) + c(i, 0) + d(0, i)$$

for some a, b, c, d in $\mathbb{R}$, so S is also a generating set.

3. Let $T_1 : V \to W$ and $T_2 : W \to U$ be F-linear transformations. Suppose the Null space $N(T_2 \circ T_1) = \{0\}$. Prove that $N(T_1) = \{0\}$. Show by example that $N(T_2)$ need not be zero. (3 pts)

Solution: Suppose that $v \in N(T_1)$, so $T_1(v) = 0$. Then $(T_2 \circ T_1)(v) = T_2(T_1(v)) = 0$ since T_2 is linear; therefore $v \in N(T_2 \circ T_1) = \{0\}$, whence $v = 0$. Since v was arbitrary in $N(T_1)$, this proves that $N(T_1) = \{0\}$.
As an example, let $F = \mathbb{R}$, $V = U = \{0\}$ and $W = \mathbb{R}$. Let T_1 and T_2 be the zero linear transformations. Then obviously $N(T_2 \circ T_1) = \{0\}$ because it is a subspace of $V = \{0\}$; but $N(T_2) = \mathbb{R} \neq \{0\}$.

4. Let $T : V \to W$ be an F-linear transformation and assume that $N(T) = 0$. Show that there exists an F-linear transformation $S : W \to V$ such that $S \circ T = id_V$. (5 pts)

Solution: Choose a basis B of V. Then $T(B) \subseteq W$ is linearly independent since $N(T) = \{0\}$. Indeed, if $v_1, \ldots, v_r \in B$ and $a_1, \ldots, a_r \in F$ such that $\sum_{i=1}^r a_i T(v_i) = 0$, then $\sum_{i=1}^r a_i v_i \in N(T)$, so $\sum_{i=1}^r a_i v_i = 0$, and since B is linearly independent, $a_i = 0$ for all i.
Next we complete $T(B)$ to a basis $C = T(B) \cup A$ of W (any linearly independent set is contained in a basis), and there is a unique linear transformation $S : W \to V$ such that $S(T(v)) = v$ for $v \in B$ and $S(w) = 0$ for $w \in A$. Now $S \circ T : V \to V$ is a linear transformation such that for all $v \in B$, $S \circ T(v) = v$. Since the identity transformation id_V has the same property, and since B is a basis for V, we conclude that $S \circ T = id_V$, as needed.