LECTURE NOTES ON STABILITY THEORY

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These are the notes for my quarter-long course on basic stability theory at UCLA (MATH 285D, Winter 2015). The presentation highlights some relations to set theory and cardinal arithmetic reflecting my impression about the tastes of the audience. We develop the general theory of local stability instead of specializing to the finite rank case, and touch on some generalizations of stability such as NIP and simplicity. The material in this notes is based on [Pil02, Pil96], [vdD05], [TZ12], [Cas11a, Cas07], [Sim15], [Poi01] and [Che12].

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1. Introduction and preliminaries

1.1. **Notation.** A *(first-order) structure* $\mathcal{M} = (M, R_1, R_2, \ldots, f_1, f_2, \ldots, c_1, c_2, \ldots)$ consists of an underlying set M, together with some distinguished relations R_i (subsets of M^{n_i}, $n_i \in \mathbb{N}$), functions $f_i : M^{n_i} \to M$, and constants c_i (distinguished elements of M). We refer to the collection of all these relations, function symbols and constants as *the signature* of $\mathcal{M}$. For example, a group is naturally viewed as a structure $\left(G, \cdot, ^{-1}, 1\right)$, as well as a ring $(R, +, \cdot, 0, 1)$, ordered set $(X, <)$, graph (X, E), etc. A *formula* is an expression of the form $\psi(y_1, \ldots, y_m) = \forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \phi(x_1, \ldots, x_n; y_1, \ldots, y_n)$, where ϕ is given by a boolean combination of (superpositions of) the basic relations and functions (and $y_1, \ldots, y_n$ are the *free variables* of ψ). We denote the set of all formulas by L. We also consider formulas with parameters, i.e. expressions of the form $\psi\left(\bar{y}, \bar{b}\right)$ with $\psi \in L$ and $\bar{b}$ a tuple of elements in M. Given a set of parameters $B \subseteq M$, we let $L(B) = \left\{\psi\left(\bar{y}, \bar{b}\right) : \psi \in L, \bar{b} \in B^{|\bar{b}|}\right\}$. If $\psi(\bar{y}) \in L(B)$ is satisfied by a tuple $\bar{a}$ of elements of M, we denote it as $\mathcal{M} \models \psi(\bar{a})$ or $a \models \psi(\bar{y})$, and we call $\bar{a}$ a solution of ψ. If $\Psi(\bar{y})$ is a set of formulas, we write $a \models \Psi(\bar{y})$ to denote that $a \models \psi(\bar{y})$, for all $\psi \in \Psi$. Given a set $A \subseteq M^{|x|}$, we denote by $\psi(A)$ the set $\left\{a \in A^{|x|} : M \models \psi(A)\right\}$ of all solutions of ψ in A. We say that $X \subseteq M^n$ is an *A-definable set* if there is some $\psi(\bar{x}) \in L(A)$ such that $X = \psi(M^n)$. If ψ has no free variables, then it is called a *sentence*, and it is either true or false in $\mathcal{M}$. By the *theory of* $\mathcal{M}$, or $\mathrm{Th}(\mathcal{M})$, we mean the collection of all sentences that are true in M.

Example 1.1.
(1) Let $\mathcal{M} = (\mathbb{C}, +, \times, 0, 1)$. Then $\mathrm{Th}(\mathcal{M})$ eliminates quantifiers, and definable subsets of $\mathcal{M}^n$ are precisely the *constructible ones*, i.e. Boolean combinations of algebraic sets. $\mathrm{Th}(\mathcal{M})$ is axiomatized as the theory of algebraically closed fields of char 0, denoted ACF_0. Note in particular that every definable subset of $\mathcal{M}$ is either finite, or cofinite. Theories satisfying this property are called *strongly minimal.*
(2) Let $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1, <)$. Then by Tarski's quantifier elimination, definable subsets of $\mathcal{M}^n$ are precisely the semialgebraic ones, i.e. Boolean combinations of polynomial equalities and inequalities. The theory $\mathrm{Th}(\mathcal{M})$ is axiomatized as the theory of ordered real closed fields and denoted RCF. In particular, all definable subsets of M are given by finite unions of points and intervals. Theories with this property are called *o-minimal.*
(3) Consider $(\mathbb{N}, +, \times, 0, 1)$. The more quantifiers we allow, the more complicated sets we can define (e.g. non-computable sets, Hilbert 10, and in fact a large part of mathematics can be encoded — "Gödelian phenomena").

In general, we call any consistent set of sentences T in a language L a *theory*, and we say that T is complete if for every L sentence ψ, either ψ or $\neg\psi$ is in T. Two structures $\mathcal{M}$ and $\mathcal{N}$ in the same language are *elementarily equivalent*, denoted by $\mathcal{M} \equiv \mathcal{N}$, if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. Given two L-structures $\mathcal{M}$ and $\mathcal{N}$ and a (partial) map $f : \mathcal{M} \to \mathcal{N}$ we say that f is an *elementary map* if for all $a \in \mathrm{Dom}(f)$ and $\phi \in L$ we have $\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(f(a))$. We say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ if the embedding map is elementary. Model theory studies complete first-order theories T, equivalently structures up to elementary equivalence, and their corresponding categories of definable sets. Note that if $\mathcal{M}$ is finite and elementarily equivalent to $\mathcal{N}$, then it is isomorphic to $\mathcal{N}$[1]. However, the situation is quite different if T admits an infinite model.

Fact 1.2. *(Löwenheim–Skolem theorem) Let $\mathcal{M} \models T$ be given, with $|\mathcal{M}| \geq \aleph_0$. Then for any cardinal $\kappa \geq |L|$ there is some $\mathcal{N}$ with $|\mathcal{N}| = \kappa$ and such that:*

- *$\mathcal{M} \preceq \mathcal{N}$ if $\kappa > |\mathcal{M}|$,*
- $\mathcal{N} \preceq \mathcal{M}$ if $\kappa < |\mathcal{M}|$.

In particular, ZFC has a countable model. To keep things interesting, from now on we will always be assuming that T admits infinite models[2].

Exercise 1.3. Let $\mathcal{M} \models T$ and $A \subseteq M$ is an infinite set. Then there is some $\mathcal{N} \preceq \mathcal{M}$ such that $A \subseteq N$ and $|N| \leq |A| + |T|$.

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In fact, most of the structural properties of T that we will consider in this course do not depend on the specifics of the language L, and are invariant up to *bi-interpretability*. Let $\mathcal{M}$ be an L-structure, and $\mathcal{N}$ an L'-structure. An *interpretation* of $\mathcal{M}$ in $\mathcal{N}$ is given by a surjective map f from a subset of N^n onto M such that for every definable relation $X \subseteq M^k$, its preimage $f^{-1}(X)$ is definable in $\mathcal{N}$ (note that in particular $f^{-1}(M)$ is definable and "=" on M). Two structures $\mathcal{M}, \mathcal{N}$ are *bi-interpretable* if there exists an interpretation of $\mathcal{M}$ in $\mathcal{N}$ and an interpretation of $\mathcal{N}$ in $\mathcal{M}$ such that the composite interpretations of $\mathcal{M}$ in itself and of $\mathcal{N}$ in itself are definable in $\mathcal{M}$ and $\mathcal{N}$, respectively.

Example 1.4.
1. (Julia Robinson) $(\mathbb{Z}, +, \times)$ is definable in $(\mathbb{Q}, +, \times)$ (which implies that $\mathbb{Q}$ is undecidable, in particular).
2. If $(G, \cdot)$ is a group and H is a definable subgroup, then G/H is interpretable in G. We will see a general construction dealing with quotients soon.
3. $(\mathbb{C}, +, \times)$ is interpretable in $(\mathbb{R}, +, \times)$, but *not* the other way around. (Why?)
4. Every structure in a finite relational language is bi-interpretable with a graph.
5. (Mekler's construction) Every structure in a finite relational language is interpretable in a pure group, and this interpretation reflects most model-theoretic properties.

Example 1.5. ["*Morleyzation*"] Starting with an arbitrary structure $\mathcal{M}_0$ in a language L_0, we can consider an expansion $\mathcal{M}_1$ in a language L_1 such that $L_1 = L_0 \cup \{R_\phi(\bar{x}) : \phi(\bar{x}) \in L\}$ and interpret $R_\phi(M) = \phi(M)$. Thus, every L_0-formula

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[1] A celebrated theorem of Keisler and Shelah shows that in fact $\mathcal{M} \equiv \mathcal{N}$ if and only if there is some ultrafilter U such that the corresponding ultrapowers $\mathcal{M}^U$ and $\mathcal{N}^U$ are isomorphic.

[2] Finite model theory (`https://en.wikipedia.org/wiki/Finite_model_theory`) studies definability in finite structures, and has close connections to computational complexity. However, first-order logic appears inadequate for that setting, it is both too weak (can't express most of the interesting properties of graphs such as connectivity, Hamiltonicity, etc.) and too strong (a single first order sentence describes a finite structure up to isomorphism). Various attempts to develop stability for finite structures were made, see e.g. notes of John Baldwin `http://www.math.uic.edu/~jbaldwin/pub/philtr.ps` and references there. Some connections between the finite and infinite model theory can be seen in the setting of pseudo-finite theories.

is equivalent to a quantifier-free L_1-formula. Repeating the same procedure for L_1 and catching our own tail, we obtain an expansion $\mathcal{M}_\infty$ of $\mathcal{M}$ in the language $L_\infty = \bigcup_{i<\omega} L_i$. It is easy to see that $\mathcal{M}_\infty$ eliminates quantifiers in the language L_∞, and that $\mathcal{M}$ and $\mathcal{M}_\infty$ have exactly the same definable sets and thus they are bi-interpretable. The conclusion is that in the development of the abstract theory, we can usually assume that T eliminates quantifiers.

Finally, recall the most important theorem of model theory.

Fact 1.6. *(Compactness theorem) Let L be an arbitrary language, and let Ψ be a set of L-sentences (of arbitrary size!). Assume that every finite subset $\Psi_0 \subseteq \Psi$ is consistent (i.e. there is some L-structure $\mathcal{M} \models \Psi_0$), then Ψ is consistent.*

It follows from Gödel's completeness theorem[3], or can be proved directly using the ultraproduct construction.

1.2. **Saturation, monster models, definable and algebraic closures.** Let A be a set of parameters in $\mathcal{M}$. By a *partial type* $\Phi(x)$ *over* A we mean a collection of formulas of the form $\phi(x)$ with parameters from A such that every finite subcollection has a common solution in $\mathcal{M}$. By a *complete type* over A we mean a partial type such that for *every* formula $\phi(x) \in L(A)$, either $\phi(x)$ or $\neg\phi(x)$ is in it. For $b \in \mathcal{M}$, we denote by $\mathrm{tp}(b/A)$ the complete type of b over A, i.e. $\mathrm{tp}(b/A) = \{\phi(x) : b \models \phi(x), \phi(x) \in L(A)\}$.

Definition 1.7. Let κ be an infinite cardinal.

(1) We say that $\mathcal{M}$ is *κ-saturated* if for any set of parameters $A \subseteq M$ with $|A| < \kappa$, every partial type $\Phi(x)$ over A with $|x| < \kappa$ can be realized in $\mathcal{M}$ (enough to verify it for 1-types).
(2) We say that $\mathcal{M}$ is *κ-homogenous* if any partial elementary map from $\mathcal{M}$ to itself with a domain of size $< \kappa$ can be extended to an automorphism of $\mathcal{M}$.

Fact 1.8. *For any T and κ, there is a κ-saturated and κ-homogeneous model $\mathcal{M}$ of T.*

Proof. (Idea) Given $\mathcal{M}$ and a complete type p over it, by compactness theorem (applied to L expanded by constants naming all elements of $\mathcal{M}$ and a new constant c) we can find some $\mathcal{N} \succeq \mathcal{M}$ and some $c \in \mathcal{N}$ such that $c \models p$. Now do this for all types over $\mathcal{M}$, and then take a union of the elementary chain catching our own tail. □

We say that $\mathcal{M}$ is *saturated* if it is $|\mathcal{M}|$-saturated. Saturated models always exist under some set-theoretic assumptions (such as GCH or inaccessible cardinals), but it is also possible that e.g. RCF has no saturated models. This has something to do with stability, in fact.

Example 1.9.
(1) $(\mathbb{C}, +, \times, 0, 1)$ is saturated (Exercise: an algebraically closed field is saturated if and only if it has infinite transcendence degree).
(2) Let $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$, and consider $\Phi(x) = \left\{0 < x < \frac{1}{n} : n \in \mathbb{N}\right\}$. It is clearly a partial type, but it is not realized in $\mathbb{R}$, thus $\mathbb{R}$ is not $\aleph_0$-saturated. Passing to some $\aleph_0$-saturated $\mathbb{R}^* \succ \mathbb{R}$, the set of solutions of $\Phi(x)$ in $\mathbb{R}^*$

[3]Both are equivalent to the Boolean Prime Ideal theorem, a weak form of the Axiom of Choice.

is the set of "infinitesimal" elements, and we can do *non-standard analysis* working in $\mathbb{R}^*$.

(3) Let $\mathcal{M} = (\mathbb{N}, +, \times, 0, 1)$ and consider the partial type

$$\Phi(x) = \{p|x : p \text{ prime in } \mathbb{N}\} \cup \{x \neq 0\}.$$

Again, it is not realized in $\mathbb{N}$, but it is realized in some non-standard extension $\mathbb{N}^*$ of $\mathbb{N}$.

