MATH 115A - Lecture 1 - Winter 2010
Midterm 1 - January 26, 2010

NAME:

STUDENT ID #:

This is a closed-book and closed-note examination.
Calculators are not allowed.
Please show all your work.
Use only the paper provided. You may write on the back if you need more space, but clearly indicate this on the front.
There are 5 problems for a total of 100 points.

POINTS:

1.

2.

3.

4.

5.

1. (20 points) Let $S = \{(1,1,2),(2,1,1),(0,0,1)\} \subseteq \mathbb{R}^3$. Is S a basis for $\mathbb{R}^3$? If so, prove it. If not, disprove it.

Solution: Since the dimension of $\mathbb{R}^3$ is 3 and S has three elements, it suffices to show either that S is linearly independent, or that S is a generating set.

1) S is linearly independent: Suppose $a(1,1,2) + b(2,1,1) + c(0,0,1) = (0,0,0)$. Then $a + b = 0$ but also $a + 2b = 0$ which implies $a = b = 0$. From this we conclude that $c = 0$, as well. Thus, S is linearly independent. (Obviously, you can use any other way to solve these linear equations...)

2) We can write $(1,0,0) = (2,1,1)-(1,1,2)+(0,0,1)$, $(0,1,0) = 2(1,1,2)-(2,1,1) - 2(0,0,1)$ and $(0,0,1) = (0,0,1)$. That is, the standard basis is in the span of S. This implies that S is a generating set. (It's just as easy to solve the general linear equations showing any (x,y,z) can be expressed, using Gauss elimination.)

Common mistakes: The set S has three elements. It *does not* have dimension three. Dimension is an invariant of vector spaces, and S is not a vector space. Conversely, $\mathbb{R}^3$ has dimension 3, but it doesn't have three elements (it has infinitely many).

2. (20 points) Let V be an F-vector space and $T : V \to V$ a linear transformation. Suppose $N(T) \cap R(T) = \{0\}$. Prove that $N(T \circ T) = N(T)$.

Solution: If $T(v) = 0$ then certainly $T(T(v)) = T(0) = 0$, so $N(T) \subseteq N(T \circ T)$. Conversely, suppose $v \in N(T \circ T)$ so $T(T(v)) = 0$. Then $T(v) \in N(T)$ (by definition!), but on the other hand $T(v) \in R(T)$, too (again, by definition!). By hypothesis $N(T) \cap R(T) = \{0\}$, so we conclude $T(v) = 0$ and therefore $v \in N(T)$. Thus, we proved $N(T \circ T) \subseteq N(T)$ and (previously) $N(T) \subseteq N(T \circ T)$. This implies that $N(T) = N(T \circ T)$, as asserted.

Common mistakes: Many of you only proved the first, easy inclusion. Since this does not need the hypothesis, this can't be the complete solution! Others made unsupported claims about T (e.g., that T was 0 or the identity).

The strategy in showing that two subsets or (in this case) subspaces are equal is *always the same:* you need to prove each is ocntained in the other. To do so, it suffices in many cases (as it does here) write down what it means to be in one of the subsets, and observe that this, together possibly with the hypothesis, implies it is in the other by definition.

Expressions of the form $N(T(v))$ and similar *are meaningless.* The Null space is a space associated with some linear transformation. Not with a vector, or a linear transformation applied to a vector.

Finally note (we'll see this again later): the range $R(T)$ of a linear transformation is *not the same as the target* (in this case V). It is only those elements in the target that are of the form $T(v)$ for some v in the source.

3. (20 points) Let $\mathcal{P}_2$ be the $\mathbb{R}$-vector space of polynomials $f(x) = a + bx + cx^2$ of degree at most two. Let $T : \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation given by $T(f) = (f(0), f'(0))$. Determine the dimensions of the Null space and range of T. (Prove your results!)

