Math 115A **Homework 1** **Due January 15th, 2010**

1. For each of the following, check if $W \subseteq V$ is an F-subspace of the F-vector space V. If yes, write out a proof. If not, prove that. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{R}^3$ and $W = \{v = (v_1, v_2, v_3) \in V | v_1 + 3v_3 = v_2\}$.

Solution: We have to check that $(0, 0, 0) \in W$, that W is closed under addition and that W is closed under scalar multiplication. First, obviously $(0, 0, 0) \in W$ as $0 + 0 = 0$. Now let $v = (v_1, v_2, v_3) \in W$, $w = (w_1, w_2, w_3) \in W$ and $c \in F$. Thus $v_1 + 3v_3 = v_2$ and $w_1 + 3w_3 = w_2$ and one checks easily that $(v_1 + w_1) + 3(v_3 + w_3) = (v_2 + w_2)$, so $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \in W$. Moreover, $cv_1 + 3cv_3 = c(v_1 + 3v_3) = cv_2$ so that $cv = (cv_1, cv_2, cv_3) \in W$. That is, $W \subseteq V$ is a subspace.

b) $F = \mathbb{Q}$, $V = \mathbb{C}$ and $W = \{x + yi \in \mathbb{C} | x \in \mathbb{Q}\}$.

Solution: Clearly $0 \in W$ as its real part is a rational number (namely, 0). If $x + yi \in W$ and $z + wi \in W$, then $x + z \in \mathbb{Q}$ as the sum of rational numbers is rational; therefore $(x + z) + (y + w)i \in W$. Finally, if $x + yi \in W$ and $c \in \mathbb{Q}$ then $cx \in \mathbb{Q}$ because the product of rational numbers is rational; therefore $c(x + yi) = cx + cyi \in W$. Since W contains 0 and is closed under addition and scalar multiplication, it is a subspace.

c) $F = \mathbb{R}$, $V = C^1(\mathbb{R})$ the space of continuously differentiable real-valued functions on the real line, and $W = \{f \in V | f'(1) = 0\}$.

Solution: The 0-vector in V is the constant function 0; since its derivative is again the zero function, $0 \in W$. If f and g are in W, then $(f + g)'(1) = f'(1) + g'(1) = 0 + 0 = 0$, so W is closed under addition. Finally, if $f \in W$ and $c \in \mathbb{R}$, then $(cf)'(1) = cf'(1) = 0$, so W is also closed under scalar multiplication and therefore a subspace.

2. For each of the following subsets $S \subseteq V$ of the F-vector space V, check if S is linearly independent. Prove your answer. (2 pts each)

a) $F = \mathbb{R}$, $V = \mathbb{C}$ and $S = \{1 + i, 1 - i\}$.

Solution: Suppose that we have real numbers a and b such that $a(1 + i) + b(1 - i) = 0$. That means $(a + b) + (a - b)i = 0$, which implies (as a and b are real) that $a = b$ and $a = -b$. But this means that $a = b = 0$. Hence S is linearly independent.

b) $F = \mathbb{C}$, $V = \mathbb{C}$ and $S = \{1 + i, 1 - i\}$.

Solution: Now (in contrast to a)) the scalars are allowed to be complex numbers. Take $a = (i - 1)$ and $b = (1 + i)$. Then $a(1 + i) + b(1 - i) = -2 + 2 = 0$. Therefore, S is linearly dependent.

c) $F = \mathbb{R}$, $V = \mathbb{R}^3$ and $S = \{(v_1, v_2, v_3) \in V | v_1^2 + v_2 = 5\}$.

Solution: The set S has infinitely many elements. Since V is of dimension 3, S must be linearly dependent.

3. Let S and T be linearly independent subsets of an F-vector space V. Assume that $S \cap T = \emptyset$. Is $S \cup T$ necessarily linearly independent? If so, give a proof. If not, give a counter-example. (3 pts)

Solution: Let $F = \mathbb{R}$ and $V = \mathbb{R}$, and let $S = \{1\}$ and $T = \{2\}$. Then S and T are linearly independent (as they each contain only one, non-zero, vector), their intersection is empty, but the union $S \cup T = \{1, 2\}$ is obviously linearly dependent.

4. Let V be an F-vector space, U and W two F-subspaces, and $S \subseteq U$ and $T \subseteq W$ linearly independent sets. Suppose that $U \cap W = \{0\}$. Is $S \cup T$ necessarily linearly independent? If so, give a proof. If not, give a counter-example. (5 pts)

Solution: I claim that $S \cup T$ must be linearly independent. Indeed, suppose we have $u_1, \dots, u_r \in S$, $w_1, \dots, w_k \in W$, $a_1, \dots, a_r \in F$ and $b_1 \dots, b_k \in F$ such that

$$a_1u_1 + \cdots + a_ru_r + b_1w_1 + \cdots + b_kw_k = 0$$

This means that

$$a_1u_1 + \cdots + a_ru_r = (-b_1)w_1 + \cdots + (-b_k)w_k$$

and since U and W are subspaces, both sides of this equation must be in $U \cap W = \{0\}$, and therefore must be zero. Since S is linearly independent, this implies that $a_1 = \cdots = a_r = 0$; and because T is linearly independent, we also conclude that $-b_1 = \cdots = -b_k = 0$ and hence $b_1 = \cdots = b_k = 0$. That is, $S \cup T$ is linearly independent, as asserted.