From now on we fix a complete L-theory T. By the completeness of T, any commutative diagram of models of T and elementary embeddings between them can be realized in a single model of T, such that the embeddings become inclusions among elementary submodels. We would like to work in a very rich model of T akin to the "universal domain" of Weil in algebraic geometry, with all the action taking place inside this fixed model. For this purpose, we fix some model $\mathbb{M}$ and some sufficiently large cardinal $\kappa(\mathbb{M})$ such that $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous, and we will refer to $\mathbb{M}$ as a *monster model.* In particular, every model of T of size $\leq \kappa(\mathbb{M})$ embeds elementarily into $\mathbb{M}$. So now whenever we say "a model", "a set of parameters", "a tuple" or "a definable set" we will mean an elementary submodel of $\mathbb{M}$, a set of parameters in $\mathbb{M}$, a tuple of elements of $\mathbb{M}$ or a definable subset of $\mathbb{M}$, respectively. Whenever we say *"small"*, we mean "of size $< \kappa(\mathbb{M})$". Given $\phi(x) \in L(\mathbb{M})$ and $a \in \mathbb{M}$ we will write "$\models \phi(a)$" to denote "$\mathbb{M} \models \phi(a)$".

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Given two sets of formulas $\Phi(x), \Psi(x)$ we will write $\Phi(x) \vdash \Psi(x)$ if for every $a \in \mathbb{M}$ such that $\models \Phi(a)$ holds, also $\models \Psi(a)$ holds. Note that, by saturation, compactness theorem translates into the following.

Fact 1.10. *Let $\phi(x)$ be an $L(\mathbb{M})$ formula, and $\Phi(x)$ a small set of $L(\mathbb{M})$-formulas. If $\Phi(x) \vdash \phi(x)$, then there is some finite $\Phi_0 \subseteq \Phi(x)$ such that $\Phi_0 \vdash \phi(x)$.*

Exercise 1.11. Let $A \subseteq \mathbb{M}$ be a small set of parameters, x is a small tuple of variables, and assume that $\bigcap_{i \in I} X_i \subseteq \bigcup_{j \in J} Y_j$, where X_i, Y_j are A-definable subsets of $\mathbb{M}^{|x|}$. Then there are finite $I_0 \subseteq I, J_0 \subseteq J$ such that $\bigcap_{i \in I_0} X_i \subseteq \bigcup_{j \in J_0} Y_j$.

Working in a monster model has several advantages, for example we can import some Galois-theoretic ideas. Given a set of parameters A, let $\mathrm{Aut}(\mathbb{M}/A)$ denote the group of automorphisms of $\mathbb{M}$ (under composition) that fix A pointwise. Note that by strong homogeneity of $\mathbb{M}$, for any small A we have $\mathrm{tp}(b/A) = \mathrm{tp}(c/A) \iff \sigma(b) = c$ for some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ (as the assumption implies that the map $bA \mapsto cA$ is elementary).

Lemma 1.12. *Let X be a definable subset of $\mathbb{M}^n$. Then X is A-definable if and only if $\sigma(X) = X$ (as a set) for all $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.*

Proof. "$\Rightarrow$". Assume that $X = \phi(\mathbb{M}, b)$ for some $\phi \in L$ and $b \in A$. Then for any $a \in \mathbb{M}$ and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ we have $a \in X \iff \models \phi(a, b) \iff \models \phi(\sigma(a), \sigma(b)) \iff \models \phi(\sigma(a), b) \iff \sigma(a) \in X$.

"$\Leftarrow$". Assume that $X = \phi(\mathbb{M}, b)$ where b is some tuple from $\mathbb{M}$, and let $p(y) := \mathrm{tp}(b/A)$.

Claim 1. $p(y) \vdash \forall x (\phi(x, y) \leftrightarrow \phi(x, b))$. Indeed, let $b' \models p(y)$ be arbitrary. Then $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$, so there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(b) = b'$. Then $\sigma(X) = \phi(\mathbb{M}, b')$, and by assumption $\sigma(X) = X$, thus $\phi(\mathbb{M}, b) = X = \phi(\mathbb{M}, b')$.

By Fact 1.10 it follows that there is some $\psi(y) \in p$ such that

$$\psi(y) \vdash \forall x (\phi(x, y) \leftrightarrow \phi(x, b)).$$

Let $\theta(x)$ be the formula $\exists y (\psi(y) \wedge \phi(x, y))$. Note that $\theta(x)$ is an $L(A)$-formula, as $\psi(y)$ is.

Claim 2. $X = \theta(\mathbb{M})$. If $a \in X$, then $\models \phi(a, b)$, and as $\psi(y) \in \mathrm{tp}(b/A)$ we have $\models \theta(a)$. Conversely, if $\models \theta(a)$, let b' be such that $\models \psi(b') \wedge \phi(a, b')$. But by the choice of ψ this implies that $\models \phi(a, b)$ holds. □

A slight generalization of the previous lemma.

Lemma 1.13. *Let $X \subseteq \mathbb{M}^n$ be definable. The following are equivalent:*

1. *X is* almost A-definable*, i.e. there is an A-definable equivalence relation E on $\mathbb{M}^n$ with finitely many classes, such that X is a union of E-classes.*
2. *The set $\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite.*
3. *The set $\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.*

Proof. (1) ⇒ (2). Again immediate, as any automorphism fixing A can only permute the classes of E.

(2) ⇒ (1). Again assume $X = \phi(\mathbb{M}, b)$ and $p(y) = \mathrm{tp}(b/A)$. By assumption there are some $b_0, \ldots, b_k$ realizing p and such that for any $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\sigma(X) = \phi(\mathbb{M}, b_i)$ for some $i \leq k$. Then, by homogeneity as before, we have $p(y) \vdash \bigvee_{i \leq k} \forall x (\phi(x, y) \leftrightarrow \phi(x, b_i))$. By compactness there is some $\psi(y) \in p$ such that $\psi(y) \vdash \bigvee_{i \leq k} \forall x (\phi(x, y) \leftrightarrow \phi(x, b_i))$. Now define $E(x_1, x_2)$ as

$$\forall y (\psi(y) \rightarrow (\phi(x_1, y) \leftrightarrow \phi(x_2, y))),$$

so it is A-definable. It is easy to check that E is an equivalence relation with finitely many classes, and that X is a union of E-classes (unwinding, $a_1 E a_2$ iff they agree on $\phi(x, b_i)$ for all $i \leq k$, and so $X = \phi(\mathbb{M}, b_0)$ is given by the union of all possible combinations intersected with it).

(3) ⇒(1). Exercise. □

Definition 1.14. Let A be a set of parameters and b a tuple.

1. We say that b is *definable over* A if there is some formula $\phi(x) \in L(A)$ such that b is the unique solution of $\phi(x)$ in $\mathbb{M}$.
2. We say that b is *algebraic over* A if there is some formula $\phi(x) \in L(A)$ such that $\models \phi(b)$ and $\phi(x)$ has only finitely many solutions in $\mathbb{M}$.
3. We denote by $\mathrm{dcl}(A)$ ($\mathrm{acl}(A)$) the set of all elements definable over A (resp. algebraic over A). Note that $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ and that both $\mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ are preserved by $\mathrm{Aut}(\mathbb{M}/A)$.

Corollary 1.15.
1. $b \in \mathrm{dcl}(A) \iff \sigma(b) = b$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ *(follows by Lemma 1.12 applied to $X = \{b\}$).*
2. $b \in \mathrm{acl}(A) \iff$ *the* $\mathrm{Aut}(\mathbb{M}/A)$*-orbit of b is finite* $\iff$ *the* $\mathrm{Aut}(\mathbb{M}/A)$*-orbit of b is small (follows by Lemma 1.13).*

Example 1.16. If T is a theory of a set, then $a \in \mathrm{acl}(B) = \mathrm{dcl}(B)$ iff $a \in B$. If T is a vector space, then acl = dcl = linear span. In ACF, $\mathrm{dcl}(A)$ is the perfect hull of the field k generated by A (so in ACF_0 it's just k itself, and in ACF_p it's given by $\bigcup_{n\in\omega} \mathrm{Frob}^{-n}(k)$). As any irreducible polynomial over k has no multiple roots, if $a \in \mathrm{dcl}(k) \subseteq \mathrm{acl}(k)$, its type is isolated by the minimal polynomial, can't have multiple roots, so has to be of degree one) and $\mathrm{acl}(A)$ coincides with the usual algebraic closure in the field sense.

Corollary 1.17. *For $A \subseteq \mathbb{M}$, $\mathrm{acl}(A) = \bigcap \{M : M \prec \mathbb{M}, A \subseteq M\}$.*

Proof. Assume that $a \in \mathrm{acl}(A)$, and let $M \supseteq A$ be arbitrary. By definition this means that there is some $\phi(x) \in L(A)$ such that $\models \phi(a)$ and $|\phi(\mathbb{M})| = n$ for some $n \in \omega$. But as $M \prec \mathbb{M}$, $|\phi(M)| = n$ as well, so in particular $a \in M$.

On the other hand, assume that $a \notin \mathrm{acl}(A)$. Then by 1.15, the $\mathrm{Aut}(\mathbb{M}/A)$-orbit of a is not small. Let $M \supseteq A$ be an arbitrary small model. Then $f(a) \notin M$ for some A-automorphism f. But then $N = f^{-1}(M)$ is a model containing A, and $a \notin N$. □

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1.3. M^{eq} and strong types.

Remark 1.18. All the notions above generalize in an obvious way to *multi-sorted structures*, i.e. the underlying set of our model is now partitioned into several sorts, and for each of the variables for each relation and function symbols we specify which sort they live in. Then formulas and other notions are defined and evaluated accordingly.

We give a construction that allows to treat definable sets and quotient objects in the same way as elements of the structure.

We start with an arbitrary L-structure M with $\mathrm{Th}(M) = T$. Let $\mathrm{ER}(T)$ be the collection of all L-formulas $E(x, y)$ that define an equivalence relation on a certain tuple of sorts in M (i.e. we have a definable relation $E(x_1, \ldots, x_n; x'_1, \ldots, x'_n)$ and x_i, x'_i live on the same sort of M, for each i). We define a new language $L^{\mathrm{eq}} := L \cup \{S_E : E \in \mathrm{ER}(T)\} \cup \{f_E : E \in \mathrm{ER}(T)\}$, where S_E is a sort and f_E is a new function symbol from the sort on which E lives into the new sort S_E. In particular, for every sort S of M there is a corresponding sort $S_=$, where $=$ is the equality on the sort S, and all the L-structure on it. Note that $|L| = |L^{\mathrm{eq}}|$.

We now enlarge M to a canonical L^{eq}-structure M^{eq}. The sorts ($S_= : S$ is a sort of M) and all the L-structure on them is identified with M, the sort S_E in M^{eq} is given by the set $\{a/E : a \in M_x\}$, and the function f_E is interpreted by $a \mapsto a/E$. With this identification we clearly have that for all $M \models T$, $\phi(x) \in L$ and $a \in M_x$, $M \models \phi(a) \iff M^{\mathrm{eq}} \models \phi(a)$.

We define the L^{eq}-theory

$$T^{\mathrm{eq}} := T \cup \{(\forall y \in S_E \exists x \in S_= f_E(x) = y) : E \in \mathrm{ER}(T)\} \cup$$
$$\cup \{(\forall x_1, x_2 (f_E(x_1) = f_E(x_2) \leftrightarrow E(x_1, x_2))) : E \in \mathrm{ER}(T)\}.$$

Clearly $M \models T$ implies $M^{\mathrm{eq}} \models T^{\mathrm{eq}}$.

Lemma 1.19. (1) *Every $M^* \models T^{\mathrm{eq}}$ is of the form M^{eq} for some $M \models T$.*
(2) *Given $E_1, \ldots, E_k \in \mathrm{ER}(T)$ and $\phi(x_1, \ldots, x_k) \in L^{\mathrm{eq}}$ (with x_i living on S_{E_i}), there is some $\psi(y_1, \ldots, y_k) \in L$ such that*
$$T^{\mathrm{eq}} \vdash (\forall y_1 \ldots y_k \in S_=)(\psi(y_1, \ldots, y_k) \leftrightarrow \phi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k))).$$
(3) *T^{eq} is complete.*

(4) $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$ *(where* $\mathrm{dcl}^{\mathrm{eq}}$ *means that the definable closure is taken in the sense of* M^{eq}*).*
(5) *Every* $X \subseteq M_x$ *definable (with parameters) in the structure* M^{eq} *is already definable in* M*.*
(6) *If* M *is* κ*-saturated (*κ*-homogeneous), then* M^{eq} *is* κ*-saturated (resp.* κ*-homogeneous).*
(7) *Every automorphism of* M *extends in a unique way to an automorphism of* M^{eq}*.*

Every definable $X \subseteq \mathbb{M}^n$ "corresponds" to an element of $\mathbb{M}^{\mathrm{eq}}$. Suppose $X = \phi(\mathbb{M}, b)$, let $E(y_1, y_2) = \forall x (\phi(x, y_1) \leftrightarrow \phi(x, y_2))$. Then $b/E \in S_E$ in $\mathbb{M}^{\mathrm{eq}}$, and we have:

(1) $\forall \sigma \in \mathrm{Aut}(\mathbb{M})$ (equivalently, $\mathbb{M}^{\mathrm{eq}}$), $\sigma(X) = X$ iff $\sigma(b/E) = b/E$.
(2) X is (b/E)-definable in $\mathbb{M}^{\mathrm{eq}}$, via $\exists y (f_E(y) = b/E \wedge \phi(x, y))$.
(3) Let $\psi(x, b/E)$ be the L^{eq}-formula defining X, then b/E is the unique element z of sort S_E such that $X = \psi(x, z)$.