Solution: If $f(x)$ is a polynomial of degree at most 2 such that $f(0) = 0$ and $f'(0) = 0$, then $f(x) = cx^2$ for some $c \in \mathbb{R}$. Therefore $\{x^2\}$ is a basis for the Null space $N(T)$, which is hence of dimension 1. By the dimension theorem, $R(T)$ is of dimension 2. Alternatively, observe that $T(1) = (1, 0)$ and $T(x) = (0, 1)$ so that $R(T)$ contains the standard basis and is therefore all of $\mathbb{R}^2$, hence of dimension 2. Apply the dimension theorem to conclude that $N(T)$ is of dimension 1.

Common mistakes: Again, you have to prove that $R(T) = \mathbb{R}^2$. This is not the case simply by definition of range; it requires argument that depends on the particular linear transformation T given in the problem.

4. (20 points) Let $F = \mathbb{R}$ and $V = \mathbb{C}^2$, viewed as an $\mathbb{R}$-vector space. Let $W = \{(z, w) \in V | z + \overline{w} \in \mathbb{R}\}$. Is W a subspace? If so, prove it; if not, disprove it. (Here for a complex number z, $\overline{z}$ is the complex conjugate.)

Solution: I claim that W is a subspace. To see that, we need to show that $0 \in W$, and that W is closed under addition and scalar multiplication.

First, $(0, 0) \in W$ as $0+\overline{0} = 0+0 = 0 \in \mathbb{R}$. Now let (z, w) and (x, y) be in W. (Note that z, w, x and y are all complex numbers.) Then $(z, w)+(x, y) = (z + x, w + y)$ and $(z + x) + \overline{(w + y)} = z + x + \overline{w} + \overline{y} = z + \overline{w} + x + \overline{y} \in \mathbb{R}$ since the sum of two real numbers is real. Finally, if $(z, w) \in W$ and $c \in \mathbb{R}$ then $c(z, w) = (cz, cw)$ and $cz + \overline{cw} = c(z + \overline{w})$ since complex conjugation is multiplicative and c is real. And the last expression is real as a product of two real numbers.

Common mistakes: There was much confusion about real numbers, complex numbers, and what $\mathbb{C}^2$ is.

5. (20 points) Suppose V and W are finite-dimensional F-vector spaces of the same dimension. Show that there is an isomorphism $T : V \to W$.

Solution: Write n for the common dimension of V and W. Choose a basis $\mathcal{A} = \{v_1, \ldots, v_n\}$ of V and a basis $\mathcal{B} = \{w_1, \ldots, w_n\}$ of W. There is (as proved in class) a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $1 \leq i \leq n$. I claim T is an isomorphism. This can be seen in various ways:

1) There is a unique linear transformation $S : W \to V$ such that $S(w_i) = v_i$ for $1 \leq i \leq n$. Note that $T \circ S(w_i) = w_i$ and $S \circ T(v_i) = v_i$. Since there's only one linear transformation $W \to W$ (resp., $V \to V$) fixing the elements of a basis, namely the identity, we conclude that S is an inverse of T. Therefore T is an isomorphism.

2) T maps the basis $\mathcal{A}$ to a basis of W. In particular, T is onto, and by the corollary to the dimension theorem, it is also one-to-one. that is, T is an isomorphism.

3) T maps a basis to a basis. In particular, T is one-to-one. By the corollary to the dimension theorem, T is also onto, and therefore an isomorphism.

Common mistakes: There were very few correct solutions. The basic problem is that you need to understand the problem! It says "show there is...". It does *not* say "show that all $T : V \to W$ are isomorphisms". The task was to find *some* T that is an isomorphism, or construct such an isomorphism. If I ask you "show there's a yellow dog" it does not mean "show every dog is yellow" - it means, find some dog that is yellow.

Many of you tried to prove that every linear transformation $T : V \to W$ has to be an isomorphism. This is false. Even if it were correct, it wouldn't solve the problem since it is still necessary to show there is such a transformation!

Finally, to repeat again, for a general linear transformation $R(T) \neq W$!