We say that b/E is a *code* for X, and can think of $\mathbb{M}^{\mathrm{eq}}$ as adjoining codes for all definable equivalence relations (as c/E' codes $E'(x, c)$ for an arbitrary equivalence relation E). If the tuple a is a code, it depends on the shape of the formula defining X, but is unique up to interdefinability: if b is another code for X, then $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$.

Definition 1.20. We say that T has *elimination of imaginaries,* or *EI*, if for any $\emptyset$-definable equivalence relation E on $\mathbb{M}_x$ and E-class X, there is some tuple a from $\mathbb{M}$ such that for some L-formula $\phi(x, y)$, X is defined by $\phi(x, a)$ and whenever $\phi(x, a')$ also defines X, then $a = a'$.

Equivalently, for any definable X and a code $e \in \mathbb{M}^{\mathrm{eq}}$ for X, there is some tuple c from $\mathbb{M}$ such that $e \in \mathrm{dcl}(c)$ and $c \in \mathrm{dcl}(e)$.

Exercise 1.21. Using compatness, show that T has EI $\iff$ every $\emptyset$-definable equivalence relation E is the kernel of some $\emptyset$-definable map f (i.e., there is some f satisfying $\forall x, y (xEy \leftrightarrow f(x) = f(y))$).

Lemma 1.22. T^{eq} *eliminates imaginaries (using Lemma 1.19(2)).*

In practice it is an important question to know if a particular theory of interest eliminates imaginaries or not, in its natural language. We will see some examples later on.

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Lemma 1.23. *Let* $X \subseteq \mathbb{M}^n$ *be definable, and* $e \in \mathbb{M}^{\mathrm{eq}}$ *a code for* X*.*

(1) X *is* A*-definable iff* $e \in \mathrm{dcl}(A)$ *in* $\mathbb{M}^{\mathrm{eq}}$ *(we may write* $\mathrm{dcl}^{\mathrm{eq}}(A)$ *to stress it) — by Lemma 1.12.*
(2) X *is almost* A*-definable iff* $e \in \mathrm{acl}^{\mathrm{eq}}(A)$ *— by Lemma 1.13.*

Lemma 1.24. *Let* X *be definable, and let* A *be a set of parameters in* $\mathbb{M}$*. Then* X *is almost* A*-definable iff* X *is* $\mathrm{acl}^{\mathrm{eq}}(A)$*-definable in* $\mathbb{M}^{\mathrm{eq}}$*.*

Proof. Let e be the code for X, so X is definable over e in $\mathbb{M}^{\mathrm{eq}}$. Then if X is almost A-definable, $e \in \mathrm{acl}^{\mathrm{eq}}(A)$ by the previous lemma, and so X is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. Conversely, suppose X is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable. As e is a code of X, it follows that $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{acl}^{\mathrm{eq}}(A)$. So e has only finitely many $\mathrm{Aut}(\mathbb{M}^{\mathrm{eq}}/A)$-images, and then the same is true for X, so X is almost A-definable by Lemma 1.13. $\square$

Definition 1.25. Let $\bar{a}, \bar{b}$ be n-tuples from $\mathbb{M}$. Then they have the same *strong type* over C, written $\operatorname{stp}(\bar{a}/C) = \operatorname{stp}(\bar{b}/C)$, if for every C-definable equivalence relation E with finitely many classes, $E(\bar{a}, \bar{b})$ holds. Note that $\operatorname{stp}(\bar{a}/C) = \operatorname{stp}(\bar{b}/C) \Rightarrow \operatorname{tp}(\bar{a}/C) = \operatorname{tp}(\bar{b}/C)$.

Example 1.26. Let T be the theory of an equivalence relation with two infinite classes, it has QE (by an easy back-and-forth). Let a, b be two elements in different classes, then it is easy to see that $\operatorname{tp}(a/\emptyset) = \operatorname{tp}(b/\emptyset)$, but their strong types are different.

Lemma 1.27. *TFAE:*

(1) $\operatorname{stp}(a/A) = \operatorname{stp}(b/A)$,
(2) *if* $X \subseteq \mathbb{M}^n$ *is almost* A*-definable, then* $a \in X$ *iff* $b \in X$,
(3) $\operatorname{tp}(a/\operatorname{acl}^{\mathrm{eq}}(A)) = \operatorname{tp}(b/\operatorname{acl}^{eq}(A))$ *(in* $\mathbb{M}^{\mathrm{eq}}$*).*

Proof. (1) $\Leftrightarrow$ (2) immediate, (2) $\Leftrightarrow$ (3) by Lemma 1.24. $\square$

1.4. **Stone duality and spaces of types.**

Fact 1.28. *(Stone duality) Let* B *be a Boolean algebra. Then there is an associated topological space* $S(B)$*, called the Stone space of* B*. Its points are ultrafilters on* B*, and the topology is generated by the basis of (clopen) sets of the form* $\langle b \rangle = \{u \in S(B) : b \in u\}$*, where* b *is an element of* B*. For every Boolean algebra* B*,* $S(B)$ *is a compact totally disconnected (every subset with more than one point is disconnected, i.e. is a disjoint union of two non-empty open sets) Hausdorff space. Stone's representation theorem says that every Boolean algebra* B *is isomorphic to the algebra of clopen subsets of its Stone space* $S(B)$ *(in fact, there is a duality between the corresponding categories).*

For $A \subseteq \mathbb{M}$, we denote by $\operatorname{Def}_x(A)$ the Boolean algebra of all A-definable subsets of $\mathbb{M}_x$. Applying this fact to $\operatorname{Def}_x(A)$, we obtain a topology on $S_x(A)$, the space of complete types (in the variable x) over A (naturally identifying types with ultrafilters on $\operatorname{Def}_x(A)$). The basis of clopens for $S_x(A)$ is given by the sets ff the form $\langle \phi(x) \rangle = \{p \in S_x(A) : \phi(x) \in p\}$ for $\phi(x) \in L_x(A)$. Elements of $S_n(\mathbb{M})$ are called *global types*.

Remark 1.29. Note that compactness theorem can be interpreted as compactness of the space $S_x(A)$.

M itself can be identified with the subset of *realized* types, i.e. $\{\operatorname{tp}(a/M) : a \in M\} \subset S_1(M)$ (note that $x = a \vdash \operatorname{tp}(a/M)$ for all $a \in M$), and then $S_1(M)$ is the topological closure of this set (take any $p \in S(M)$, take any open set $X \subseteq S(M)$ containing p, by the definition of topology there is some $\phi(x) \in L(M)$ such that $p \in \langle \phi(x) \rangle \subseteq X$, hence $\phi(x) \in p$. But as p is a type, $\phi(x)$ is realized in M, say by b, and then $\operatorname{tp}(b/M) \in X$). So one can think of the space of types as a "compactification of the model".

Example 1.30.

(1) Let T be the theory of an infinite set, in the language of equality. By QE we have $S(M) = M \cup \{p^*\}$, where $p^* = \{x \neq a : a \in M\}$ is the type of a new element.
(2) Let T be the theory of dense linear orders without end points (DLO), it has QE by an easy back-and-forth argument. Let $M \models T$, and let $C = (A, B)$ be a Dedekind cut in M (i.e. $M = A \cup B, A \cap B = \emptyset$ and $a < b$ for all

$a \in A, b \in B$). Let $p_C := \{a < x < b : a \in A, b \in B\}$. This set of formulas is consistent by density of the order, and defines a complete type by QE, thus non-realized types correspond to Dedekind cuts.

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The topology on $S_{x,y}(A)$ is *not* the product topology. For each $p(x,y) \in S_{x,y}(A)$, let $p_x(y) \in S_y(A)$ be the type such that if $(a,b) \models p$ then $b \models p_x(y)$. Note that also for any $b \models p_x(y)$ there is *some* $a \in \mathbb{M}$ such that $(a,b) \models p$ (easy to see via homogeneity). For each $q(y) \in S_y(A)$, let b_q in $\mathbb{M}$ be a realization of q.

Lemma 1.31. *Define the map $S_{x,y}(A) \to \bigcup_{q \in S_y(A)} S_x(Ab_q)$. Given $p(x,y)$, take a realization $(a,b) \models p(x,y)$ and $b = b_q$ for $q = p_x(y)$. Map p to the element $\mathrm{tp}(a/Ab) \in S_x(Ab_q)$. This map is* injective.

The conclusion is that usually one can bound the size of the space of types in several variables by the size of the space of types in one variable.

In view of the Stone duality, we know that this space of types reflects the complexity of definable sets. For many questions, working in the topological setting is more intuitive and allows to use certain results from general topology.

References. Most of the material of this section is based on [Pil02, Section 1], [vdD05, Sections 3,4,5] and [Pil96, Section 1.1]. For more details on basic model theory see [Mar02] or [TZ12].

2. Stability

2.1. **Historic remarks and motivations.** Any theory with an infinite model has models of arbitrary infinite cardinalities (larger then the size of the language). The next question one can ask is, for a fixed infinite cardinal, how many models of this cardinality can T have? More precisely, consider the function $I_T(\kappa)$ giving the number of models of T of size κ, up to isomorphism. Note that $1 \leq I_T(\kappa) \leq 2^\kappa$ for all infinite κ bigger than the cardinality of the language. A fundamental result of Morley that started modern model theory (confirming a conjecture of Vaught):

Fact 2.1. *Let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.*

For example, consider the theory of an infinite set — it is uncountably categorical, the isomorphism type of its model is completely determined by its size. Consider the theory of a vector spaces over a fixed field — then the isomorphism type of its model is completely determined by the dimension. For algebraically closed fields, a model is determined up to isomorphism by the characteristic and transcendence degree, if the size is κ uncountable, then the transcendence degree has to be κ too).

Stability theory developed historically in Shelah's work as a chunk of machinery intended to generalize Morley's theorem to a computation of the possible "spectra" of complete first order theories, in particular to prove the following conjecture.

Conjecture 2.2. *(Morley) Let T be countable, then function $I_T(\kappa)$ is non-decreasing on uncountable cardinals.*

Also, there is a version of Morley's theorem for theories of arbitrary cardinality (with "uncountable" replace by "sufficiently large relatively to $|T|$").

This project was essentially completed by Shelah in the early 80's with the "Main gap theorem" [She90]. Shelah isolated a bunch of "dividing lines" on the space of first-order theories, showing that all theories on the non-structure side of the dividing line have as many models as possible, and on the structure side developing some kind of dimension theory and showing that the isomorphism type of a model can be described by some "small" invariants implying e.g. that there are few models (so in the case of a vector space we only need one cardinal invariant, but if we have an equivalence relation then we need to know the size of each of its classes). One of such dividing lines is stability. A complete classification of the possible functions $I_T(\kappa)$ for countable theories was given by Hart, Hrushovski, Laskowski [HHL00] (required some descriptive set-theoretic ideas in order to prove a "continuum hypothesis" for a certain notion of dimension).

Later on, other perspectives developed, in the work of Macintyre, Zilber, Cherlin, Poizat, Hrushovski, Pillay and many others, in which stability theory is seen rather as a way of classifying definable sets in a structure and describing the interaction between definable sets. Eventually this theory started to be seen as having a "geometric meaning". Moreover, more recently it was realized that a lot of techniques from stability can still be developed in larger contexts, and the so-called "generalized stability" is currently an active area of research.

2.2. **Counting types and stability.** Recall that the topological complexity of type spaces reflects the complexity of definable sets. We consider the most basic property of type spaces — their size.

Definition 2.3. For a complete first order theory T, let $f_T : \mathrm{Card} \to \mathrm{Card}$ be defined by $f_T(\kappa) = \sup\{|S_1(M)| : M \models T, |M| = \kappa\}$, for κ an infinite cardinal.

Exercise 2.4. Show that taking $f_T(\kappa) = \sup\{|S_n(M)| : M \models T, |M| = \kappa, n \in \omega\}$ gives an equivalent definition (hint: use Lemma 1.31 inductively).

It is easy to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$ (every $p \in S_x(M)$ is determined by the collection of all sets of the form $\{a \in M_y : \phi(x,a) \in p\}$, where $\phi(x,y)$ varies over all formulas in L, which gives the upper bound; the lower bound is given by the set of realized types of the form $\{\mathrm{tp}(a/M) : a \in M\}$).

Fact 2.5. *(Keisler, Shelah* [Kei76]*) Let T be an arbitrary complete theory in a countable language. Then $f_T(\kappa)$ is one of the following functions (and all of these options occur for some T):*

$\kappa, \kappa + 2^{\aleph_0}, \kappa^{\aleph_0}$, $\mathrm{ded}\,\kappa$, $(\mathrm{ded}\,\kappa)^{\aleph_0}$, 2^{κ}.

Here $\mathrm{ded}\,\kappa = \sup\{|I| : I$ is a linear order with a dense subset of size $\kappa\}$, equivalently $\sup\{\lambda :$ there is a linear order of size κ with λ cuts$\}$ (it is enough to consider dense linear orders).

Lemma 2.6. $\kappa < \mathrm{ded}\,\kappa \leq 2^{\kappa}$.

Proof. To see $\kappa < \mathrm{ded}\,\kappa$: let μ be minimal such that $2^{\mu} > \kappa$, and consider the tree $2^{<\mu}$. Take the lexicographic ordering I on it, then $|I| \leq \kappa$ by the minimality of μ, but there are at least $2^{\mu} > \kappa$ cuts.

To see $\mathrm{ded}\,\kappa \leq 2^{\kappa}$, note that every cut is *uniquely* determined by the subset of elements in its lower half. □

Remark 2.7. (1) $\mathrm{ded}\,\aleph_0 = 2^{\aleph_0}$ (as $\mathbb{Q} \subset \mathbb{R}$), and under GCH, $\mathrm{ded}\,\kappa = 2^{\kappa}$ for all κ.

(2) (Mitchell [Mit72]) For any κ of uncountable cofinality, there is a cardinal preserving Cohen extension such that $\operatorname{ded}\kappa < 2^{\kappa}$.
(3) (C., Kaplan, Shelah [CKS12]) It is consistent with ZFC that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ for certain κ. E.g., starting with a model with GCH, we force $\aleph_{\omega+\omega} = \operatorname{ded}\aleph_\omega < (\operatorname{ded}\aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$. It is open if both inequalities $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0} < 2^{\kappa}$ can be strict simultaneously for some κ.
(4) (C., Shelah [CS13]) On the other hand one can prove (in ZFC) that for any κ we have $2^{\kappa} \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded}\kappa)))$. Again, the optimality of the required number of iterations is open.

Exercise 2.8. Find an example of T for each of the possible values of f_T listed above.

Definition 2.9. Let $M \models T$.

(1) A formula $\phi(x, y)$, with its variables partitioned into two groups x, y, has the *k-order property*, $k \in \omega$, if there are some $a_i \in M_x, b_i \in M_y$ for $i < k$ such that $M \models \phi(a_i, b_j) \iff i < j$.
(2) $\phi(x, y)$ has the *order property* if it has the k-order property for all $k \in \omega$.
(3) We say that a formula $\phi(x, y) \in L$ is *stable* if there is some $k \in \omega$ such that it does not have the k-order property.
(4) A theory is *stable* if it implies that all formulas are stable (note that this is indeed a property of a theory, if $\mathcal{M} \equiv \mathcal{N}$ then $\phi(x, y)$ has the k-order property in $\mathcal{M}$ if and only if it has the k-order property in $\mathcal{N}$).

Proposition 2.10. *Assume that T is unstable, then $f_T(\kappa) \geq \operatorname{ded}\kappa$ for all cardinals $\kappa \geq |T|$.*

Proof. Fix a cardinal κ. Let $\phi(x, y) \in L$ be a formula that has the k-order property for all $k \in \omega$. Then by compactness we have:

Claim. Let I be an arbitrary linear order. Then we can find some $\mathcal{M} \models T$ and $(a_i, b_i : i \in I)$ from M such that $\mathcal{M} \models \phi(a_i, b_j) \iff i < j$, for all $i, j \in I$.

Let I be an arbitrary dense linear order of size κ, and let $(a_i, b_i : i \in I)$ in $\mathcal{M}$ be as given by the claim. By Löwenheim–Skolem (Exercise 1.3) we can assume that $|\mathcal{M}| = \kappa$.

Given a cut $C = (A, B)$ in I, consider the set of $L(M)$-formulas

$$\Phi_C = \{\phi(x, b_j) : j \in B\} \cup \{\neg\phi(x, b_j) : j \in A\}.$$

Note that by compactness it is a partial type (consistency of finite subtypes is witnessed by the appropriate a_i's), let $p_C \in S_x(M)$ be a complete type over M extending $\Phi_C(x)$. Given two cuts C_1, C_2, we have $p_{C_1} \neq p_{C_2}$ (say $B_1 \subsetneq B_2$, then take $j \in B_2 \setminus B_1$, it follows that $\phi(x, b_j) \in p_{C_2}, \phi(x, b_j) \notin p_{C_1}$). As I was arbitrary, this shows that $\sup\{|S_x(M)| : M \models T, |M| = \kappa\} \geq \operatorname{ded}\kappa$. Note that we may have $|x| > 1$, however using Exercise 2.4 we get $f_T(\kappa) \geq \operatorname{ded}\kappa$ as well. □

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Recall:

Fact 2.11. *(Ramsey theorem) $\aleph_0 \to (\aleph_0)^n_k$ holds for all $n, k \in \omega$ (i.e. for any coloring of subsets of $\mathbb{N}$ of size n in k colors, there is some infinite subset I of $\mathbb{N}$ such that all n-element subsets of I have the same color).*

Lemma 2.12. *Let $\phi(x,y), \psi(x,z)$ be stable formulas (where y,z are not necessarily disjoint tuples of variables). Then:*

1. *Let $\phi^*(y,x) := \phi(x,y)$, i.e. we switch the roles of the variables. Then $\phi^*(y,x)$ is stable.*
2. *$\neg\phi(x,y)$ is stable.*
3. *$\theta(x,yz) := \phi(x,y) \wedge \psi(x,z)$ and $\theta'(x,yz) := \phi(x,y) \vee \psi(x,z)$ are stable.*
4. *If $y = uv$ and $c \in M_v$ then $\theta(x,u) := \phi(x,uc)$ is stable.*
5. *It T is stable, then every L^{eq}-formula is stable as well.*

Proof. (3) Suppose that $\theta'(x,yz) = \phi(x,y) \vee \psi(x,z)$ is unstable, i.e. there are $(a_i, b_i b_i' : i \in \mathbb{N})$ such that $\models \phi(a_i, b_j) \vee \psi(a_i, b_j') \iff i < j$, for all $i,j \in \mathbb{N}$.

Let

$$P := \left\{(i,j) \in \mathbb{N}^2 : i < j \& \models \phi(a_i, b_j)\right\}, Q := \left\{(i,j) \in \mathbb{N}^2 : i < j \& \models \psi(a_i, b_j')\right\},$$

then $P \cup Q = \left\{(i,j) \in \mathbb{N}^2 : i < j\right\}$. By Ramsey there is an infinite $I \subseteq \mathbb{N}$ such that either all increasing pairs from I belong to P (in which case ϕ is unstable), or all increasing pairs from I belong to Q (in which case ψ is unstable). $\square$

Theorem 2.13. *(Erdős-Makkai) Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(b_i : i < \omega)$ of elements of B and $(S_i : i < \omega)$ of elements of $\mathcal{F}$ such that one of the following holds:*

1. *$b_i \in S_j \iff j < i$ for all $i,j \in \omega$,*
2. *$b_i \in S_j \iff i < j$ for all $i,j \in \omega$.*

Proof. Choose $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = |B|$, such that any two finite subsets of B that can be separated by an element of $\mathcal{F}$, can already be separated by an element of $\mathcal{F}'$ (possible as there are at most $|B|$-many pairs of finite subsets of B).

By assumption there is some $S^* \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$ (again there are at most $|B|$-many different Boolean combinations of sets from $\mathcal{F}'$).

We choose by induction sequences $(b_i' : i < \omega)$ in S^*, $(b_i'' : i < \omega)$ in $B \setminus S^*$ and $(S_i : i < \omega)$ in $\mathcal{F}'$ such that:

- S_n separates $\{b_0', \dots, b_n'\}$ and $\{b_0'', \dots, b_n''\}$,
- $b_n' \in S_i \iff b_n'' \in S_i$ for all $i < n$.

Assume $(b_i' : i < n)$, $(b_i'' : i < n)$ and $(S_i : i < n)$ have already been constructed. Since S^* is not a Boolean combination of $S_0, \dots, S_{n-1}$, there are $b_n' \in S^*, b_n'' \in B \setminus S^*$ such that for all $i < n$,

$$b_n' \in S_i \iff b_n'' \in S_i.$$

Choose S_n as any set in $\mathcal{F}'$ separating $\{b_0', \dots, b_n'\}$ and $\{b_0'', \dots, b_n''\}$.

Now by Ramsey theorem we may assume that either: $b_n' \in S_i$ for all $i < n < \omega$, or $b_n' \notin S_i$ for all $i < n < \omega$ (color the set of pairs (i,n) for $i < n$ with two colors according to whether $b_n' \in S_i$ or $b_n' \notin S_i$). In the first case we set $b_i = b_i''$ and get (1), in the second case we set $b_i = b_{i+1}'$ and get (2). $\square$

Definition 2.14. Let $\phi(x,y)$ be a formula, by a *complete ϕ-type* over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x,b), \neg\phi(x,b)$ where b ranges over A. Let $S_\phi(A)$ be the space of all complete ϕ-types over A.

Proposition 2.15. *Assume that $|S_\phi(B)| > |B|$ for some infinite set of parameters B. Then $\phi(x,y)$ is unstable.*

Proof. For any $a \in \mathbb{M}_x$, $\mathrm{tp}_\phi(a/B)$ is given by $S_a = \{b \in B : \models \phi(a,b)\} \subseteq B$.

Applying the Erdős-Makkai theorem to B and $\mathcal{F} = \{S_a : a \in \mathbb{M}_x\}$ we obtain a sequence $(b_i : i < \omega)$ of elements of B and a sequence $(a_i : i < \omega)$ of elements of $\mathbb{M}_x$ such that either $b_i \in S_{a_j} \iff j < i$ or $b_i \in S_{a_j} \iff i < j$ for all $i,j \in \omega$. In the first case $\phi(x,y)$ is unstable by definition, in the second case by Lemma 2.12(1). □

2.3. **Local ranks and definability of types.** We will see several (ordinal-valued) ranks playing a role in model theory.

Definition 2.16. We define *Shelah's local* 2*-rank* taking values in $\{-\infty\}\cup\omega\cup\{+\infty\}$ by induction on $n \in \omega$ (there are many other related ranks). Let Δ be a set of L-formulas, and $\theta(x)$ a partial type over $\mathbb{M}$.

- $R_\Delta(\theta(x)) \geq 0$ iff $\theta(x)$ is consistent (and $-\infty$ otherwise).
- $R_\Delta(\theta(x)) \geq n+1$ if for some $\phi(x,y) \in \Delta$ and $a \in \mathbb{M}_y$ we have both $R_\Delta(\theta(x) \wedge \phi(x,a)) \geq n$ and $R_\Delta(\theta(x) \wedge \neg\phi(x,a)) \geq n$.
- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and$R_\Delta(\theta(x)) \ngeq n+1$, and $R_\Delta(\theta(x)) = \infty$ if $R_\Delta(\theta(x)) \geq n$ for all $n \in \omega$.

If $\phi(x,y)$ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$.

Proposition 2.17. *$\phi(x,y)$ is stable if and only if $R_\phi(x=x)$ is finite (and so also $R_\phi(\theta(x))$ is finite for any partial type θ). Here $x = (x_i : i \in I)$ is a tuple of variables, and $x = x$ is an abuse of notation for $\bigwedge_{i\in I} x_i = x_i$.*

Proof. Assume that $\phi(x,y)$ is unstable, i.e. it has the k-order property for all $k \in \omega$. By compactness we find $(a_i b_i : i \in [0,1])$ such that $\models \phi(a_i, b_j) \iff i < j$. We know that both $\phi\left(x, b_{\frac{1}{2}}\right)$ and $\neg\phi\left(x, b_{\frac{1}{2}}\right)$ contain dense subsequences of a_i's. Each of these sets can be split again, by $\phi\left(x, b_{\frac{1}{4}}\right)$ and $\phi\left(x, b_{\frac{3}{4}}\right)$, resp., etc.

Conversely, assume that the rank is infinite, then we can find an infinite tree of parameters $B = (B_\eta : \eta \in 2^{<\omega})$ such that for every $\eta \in 2^\omega$ the set of formulas $\left\{\phi^{\eta(i)}\left(x, b_{\eta|i}\right) : i < \omega\right\}$ is consistent (rank being $\geq k$ guarantees that we can find such a tree of height k, and then use compactness to find one of infinite height). This gives us that $|S_\phi(B)| > |B|$, which by Proposition 2.15 implies that $\phi(x,y)$ is unstable. □

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Definition 2.18. (1) Let $\phi(x,y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is *definable over* B if there is some $L(B)$-formula $\psi(y)$ such that for all $a \in A$,
$$\phi(x,a) \in p \iff \models \psi(a).$$
(2) A type $p \in S_x(A)$ is definable over B if $p|_\phi$ is definable over B for all $\phi(x,y) \in L$.
(3) A type is *definable* if it is definable over its domain.
(4) We say that types in T are *uniformly definable* if for every $\phi(x,y)$ there is some $\psi(y,z)$ such that every type can be defined by an instance of $\psi(y,z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that $\phi(x,a) \in p \iff \models \psi(a,b)$, for all $a \in A$.

Remark 2.19. Another way to think about it:

Given a set $A \subseteq \mathbb{M}_x$, we say that a subset $B \subseteq A$ is *externally definable* (as a subset of A) if there is some definable (over $\mathbb{M}$) set X such that $B = X \cap A$.

Assume moreover that we have $X = \phi(\mathbb{M}, c)$ above. Then $\mathrm{tp}_\phi(c/A)$ is definable if and only if B is in fact internally definable, i.e. $B = A \cap Y$ for some A-definable set Y. A set is called *stably embedded* if every externally definable subset of it is internally definable.

Example 2.20. Consider $(\mathbb{Q}, <) \models \mathrm{DLO}$, and let $p = \mathrm{tp}(\pi/\mathbb{Q})$ $(\pi \in \mathbb{R} \succ \mathbb{Q})$. It is easy to check by QE that p is not definable.

Lemma 2.21.
(1) *The set $\{e : R_\phi(\theta(x, e)) \geq n\}$ is definable, for all $n \in \omega$.*
(2) *If $R_\phi(\theta(x)) = n$, then for any $a \in M_y$, at most one of $\theta(x) \wedge \phi(x, a)$, $\theta(x) \wedge \neg\phi(x, a)$ has R_ϕ-rank n.*

Proposition 2.22. *Let $\phi(x, y)$ be a stable formula. Then all ϕ-types are uniformly definable.*

Proof. Given $p \in S_\phi(A)$, call a subtype $p_i \subseteq p$ one-element minimal if $R_\phi(q) = R_\phi(p_i)$ for all $p_i \subseteq q \subseteq p$ with $|\mathrm{dom}(q) \setminus \mathrm{dom}(p_i)| = 1$.

Claim. For any $p \in S_\phi(A)$ there is a one-element minimal $p_i \subseteq p$ with $|p_i| \leq R_\phi(x = x)$.

Why? Let $p_0 = \emptyset$, and given p_i let $p_{i+1} \subseteq p$ be any one-element extension of p_i of smaller R_ϕ-rank, if one exists.

Claim. For any $p \in S_\phi(A)$, if $p_i \subseteq p$ is one-element minimal then p is defined by the formula "$R_\phi(p_i(x) \wedge \phi(x, a)) = R_\phi(p_i)$".

Why? For $a \in A$, $\phi(x, a) \in p$ implies $R_\phi(p \cup \{\phi(x, a)\}) = R_\phi(p_i)$ by minimality of p_i. And

$$\phi(x, a) \notin p \Rightarrow \neg\phi(x, a) \in p \Rightarrow R_\phi(p_i \cup \{\neg\phi(x, a)\}) = R_\phi(p_i) \Rightarrow$$

$$R_\phi(p_i \cup \{\phi(x, a)\}) \neq R_\phi(p_i)$$

by the previous lemma. ☐

Summarizing the results of the last sections we have the following characterization of stability for formulas.

Theorem 2.23. *The following are equivalent for a formula $\phi(x, y)$.*
(1) *$\phi(x, y)$ is stable.*
(2) *$R_\phi(x = x) < \omega$.*
(3) *All ϕ-types are uniformly definable.*
(4) *All ϕ-types over models are definable.*
(5) *$|S_\phi(M)| \leq \kappa$ for all $\kappa \geq |L|$ and $M \models T$ with $|M| = \kappa$.*
(6) *There is some κ such that $|S_\phi(M)| < \mathrm{ded}\,\kappa$ for all $M \models T$ with $|M| = \kappa$.*

Proof. (1) $\Leftrightarrow$ (2) by Proposition 2.17, (1) $\Rightarrow$ (3) by Proposition 2.22, (3) $\Rightarrow$ (4) is obvious, (4) implies (5) since over a model M of size κ, there can be at most $\kappa + L$ definitions for types and all types over M are definable, (5) $\Rightarrow$ (6) is obvious, and (6) $\Rightarrow$ (1) by Proposition 2.10. ☐

For a complete theory, this translates into the following corollary.

Theorem 2.24. *Let T be a complete first-order theory. Then the following are equivalent.*

1. *T is stable, i.e. it implies that all formulas are stable.*
2. *There is no sequence of tuples $(a_i : i \in \omega)$ from $\mathbb{M}$ and formula $\phi(z_1, z_2) \in L(\mathbb{M})$ such that $\models \phi(c_i, c_j) \iff i < j$.*
3. *The inequality $f_T(\kappa) \leq \kappa^{|T|}$ holds for all infinite cardinals κ.*
4. *There is some κ such that $f_T(\kappa) \leq \kappa$.*
5. *There is some κ such that $f_T(\kappa) < \operatorname{ded} \kappa$.*
6. *All formulas of the form $\phi(x, y)$ where x is a singleton variable, are stable.*
7. *All types over models are definable.*

Proof. (1) $\Leftrightarrow$ (2): any $\phi(x_1, x_2)$ as in (2) has the order property witnessed by the sequence (a_i, b_i) with $b_i = a_i$. Conversely, let $\phi(x, y)$ be an arbitrary formula with the order property witnessed by the sequence (a_i, b_i). Consider $\phi^*(x_1y_1, x_2y_2) := \phi(x_1, y_2)$, and let $c_i := a_ib_i$. Then $\models \phi^*(c_i, c_j) \iff i < j$.

(1) $\Rightarrow$ (3): Let $M \models T$ with $|M| = \kappa \geq |L|$ be given. By Theorem 2.23(5) we have $|S_\phi(M)| \leq \kappa$ for all $\phi \in L$, and every $p \in S_x(M)$ is determined by $\{p|_\phi \in S_\phi(M) : \phi(x, y) \in L\}$.

For (3) $\Rightarrow$ (4) take any $\kappa = \kappa^{|T|}$, (4) $\Rightarrow$ (5) is obvious, and (5) $\Rightarrow$ (1) by 2.23(6).

Next, (6) is equivalent to the other conditions as we can bound the number of types in a tuple of variable by the number of types in a single variable (by Exercise 2.4).

Finally, (7) is equivalent to (1) by the corresponding equivalence for ϕ-types in Theorem 2.23. $\square$

***

Example 2.25. Note that stability of T is characterized by the definability of types over *all* models of T. Some unstable theories have certain special models over which all types are definable.

1. All types over $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$ are (uniformly) definable (easy to check for types in a single variable by *o*-minimality as the order is Dedekind-complete. E.g., let $\varepsilon \in \mathbb{R}$ be arbitrary, and let $\varepsilon^+ \in \mathbb{M}$ realize the cut)
   There is a theorem of Marker-Steinhorn [MS94] which says that in *o*-minimal theories this is sufficient. In fact, this is the only model of the theory RCF with this property.
2. All types over $(\mathbb{Q}_p, +, \times, 0, 1)$ are (uniformly) definable [Del89].

2.4. **Indiscernible sequences and stability.** Model theory has a convenient way of doing all the Ramsey theory that you may need in advance of doing anything else.

Definition 2.26. Given a linear order I, a sequence of tuples $(a_i : i \in I)$ with $a_i \in \mathbb{M}_x$ is *indiscernible* over a set of parameters A if $a_{i_0} \ldots a_{i_n} \equiv_A a_{j_0} \ldots a_{j_n}$ for all $i_0 < \ldots < i_n$ and $j_0 < \ldots < j_n$ from I and all $n \in \omega$.

Example 2.27.
1. A constant sequence (in any theory) is indiscernible over any set.
2. Of course, any subsequence of an A-indiscernible sequence is A-indiscernible.
3. In the theory of equality, any sequence of distinct singletons is indiscernible.
4. Any increasing (or decreasing) sequence of singletons in a dense linear order is indiscernible.

(5) Any basis in a vector space is an indiscernible sequence.

The following is a standard method of finding indiscernible sequences in an arbitrary theory.

Definition 2.28. For any sequence $\bar{a} = (a_i : i \in I)$ and a set of parameters B, we define $\mathrm{EM}(\bar{a}/B)$, the *Ehrenfeucht-Mostowski type* of the sequence $\bar{a}$ over B, as a partial type over B in countably many variables indexed by ω and given by the following collection of formulas

$$\{\phi(x_0, \ldots, x_n) \in L(A) : \forall i_0 < \ldots < i_n, \models \phi(a_{i_0}, \ldots, a_{i_n}), n \in \omega\}.$$

Proposition 2.29. *Let $\bar{a} = (a_i : i \in J)$ be an arbitrary sequence in $\mathbb{M}$, where J is an arbitrary linear order and A is a small set of parameters. Then for any small linear order I we can find (in $\mathbb{M}$) an A-indiscernible sequence $(b_i : i \in I)$* based on *$\bar{a}$, i.e. such that:*

for any $i_0 < \ldots < i_n$ in I and a finite set of formulas $\Delta \subseteq L(A)$ there are some $j_0 < \ldots < j_n$ in ω such that $\models \phi(b_{i_0}, \ldots, b_{i_n}) \iff \models \phi(a_{j_0}, \ldots, a_{j_n})$ for all $\phi \in \Delta$.

Proof. Let $(c_i : i \in I)$ be a new set of constants, and consider the theory $T' \supseteq T$ in the language $L' = L \cup \{c_i : i \in I\}$ which in addition contains the following axioms:

- $\phi(c_{i_0}, \ldots, c_{i_n})$ for all $i_0 < \ldots < i_n \in I$ and $\phi \in \mathrm{EM}(\bar{a}/A)$,
- $\psi(c_{i_0}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_0}, \ldots, c_{j_n})$ for all $i_0 < \ldots < i_n, j_0 < \ldots < j_n$ in I and $\psi \in L(A)$.

It is enough to show that T' is consistent. By compactness it is enough to show that every finite $T_0 \subseteq T'$ is consistent. Say T_0 only involves formulas from some finite $\Delta \subseteq L(A)$ with at most n free variables. Applying Ramsey theorem we can find some infinite subsequence of $\bar{a}' = (a_i : i \in I')$, $I' \subseteq \omega$ such that all the increasing n-tuples from $\bar{a}'$ agree on all formulas from Δ, which is enough. □

Corollary 2.30. *If $(a_i : i \in I)$ is an A-indiscernible sequence and $J \supseteq I$ is an arbitrary linear order, then (in $\mathbb{M}$) there is an A-indiscernible sequence $(b_j : j \in J)$ such that $b_j = a_j$ for all $j \in I$ (everything involved is small).*

Proof. First let $(b_j : j \in J)$ be an arbitrary A-indiscernible sequence in $\mathbb{M}$ based on I, exists by Proposition 2.29. It is easy to see that in particular $(b_j : j \in I) \equiv_A (a_j : j \in I)$, which by homogeneity of $\mathbb{M}$ implies that there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(b_j) = a_j$. Then define $b'_j = \sigma(b_j)$ for all $j \in J$. As indiscernibility of a sequence is clearly preserved by an elementary map, $\left(b'_j : j \in J\right)$ is as wanted. □

Lemma 2.31. *If $\bar{a} = (a_i : i \in I)$ is an infinite A-indiscernible sequence, then for all $S \subseteq I$ and $i \in I \setminus S$, $a_i \notin \mathrm{acl}(A(a_j : j \in S))$.*

Proof. By definition, we want to show that for any formula $\phi(x) \in L\left(A(a_j)_{j \in S}\right)$, if $\models \phi(a_i)$ holds then $\phi(x)$ has infinitely many solutions in $\mathbb{M}$. Let $S' < i < S''$ be two finite subsets of S such that $A \cup S' \cup S''$ contains all the parameters of $\phi(x)$. First assume that the indexing order I is dense, then there is an infinite set $S^* \subset I$ such that $S' < S^* < S''$. By the A-indiscernibility of $\bar{a}$ it follows that $\models \phi(a_j)$ for all $j \in S^*$, and we are done.

For an arbitrary I, extend the sequence $\bar{a}$ to a larger A-indiscernible sequence $\bar{a}'$ indexed by an arbitrary dense order $J \supseteq I$ using Corollary 2.30, and apply the previous argument (recall that $C \subseteq D \Rightarrow \operatorname{acl}(C) \subseteq \operatorname{acl}(D)$ for arbitrary sets of parameters). $\square$

Exercise 2.32. Start with the sequence $(1, 2, 3, \ldots)$ in $(\mathbb{C}, +, \times, 0, 1) \models \mathrm{ACF}_0$. Give an explicit example of an indiscernible sequence based on it.

If we start with a very long sequence, we can "extract" an indiscernible sequence preserving the full type of finite subtuples (rather than formula-by-formula as in the previous proposition).

Proposition 2.33. *Let κ, λ be small cardinals and let $(a_i)_{i \in \lambda}$ be a sequence of tuples with $|a_i| < \kappa$ and a set B be given. If $\lambda \geq \beth_{\left(2^{\kappa+|B|+|T|}\right)^+}$ there is a B-indiscernible sequence $(a'_i)_{i \in \omega}$ such that for every $n \in \omega$ there are $i_0 < \ldots < i_n \in \kappa$ such that $a'_0 \ldots a'_n \equiv_B a_{i_0} \ldots a_{i_n}$.*

Proof. The same argument as above, but using the Erdős-Rado theorem ($\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$ for all $n \in \omega$) instead of Ramsey — this time we color tuples from the sequence by their complete types over B (see e.g. [BY03, Lemma 1.2] for the details). $\square$

Remark 2.34.
(1) In general (without any set-theoretic assumptions), it is not possible to find an *infinite* indiscernible subsequence, no matter how long is the sequence that we start with (but is always possible if its length is a weakly compact cardinal, for example). In stable theories, if a sequence is sufficiently long, then one can actually find an infinite indiscernible subsequence.
(2) Another way to phrase the definition of an indiscernible sequence is to require that the type of a tuple from the sequence is determined by the quantifier-free type of its indices in the indexing structures. Then our notion of indiscernibility can be generalized to arbitrary indexing structures, and an analog of Proposition 2.29 holds precisely when the indexing structure satisfies the Ramsey property (see e.g. [TN14]).

Definition 2.35. A sequence $(a_i : i \in I)$ is *totally indiscernible* over A if $a_{i_0} \ldots a_{i_n} \equiv_A a_{j_0} \ldots a_{j_n}$ for any $i_0 \neq \ldots \neq i_n, j_0 \neq \ldots \neq j_n$ from I (so the order of the indices doesn't matter any longer).

***

Theorem 2.36. *T is stable if and only if every indiscernible sequence is totally indiscernible.*

Proof. Assume that T is unstable, then by Theorem 2.24(2) there is some sequence of tuples $(a_i : i \in \omega)$ and a formula $\phi(x_1, x_2)$ such that $\models \phi(a_i, a_j) \iff i < j$. Let $(a'_i : i \in \omega)$ be an indiscernible sequence based on (a_i), we still have $\models \phi\left(a'_i, a'_j\right) \iff i < j$, so it is not totally indiscernible.

Conversely, assume that we have an indiscernible sequence $\bar{a} = (a_i : i \in I)$ that is not totally indiscernible (by Proposition 2.29 we may assume that $I = \mathbb{Q}$). This means that there is some formula $\phi(x_1, \ldots, x_n) \in L(A)$, some indices $r_1 < \cdots < r_n$ and some permutation $\sigma \in \mathrm{Sym}(n)$ such that

$$\models \phi(a_{r_1}, \ldots, a_{r_n}) \wedge \neg\phi\left(a_{r_{\sigma(1)}}, \ldots a_{r_{\sigma(n)}}\right).$$

Say, we have $\sigma = \tau_1 \cdots \tau_k$, where each τ_i is a transposition of two consecutive elements, and let $\sigma_i := \prod_{1 \leq j \leq i} \tau_j$. Consider the sequence of the truth values of

$$\models \phi(a_{r_1}, \ldots, a_{r_n}),$$

$$\models \phi(a_{r_{\sigma_1(1)}}, \ldots, a_{r_{\sigma_1(n)}}),$$

$$\models \phi(a_{r_{\sigma_2(1)}}, \ldots, a_{r_{\sigma_2(n)}}),$$

$$\ldots$$

Then we find the first place where it switches from true to false, say we have: $\models \phi(a_{r_{\sigma_{j-1}(1)}}, \ldots, a_{r_{\sigma_{j-1}(n)}}) \wedge \neg\phi(a_{r_{\sigma_j(1)}}, \ldots, a_{r_{\sigma_j(n)}})$, which used the transposition $\tau_j = (s, s+1)$ for some $1 \leq s < n$. Then we define $\phi^*(x_1, \ldots, x_n) := \phi(x_{\sigma_{j-1}(1)}, \ldots, x_{\sigma_{j-1}(n)})$ and $\psi(x_1, x_2) = \phi^*(a_{r_1}, \ldots, a_{r_{s-1}}, x_1, x_2, a_{r_{s+2}}, \ldots, a_{r_n})$. By indicsernibility of $\bar{a}$, $\psi(x_1, x_2)$ defines the ordering on the infinite subsequence $(a_i : r_{s-1} < i < r_{s+2})$. By Theorem 2.24(2) this contradicts stability. □

Proposition 2.37. *For any stable formula $\phi(x, y)$, in an arbitrary theory, there is some $k_\phi \in \omega$ depending just on ϕ such that for any indiscernible sequence $I \subseteq \mathbb{M}_x$ and any $b \in \mathbb{M}_y$, either $|\phi(I, b)| \leq k_\phi$ or $|\neg\phi(I, b)| \leq k_\phi$.*

Proof. Let a stable $\phi(x, y)$ be given, say $\phi(x, y)$ does not have the k-order property. Let I be some indiscernible sequence and b a parameter. We claim that either $\phi(I, b)$ or $\neg\phi(I, b)$ is of size smaller than k.

Assume not, then one of the following happens: we can find some $i_0 < \ldots < i_{k-1} < i_k < i_{k+1} < \ldots < i_{2k-1}$ such that

$$\models \bigwedge_{0 \leq j < k} \neg\phi\left(a_{i_j}, b\right) \wedge \bigwedge_{k \leq j < 2k} \phi\left(a_{i_j}, b\right),$$

or the same with the roles of ϕ and $\neg\phi$ reversed. Let's do the first case, the other is analogous. So, in the first case we have in particular

$$\models \exists y \bigwedge_{0 \leq j < k} \neg\phi\left(a_{i_j}, y\right) \wedge \bigwedge_{k \leq j < 2k} \phi\left(a_{i_j}, y\right).$$

By indiscernibility of the sequence this implies that for any $0 \leq i < k$,

$$\models \exists y \bigwedge_{0 \leq j < i} \neg\phi\left(a_{i_j}, y\right) \wedge \bigwedge_{i \leq j < k} \phi\left(a_{i_j}, y\right)$$

holds. Let

$$b_i \models \bigwedge_{0 \leq j < i} \neg\phi\left(a_{i_j}, y\right) \wedge \bigwedge_{i \leq j < k} \phi\left(a_{i_j}, y\right)$$

be arbitrary, and let $c_j := a_{i_j}$, for $i, j < k$. Then the sequence $(b_i, c_i : i < k)$ with the order reversed shows that $\phi(x, y)$ has the k-order property, a contradiction. □

Corollary 2.38. *In a stable theory, we can define the* average type *of an indiscernible sequence $\bar{b} = (b_i)$ over a set of parameters A as*

$$\mathrm{Av}\left(\bar{b}/A\right) = \left\{\phi(x, a) \in L(A) : \models \phi(b_i, a) \text{ for all but finitely many } i \in I\right\}.$$

By Proposition 2.37 it is a complete consistent type over A.

***

2.5. **Stable = NIP∩NSOP and the classification picture.** The failure of stability can occur in one of the following two "orthogonal" ways.

Definition 2.39. (1) A (partitioned) formula $\phi(x, y) \in L$ has the *strict order property*, or *SOP*, if there is an infinite sequence $(b_i)_{i \in \omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j \in \omega$.

(2) A theory T has SOP if some formula does.

(3) T is *NSOP* if it doesn't have the strict order property.

Remark 2.40.

- If $\phi(x, y)$ has SOP, then by Proposition 2.29 we can choose an indiscernible sequence (b_i) satisfying the condition above. If we have arbitrary long finite sequences $(b_i)_{i<n}$ satisfying the condition above for a fixed formula $\phi(x, y)$, then it has SOP by compactness.
- A typical example of an SOP theory is given by DLO.
- T is NSOP if and only if all formulas with parameters are NSOP (can incorporate the parameters into the sequence of b_i's), if and only if all formulas $\phi(x, y)$ with x singleton are NSOP [Lac75].

Exercise 2.41. T has SOP if and only if there is a definable partial order with infinite chains (on some sort in the monster model).

Definition 2.42. (1) A (partitioned) formula $\phi(x, y)$ has the *independence property*, or *IP*, if (in $\mathbb{M}$) there are infinite sequences $(b_i)_{i \in \omega}$ and $(a_s : s \subseteq \omega)$ such that $\models \phi(a_s, b_i) \iff i \in s$.

(2) A theory T has IP if some formula does, otherwise T is *NIP*.

Remark 2.43.

- If we have arbitrary long finite sequences $(b_i)_{i<n}$ satisfying the condition above for a fixed formula $\phi(x, y)$, then by compactness we can find an infinite sequence satisfying the condition above, hence $\phi(x, y)$ it has IP. And if $\phi(x, y)$ has IP, then by Proposition 2.29 and compactness we can choose an indiscernible sequence (b_i) in the definition above.
- A typical example of a theory with IP is given by the theory of the countable random graph, i.e. the theory of a single (symmetric, irreflexive) binary relation $E(x, y)$ axiomatized by the following list of "extension axioms", for each $n \in \omega$:

$$\forall a_0 \neq \ldots \neq a_{n-1} \neq b_0 \neq \ldots \neq b_{n-1} \exists c \left(\bigwedge_{i<n} E(c, a_i) \wedge \bigwedge_{i<n} \neg E(c, b_i) \right).$$

- T is NIP if and only if all formulas with parameters are NIP, if and only if all formulas $\phi(x, y)$ with x singleton are NIP. Also $\phi(x, y)$ is NIP if and only if $\phi^*(y, x) = \phi(x, y)$ is NIP (see e.g. [Adl08]).

Proposition 2.44. *A formula $\phi(x, y)$ is NIP if and only if for any indiscernible sequence $\bar{b} = (b_i : i \in I)$ and a parameter a, the alternation of $\phi(a, y)$ on $\bar{b}$ is finite, bounded by some number $n \in \omega$ depending just on ϕ. That is, there are at most n increasing indices $i_0 < \ldots < i_{n-1}$ such that $\models \phi(a, b_i) \leftrightarrow \neg\phi(a, b_{i+1})$ for all $i < n - 1$.*

Proof. If $\phi(x, y)$ has IP, then as remarked we can choose an indiscernible sequence $\bar{b} = (b_i)_{i \in \omega}$ in the definition above. But then taking $s \subseteq \omega$ to be the set of even number, $\phi(a_s, y)$ has infinite alternation on $\bar{b}$.

Conversely, assume $\phi(a, y)$ has infinite alternation on an indiscernible sequence $\bar{b}$, without loss of generality $\bar{b} = (b_i : i \in \omega)$ and $\models \phi(a, b_i) \iff i \equiv 0\,(\mathrm{mod} 2)$. Let

$J \subseteq n$ be arbitrary. Choose indices $i_0 < i_1 < \ldots < i_{n-1}$ such that i_k is even if and only if $k \in J$, hence $\models \phi(a, b_{i_k}) \iff k \in J$. But $b_{i_0} b_{i_1} \ldots b_{i_{n-1}} \equiv b_0 b_1 \ldots b_{n-1}$ by indiscernibility, and so there is some a_J in $\mathbb{M}$ such that $\models \phi(a_J, b_k) \iff k \in J$, for all $k < n$. As n was arbitrary, conclude by compactness. □

Remark 2.45. Working in an NIP theory and given an indiscernible sequence $\bar{b} = (b_i : i \in I)$ with I an endless order, and A an arbitrary set of parameters, Proposition 2.44 allows us to define a complete consistent type

$$\operatorname{Av}\left(\bar{b}/A\right) = \{\phi(x) \in L(A) : \text{ the set } \{i \in I : \models \phi(a, b_i)\} \text{ is cofinal}\}.$$

In the case of a stable theory this coincides with the definition from Corollary 2.38.

Theorem 2.46. *[Shelah] T is unstable if and only if it has the independence property or the strict order property.*

Proof. We leave it as an exercise to show that if $\phi(x, y)$ has IP or SOP, then it is unstable.

Now assume that $\phi(x, y)$ is unstable and T is NIP. Applying Proposition 2.29 we can find an indiscernible sequence $(a_i b_i : i \in \mathbb{Q})$ such that $\models \phi(a_i, b_j) \iff i < j$.

$(*)$ As T is NIP, by Proposition 2.44 there exists some k such that

$$\left\{\phi^{i(\mathrm{mod} 2)}(x, b_i) : i \in \mathbb{N}, i < k\right\}$$

is inconsistent.

$(**)$ On the other hand by instability, for every $l < k$ we have

$$\{\neg\phi(x, b_i) : i < l\} \cup \{\phi(x, b_i) : i \geq l\}$$

is consistent (witnessed by $a_{l-\frac{1}{2}}$).

We can get from $(*)$ to $(**)$ by replacing $\phi(x, b_i) \wedge \neg\phi(x, b_{i+1})$ with $\neg\phi(x, b_i) \wedge \phi(x, b_{i+1})$ one at a time.

This means that there is some $\eta : k \to 2$ and $l < k$ such that

$$\left\{\phi^{\eta(i)}(x, b_i) : i \neq l, l+1\right\} \cup \{\phi(x, b_l), \neg\phi(x, b_{l+1})\}$$

is inconsistent, but

$$\left\{\phi^{\eta(i)}(x, b_i) : i \neq l, l+1\right\} \cup \{\neg\phi(x, b_l), \phi(x, b_{l+1})\}$$

is consistent.

Let $\psi_1(x) := \bigwedge_{i \neq l, l+1} \phi^{\eta(i)}(x, b_i)$. By indiscernibility of the sequence (b_i), for any $i < j \in \mathbb{Q} \cap (l, l+1)$ we have the following:

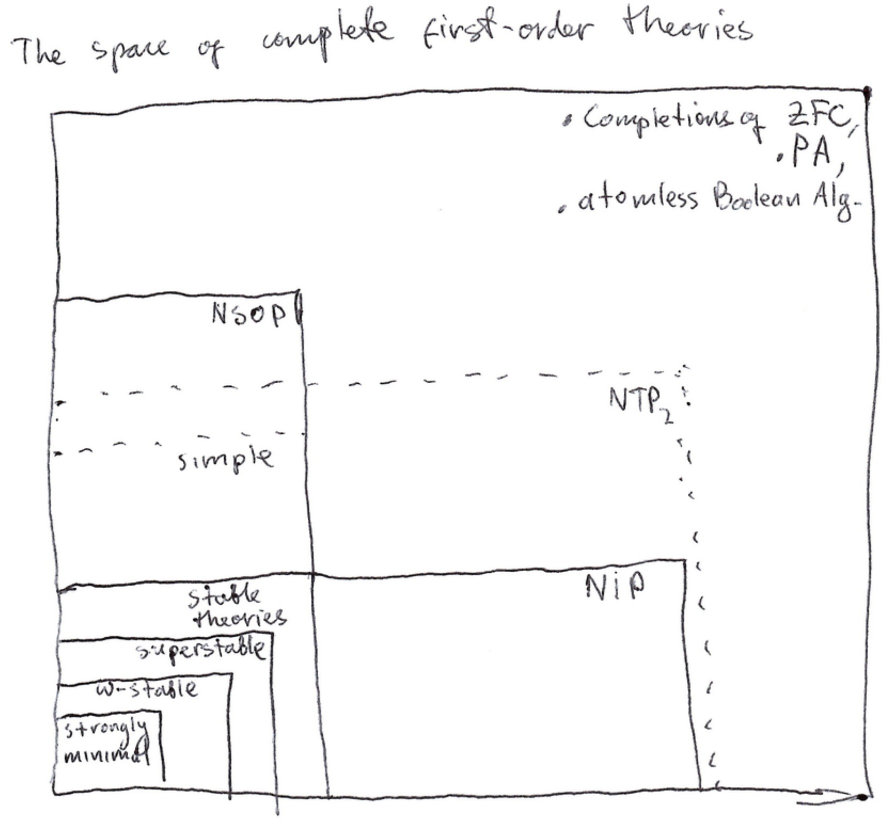
- $\psi_1(x) \wedge \{\phi(x, b_i), \neg\phi(x, b_j)\}$ is inconsistent,
- $\psi_1(x) \wedge \{\neg\phi(x, b_i), \phi(x, b_i)\}$ is consistent.

Define $\psi(x, y) := \psi_1(x) \wedge \phi(x, y)$ and $\bar{b}' := (b_i : i \in \mathbb{Q} \cap (l, l+1))$. Then on $\bar{b}'$ we have $\exists x \neg\psi(x, b_i) \wedge \psi(x, b_j) \iff i < j$, which shows that $\psi(x, y)$ has the strict order property. □

Exercise 2.47. Show that DLO is NIP, and that the theory of a random graph is indeed NSOP.

***

We have the following classification picture for the space of all complete first-order theories.



An interactive map with more details and examples can be found at `http://www.forkinganddividing.com/`.

Example 2.48. Examples of stable theories.

(1) The theory of a countable number of equivalence relations E_n for n a natural number such that each equivalence relation has an infinite number of equivalence classes and each equivalence class of E_n is the union of an infinite number of different classes of E_{n+1} (it has QE, so types are determined by specifying the class with respect to each of the equivalence relations, which implies that over any set A there are at most $|A|^{\aleph_0}$-many types).

(2) Modules are stable (so in particular vector spaces and abelian groups are stable).
We consider a module $\mathcal{M} = (M, 0, +, -, (r(x) : r \in R))$ where R is a ring, and $r(x)$ is a function $x \mapsto rx$. Any theory of a module in this language admits QE down to the *pp-formulas* [Bau76]. Namely, every formula is equivalent to a Boolean combination of pp-formulas, where a pp-formula is a formula of the form $\exists \bar{y} (\gamma_1 \wedge \ldots \wedge \gamma_n)$, where $\gamma_i(\bar{x}\bar{y})$ is an equation of the form $r_1 x_1 + \ldots + r_n x_n = 0$. A pp-formula defines a subgroup of M^n, and if $\phi(x, y)$ is a pp-formula and $a \in M$, then $\phi(x, a)$ is either empty or defines a coset of $\phi(x, 0)$. Thus, given a *pp*-formula $\phi(x, y)$ and a, b in M, $\phi(x, a), \phi(x, b)$ are either equivalent or contradictory. Since types are

determined by pp-formulas, there are few of them, see e.g. [TZ12, Example 8.6.6]. Note that the ring R is incorporated into the language, and is not a definable as a part of the structure. This is essential to obtain stability, as for example every infinite finitely generated ring interprets arithmetic ([AKNS16]).

(3) Free groups are stable, in the pure group language (more generally, torsion-free hyperbolic groups are stable). This is a deep theorem of Sela (`https://en.wikipedia.org/wiki/Zlil_Sela`). Furthermore, if F_n is a free group on n generators, then we have $F_2 \prec F_3 \prec \ldots$, in particular they all have the same first-order theory.

(4) Algebraically closed fields, ACF_0 and ACF_p, are stable. Let K be a small subfield, by QE the type of an element a over K is determined by the isomorphism type of the extension $K[a]/K$. If a is transcendental over K, $K[a]$ is isomorphic to the polynomial ring $K[x]$. If a is algebraic with a minimal polynomial $f(x) \in K[x]$, then $K[a]$ is isomorphic to $K[x]/(f)$. Thus there are at most $(|K| + \aleph_0)$-many 1-types over K. In fact, we saw that ACF_0 is a strongly minimal theory (as well as ACF_p), and all strongly minimal theories are stable (**Exercise**).

(5) Separably closed fields are stable. Recall that a field K is separably closed if every non-constant separable polynomial over K has a zero in K. Any separably closed perfect field is algebraically closed, so we restrict to positive characteristic, in which case K^p is a subfield. If the degree of $[K : K^p]$ is finite, it is of the form p^e, and e is called the *degree of imperfection* of K. For any $e \in \mathbb{N}$, let $\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of char p with the degree of imperfection e, and let $\mathrm{SCF}_{p,\infty}$ be the theory of separably closed fields of char p with infinite degree of imperfection. These theories are complete, and they eliminate quantifiers after naming a basis and adding some function symbols to the language. See e.g. [TZ12, Example 8.6.7].

Open problem. Is every field K with $\mathrm{Th}(K)$ stable is separably closed? (a positive answer is known for superstable fields [CS80] and some other special cases).

(6) Differentially closed fields are stable. A differential field is a field K equipped with a function symbol $d : K \to K$ for a derivation d, i.e. d is an additive map such that $d(r_1 r_2) = d(r_1) r_2 + r_1 d(r_2)$ (Leibniz rule). The theory of differentially closed fields DCF_0 is the theory of differential fields of characteristic 0 satisfying the following property: for $f \in K[x_0, \ldots, x_n] \setminus K[x_0, \ldots, x_{n-1}]$ and $g \in K[x_0, \ldots, x_{n-1}]$, $g \neq 0$, there is some $a \in K$ such that $f(a, da, \ldots d^n a) = 0$ and $g\left(a, da, \ldots, d^{n-1} a\right) \neq 0$. Any differential field can be extended to a model of DCF_0, and DCF_0 has QE. The theory DCF_0 is stable (using QE one can establish a bijection between n-types over F and the so-called *prime δ-ideals* in $F\{x_1, \ldots, x_n\}$, the ring of differential polynomials, and such ideals are always generated by finitely many differential polynomials — a form of Noetherianity. Thus there are few types.) There is a positive characteristic analogue. See e.g. [Mar00].

(7) [PZ78] Let G be a planar graph, in the language with only the edge relation $\{E(x, y)\}$. Then G is stable (it is easy to see that e.g. if the relation $E(x, y)$

has the order property, then E is not planar, but one has to take care of formulas with quantifiers as well).

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2.6. **Stability in continuous logic.** A good reference for continuous logic is Ben Yaacov, Berenstein, Henson, Usvyatsov "Model theory for metric structures" [BYBHU08].

Basically, every structure M now is a complete metric space of bounded diameter, with metric d. Signatures are given by:

- function symbols with given moduli of uniform continuity (interpreted as uniformly continuous functions from M^n to M),
- predicate symbols with given moduli of uniform continuity (uniformly continuous functions from M to $[0,1]$).

Connectives are given by the set of all continuous functions from $[0,1] \to [0,1]$, or any subfamily which generates a dense subset (e.g. $\left\{\neg, \frac{x}{2}, \dot{-}\right\}$, where $\neg x = 1-x$ and $x \dot{-} y = \max\{0, x-y\}$, compare to the standard Boolean operations that generate all possible functions to $\{0,1\}$). Quantifiers are given by sup for $\forall$ and inf for $\exists$ (so truth is 0 and false is 1, but formulas can also have any truth value in the interval $[0,1]$). By induction one defines formulas with truth values in $[0,1]$, and using the assumptions one shows that they all define uniformly continuous functions. This logic admits a compactness theorem and basic stability can be developed in a manner analogous to the discrete case. Of course, modulo some natural changes: cardinality is replaced by the density character, in acl "finite" is replaced by "compact", some equivalences are replaced by the ability to approximate uniformly up to any $\varepsilon > 0$, etc. Recently it was realized that in fact basic stability theory as developed by Shelah follows from some classical results of Grothendieck in functional analysis (stable formulas correspond to weakly almost periodic functions, etc.), which provides a natural generalization to continuous logic, but also covers the discrete case (this gives no bounds however) — see [BY14].

Example 2.49. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, let $L^1((\Omega, \mathcal{F}, \mu); [0,1])$ be the space of $[0,1]$-valued random variables on it. We consider it as a continuous structure in the language $L_{\mathrm{RV}} = \left\{0, \neg, \frac{x}{2}, \dot{-}\right\}$ with the natural interpretation of the connectives (e.g. $\left(X \dot{-} Y\right)(\omega) = X(\omega) \dot{-} Y(\omega)$) and the distance $d(X,Y) = \mathbf{E}[|X-Y|] = \int_{\Omega} |X-Y| \, d\mu$. Consider the following continuous theory RV in the language L_{RV}, we write 1 as an abbreviation for $\neg 0$, $E(x)$ for $d(0,x)$ and $x \wedge y$ for $x \dot{-} \left(x \dot{-} y\right)$:

- $E(x) = E\left(x \dot{-} y\right) + E(y \wedge x)$
- $E(1) = 1$
- $d(x,y) = E\left(x \dot{-} y\right) + E\left(y \dot{-} x\right)$
- $\tau = 0$ for every term τ which can be deduced in the propositional continuous logic.

The theory ARV of atomless random variable is defined by adding:

- Atomlessness: $\inf_y \left(E(y \wedge \neg y) \vee \left|E(y \wedge x) - \frac{E(x)}{2}\right|\right) = 0$.

The following is demonstrated in [BY13].

(1) $M \models \mathrm{ARV} \Leftrightarrow$ it is isomorphic to $L^1(\Omega, [0,1])$ for *some* atomless probability space $(\Omega, \mathcal{F}, \mu)$.

(2) ARV is the model completion of the universal theory RV (so every probability space embeds into a model of ARV).
(3) ARV eliminates quantifiers, and two tuples have the same type over a set $A \subseteq M$ if and only if they have the same joint conditional distribution as random variables over $\sigma(A)$, where $\sigma(A) \subseteq \mathcal{F}$ denotes the minimal complete subalgebra with respect to which every random variable $X \in A$ is measurable.
(4) The theory ARV is stable. In fact, by Theorem 2.36 stability of ARV is equivalent to the following classical theorem of Ryll-Nardzewski from probability theory: a sequence of random variables is exchangeable iff it is spreadable (see e.g. [Kal88]), which in view of (3) means precisely that any infinite indiscernible sequence is totally indiscernible.

2.7. **Number of types and definability of types in NIP.**

Lemma 2.50. *If $F \subseteq 2^{\lambda}$ and $|F| > \operatorname{ded}\lambda$, then for each $n < \omega$ there is some $I \subseteq \lambda$ such that $|I| = n$ and $F \restriction I = 2^{I}$.*

Proof. Assume F, λ are a counterexample, with λ minimal. Note that F can be naturally identified with a set of branches of the tree $\bigcup_{i<\lambda} F \restriction i$. By minimality of λ we may assume that for each $i < \lambda$, $|F \restriction i| \leq \operatorname{ded}\lambda$.

For each $f \in F \restriction i$, let $F(f) := \{g \in F : f \subseteq g\}$, $G_i := \{f \in F \restriction i : |F(f)| > \operatorname{ded}\lambda\}$ and $G = \{f \in F : f \restriction i \in G_i \text{ for all } i < \lambda\}$. Then $G \subseteq F$ is a set of branches of the tree $\bigcup_{i<\lambda} G_i$. Note that $F \setminus G = \bigcup_{i<\lambda} \bigcup_{g \in (F\restriction i) \setminus G_i} F(g)$, hence $|F \setminus G| \leq \lambda \times \operatorname{ded}\lambda \times \operatorname{ded}\lambda = \operatorname{ded}\lambda$, and so $|G| > \operatorname{ded}\lambda$. Therefore we may assume that $G = F$, i.e. we can assume that for each $i < \lambda$ and $f \in F \restriction i$, $|F(f)| > \operatorname{ded}\lambda$.

Now we prove by induction on n, that for each $n < \omega$, for each $f \in \bigcup_{i<\lambda} F \restriction i$ there is some $I \subseteq \lambda$ such that $|I| = n$ and $F(f) \restriction I = 2^{I}$. This is clear for $n = 0$ since $F(f) \neq \emptyset$, and consider the case of $n+1$. By definition of ded, since $F(f)$ is a set of branches of the tree $\bigcup_{i<\lambda} F(f) \restriction i$, this tree has cardinality $> \lambda$ and therefore $|F(f) \restriction i| > \lambda$ for some $i < \lambda$. By the induction hypothesis, for each $g \in F(f) \restriction i$ there is some $I_g \subseteq \lambda$ such that $|I_g| = n$ and $F(g) \restriction I_g = 2^{I_g}$. By pigeonhole, there are two different $g, h \in F(f) \restriction i$ such that $I := I_g = I_h$. Choose some $j < i$ such that $h(j) \neq g(j)$. Then $j \notin I$. If $J = I \cup \{j\}$, then $F(f) \restriction J = 2^{J}$. $\square$

Proposition 2.51. (1) *If $\phi(x,y)$ has IP, then for each cardinal κ there is a set A of cardinality κ such that $|S_{\phi}(A)| = 2^{\kappa}$.*
(2) *If $\phi(x,y)$ is NIP, then for each cardinal κ and a set of parameters A, if $|A| = \kappa$ then $|S_{\phi}(A)| \leq \operatorname{ded}\kappa$.*

Proof. (1) If $\phi(x,y)$ has IP, then so does $\phi^{*}(y,x) := \phi(x,y)$. Thus by compactness for any κ we can find a set A of size κ, such that for any $S \subseteq A$ there is some a_S with $\models \phi(a_S, b) \iff b \in S$, for all $b \in S$.

(2) Assume $|A| = \kappa$ and $\left|S_{\phi(x,y)}(A)\right| > \operatorname{ded}\kappa$. Fix an enumeration $A = (a_i : i < \kappa)$. For each $p \in S_{\phi}(A)$ let $f_p \in 2^{\kappa}$ be defined by $f_p(i) = 0 \iff \phi(x, a_i) \in p$. Let $F = \{f_p : p \in S_{\phi}(A)\}$. Since $|F| > \operatorname{ded}\kappa$, for each $n \in \omega$ there is some $I \subseteq \kappa$ such that $|I| = n$ and $F \restriction I = 2^{I}$. Then for each $X \subseteq I$, $\{\phi(x, a_i) : i \in X\} \cup \{\neg\phi(x, a_i) : i \in I \setminus X\}$ is contained in one of the types over A, and so consistent. Hence $\phi(x,y)$ has IP. $\square$

Thus, the NIP property is precisely the dividing line between the last two cases in Fact 2.5.

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If we consider ϕ-types over finite sets, this translates into the following lemma of Sauer/Shelah/Perles/Vapnik, Chervonenkis (the ded function becomes a polynomial over finite sets).

Fact 2.52. *A formula $\phi(x,y)$ is NIP if and only if there are some $d,c \in \omega$ such that for any finite set A with $|A| = n$ we have $|S_\phi(A)| \leq cn^d$. In fact, d can be taken to be the maximal size of a set that can be shattered by instances of $\phi(x,y)$.*

So, over finite sets the bound on the number of types in stable theories is not better than in NIP theories. Recall that uniform definability of types is a characteristic property of stability (Theorem 2.23). Let's consider definability of types again. We have already observed that if we consider a type given by a non-realized cut over $(\mathbb{Q},<)$, then it is not definable. However, all cuts over *finite* subsets actually have an endpoint, which gives a uniform definability procedure. More generally:

Fact 2.53. [CS15b] *Let T be NIP. Then types over* ***finite*** *sets are uniformly definable. I.e., for every formula $\phi(x,y)$ there is a formula $\psi(y,z)$ such that for every finite set $A \subseteq \mathbb{M}_y$ (with $|A| \geq 2$) and every $p(x) \in S_\phi(A)$ there is some tuple b from A such that $\phi(x,a) \in p \iff \models \psi(a,b)$ for all $a \in A$.*

This is related to a conjecture of Warmuth on the existence of compression schemes for families of sets of finite VC-dimension (note that a formula $\phi(x,y)$ is NIP iff the family $\mathcal{F} = \{\phi(\mathbb{M},a) : a \in \mathbb{M}\}$ has finite VC-dimension). Fact 2.53 establishes a weaker form of this conjecture (it requires that not only the family given by $\phi(x,y)$ has finite VC-dimension, but also that certain families defined from it using quantifiers have finite VC-dimension). Very recently the conjecture was proved in [MY15].

3. Forking calculus

Now we begin moving towards some more "geometric" parts of the theory. First we define some notions of "small" and "large" definable sets, then we consider various "generic" extensions of types and show how in an arbitrary stable theory one can use it to define a notion of "independence" for subsets of the monster, generalizing for example linear independence in vector spaces and algebraic independence in algebraically closed fields.

3.1. Keisler measures and generically prime ideals.

Definition 3.1.
1. A *Keisler measure (over a set of parameters A)* is a finitely-additive probability measure on the Boolean algebra of A-definable subsets of $\mathbb{M}_x$. That is, a Keisler measure over A is a map $\mu : \mathrm{Def}_x(A) \to [0,1]$ such that
   (a) $\mu(\mathbb{M}_x) = 1$,
   (b) $\mu(P \cup Q) = \mu(P) + \mu(Q)$ for all disjoint $P, Q \in \mathrm{Def}_x(A)$.
2. A Keisler measure μ is *invariant over A* if $a \equiv_A b$ implies $\mu(\phi(x,a)) = \mu(\phi(x,b))$.

Exercise 3.2. One can think of a Keisler measure over A as defined on the clopen subsets of the space of types $S_x(A)$ (recall Section 1.4). Then every Keisler measure over A can be extended in a unique way to a regular countable additive Borel probability measure on the space of types over A (regular means that for all Borel sets X and $\varepsilon > 0$ there are some closed Y and open Z such that $Y \subseteq X \subseteq Z$ and $\mu(Z) - \mu(Y) < \varepsilon$).

From now on by a measure we always mean a Keisler measure. A type can be thought of as a $\{0,1\}$-measure.

Definition 3.3. Recall that a set $I \subseteq \mathrm{Def}_x(A)$ is an *ideal* if:

(1) $\emptyset \in I$,
(2) $\phi(x,a) \vdash \psi(x,b)$ and $\psi(x,b) \in I$ implies $\phi(x,a) \in I$,
(3) $\phi(x,a) \in I$ and $\psi(x,b) \in I$ implies $\phi(x,a) \vee \psi(x,b) \in I$.

An ideal I is invariant over A if $\phi(x,a) \in I$ and $a \equiv_A b$ implies $\phi(x,b) \in I$. As usual, an ideal I in $\mathrm{Def}(\mathbb{M})$ is *prime* if whenever $\phi(x,a) \wedge \psi(x,b) \in I$, then either $\phi(x,a) \in I$ or $\psi(x,b) \in I$. However, in the Boolean algebra $\mathrm{Def}_x(\mathbb{M})$, prime ideals correspond to complete types in $S_x(\mathbb{M})$ (as for any $\phi(x,b)$, $\phi(x,b) \wedge \neg\phi(x,b) = \emptyset \in I$, so either $\phi(x,b)$ or $\neg\phi(x,b)$ has to belong to I, hence the complement of I is a complete type). We introduce a weaker notion.

Definition 3.4. Given a cardinal κ, we say that an ideal $\mathcal{I}$ in $\mathrm{Def}_x(A)$ is *κ-prime* if for any family $(S_i)_{i<\kappa}$ of A-definable sets with $S_i \notin \mathcal{I}$ for all $i < \kappa$, there are some $i < j < \kappa$ such that $S_i \cap S_j \notin \mathcal{I}$. We say that an ideal $\mathcal{I}$ is *generically prime* if it is κ-prime for some κ.

Example 3.5. (1) With this definition an ideal is prime if and only if it is 2-prime.
(2) Let μ be an arbitrary finitely-additive probability measure on X, and let 0_μ be its 0-ideal. Then 0_μ is $\aleph_1$-prime. Indeed, take $J = \aleph_1$ and assume we are given a family $(S_i : i \in J)$ of sets of positive measure, say $\mu(S_i) > \frac{1}{n_i}$ for some $n_i \in \omega$. Then by pigeon-hole there is some $n \in \omega$ and some infinite $J' \subseteq J$ such that $\mu(S_i) > \frac{1}{n}$ for all $i \in J'$. Then it follows from basic probability theory that we can find an infinite subsequence $J'' \subseteq J$ such that $\mu(S_{i_0} \cap \ldots \cap S_{i_m}) > 0$ for any $m \in \omega$ and $i_0 < \ldots < i_n$ from J''.

Generically prime ideals are closely related to Hrushovski's notion of $S1$-ideals (see [Hru12]).

Proposition 3.6. *Let I be an A-invariant ideal in $\mathrm{Def}_x(\mathbb{M})$. Then the following are equivalent:*

(1) *I is $S1$, i.e. for any A-indiscernible sequence $(b_i)_{i\in\omega}$ and any formula $\phi(x,y)$, if $\phi(x,b_0) \notin I$ then $\phi(x,b_0) \wedge \phi(x,b_1) \notin I$.*
(2) *I is generically prime.*
(3) *I is $\left(2^{|A|+|T|}\right)^+$-prime.*

Proof. Assume that we have an A-indiscernible sequence $(a_i)_{i\in\omega}$ such that $\phi(x,a_0) \wedge \phi(x,a_1) \in I$ but $\phi(x,a_0) \notin I$. By compactness, indiscernibility and invariance of I, for any κ we can find a sequence $(a_i)_{i\in\kappa}$ such that $\phi(x,a_i) \notin I$ and $\phi(x,a_i) \wedge \phi(x,a_j) \in I$ for all $i \neq j \in \kappa$, thus I is not generically prime.

Conversely, assume that I is not generically prime. Then for any κ we can find $(\phi_i(x,a_i))_{i\in\kappa}$ with $\phi_i(x,a_i) \notin I$ and $\phi_i(x,a_i) \wedge \phi_j(x,a_j) \in I$. Taking κ large enough and applying Fact 2.33 we find an A-indiscernible sequence starting with a_i, a_j for some $i \neq j$ and such that $\phi_i = \phi_j$. $\square$

In fact a refinement of this proof shows that if an ideal I is generically prime and invariant over A, then one can always take $\kappa = \left(2^{|A|+|T|}\right)^+$ in the definition.

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3.2. **Dividing and forking.**

Definition 3.7. (1) A formula $\phi(x,a)$ *divides* over B if there is a sequence $(a_i)_{i\in\omega}$ and $k \in \omega$ such that $a_i \equiv_B a$ and $\{\phi(x,a_i)\}_{i\in\omega}$ is k-inconsistent. Equivalently, if there is a B-indiscernible sequence $(a_i)_{i\in\omega}$ starting with a and such that $\{\phi(x,a_i)\}_{i\in\omega}$ is inconsistent (by compactness and Fact 2.33).

(2) A formula $\phi(x,a)$ *forks* over B if it belongs to the ideal generated by the formulas dividing over B, i.e. if there are $\psi_i(x,c_i)$ dividing over B for $i<n$ and such that
$$\phi(x,a) \vdash \bigvee_{i<n} \psi_i(x,c_i).$$

(3) We denote by $\mathbf{F}(B)$ the ideal of formulas forking over B. It is invariant over B.

Example 3.8. Let T be DLO, then $a<x$ does not divide over $\emptyset$, but $a<x<b$ does (easy to check using QE).

Example 3.9. In general there are formulas which fork, but do not divide. Consider the unit circle around the origin on the plane, and a ternary relation $R(x,y,z)$ on it which holds if and only if y is between x and z, ordered clock-wise. Let T be the theory of this relation. Check:

(1) This theory eliminates quantifiers.
(2) There is a unique 2-type $p(x,y)$ over $\emptyset$ consistent with “$x \neq y$”.
(3) $R(a,y,c)$ divides over $\emptyset$ for any a,c.
(4) The formula “$x=x$” forks over $\emptyset$ (but it does not divide, of course — no formula can divide over its own parameters).

Definition 3.10. A (partial) type does not divide (fork) over B if it does not imply any formula which divides (resp. forks) over B.

Note: if $a \notin \operatorname{acl}(A)$ then $\operatorname{tp}(a/Aa)$ divides over A. Also, if $\pi(x)$ is consistent and defined over $\operatorname{acl}(A)$, then it doesn't divide over A.

Exercise 3.11. Let $p \in S_x(\mathbb{M})$ be a global type, and assume that it doesn't divide over a small set A. Then it doesn't fork over A.

Proposition 3.12. $\mathbf{F}(B)$ *is contained in every generically prime B-invariant ideal.*

Proof. It is enough to show that if $\varphi(x,a)$ divides over B and I is a generically prime ideal, then $\varphi(x,a) \in I$. We use the equivalence from Proposition 3.6. Let $(a_i)_{i\in\omega}$ be indiscernible over B with $a_0 = a$ and $\{\varphi(x,a_i)\}_{i\in\omega}$ inconsistent. If $\varphi(x,a_0) \notin I$, then by induction using that I is generically prime (and that if $(a_i)_{i\in\omega}$ is indiscernible over B, then $(a_{2i}a_{2i+1})_{i\in\omega}$ is indiscernible over B), we see that $\bigwedge_{i<k}\varphi(x,a_i) \notin I$ for all $k\in\omega$. But as $\emptyset \in I$ this would imply that $\{\varphi(x,a_i)\}$ is consistent, a contradiction. $\square$

Notice that any intersection of B-invariant generically prime ideals is still B-invariant and generically prime.

Definition 3.13. (1) Let $\mathbf{GP}(A)$ be the smallest generically prime ideal invariant over A.
(2) Let $\mathbf{0}(A)$ be the ideal of formulas which have measure 0 with respect to every A-invariant Keisler measure.

Summing up the previous observations, we have the following picture:

Proposition 3.14. *In any theory and for any set A, $\mathbf{F}(A) \subseteq \mathbf{GP}(A) \subseteq \mathbf{0}(A)$.*

Example 3.15. There are theories with $\mathbf{F}(A) \subsetneq \mathbf{GP}(A)$, equivalently theories in which the forking ideal is not generically prime. Look at the triangle-free random graph (i.e. the model completion of the theory of graphs saying that there are no triangles — it exists and eliminates quantifiers, an important property for us is that it embeds any finite graph without triangles). Then we have:

(1) $R(x,a)$ does not divide for any a (as any indiscernible sequence of singletons has to be an anti-clique).
(2) $R(x,a) \wedge R(x,b)$ divides for any $a \neq b$ (witnessed by a sequence $(a_i b_i)$ such that $R(a_i, b_j) \Leftrightarrow i \neq j$).
(3) Thus for any infinite indiscernible sequence of singletons (a_i), $R(x,a_0)$ does not divide while $R(x,a_0) \wedge R(x,a_1)$ does.

Problem 3.16. Hrushovski suggested an example of a (simple) theory in which $\mathbf{F}(\emptyset) = \mathbf{GP}(\emptyset) \subsetneq \mathbf{0}(\emptyset)$. I don't known any examples of $\mathbf{F}(A) \subsetneq \mathbf{GP}(A) \subsetneq \mathbf{0}(A)$ and of $\mathbf{F}(A) \subsetneq \mathbf{GP}(A) = \mathbf{0}(A)$. In NIP theories we have that $\mathbf{F}(M) = \mathbf{0}(M)$.

3.3. Special extensions of types.

- Let $A \subseteq B$ and $p \in S_x(A)$. Then of course there is some $q \in S_x(B)$ with $p \subseteq q$ (as p is a filter in $\mathrm{Def}_x(B)$, so extends to an ultrafilter).
- We would like to be able to choose a "generic" extension q of p, such that it doesn't add any new conditions on q with respect to the new parameters from B which were not already present with respect to the parameters from A (as opposed to something like "$(x = b) \in q$", this can be also thought of as a far-reaching generalization of taking a generic point on an algebraic variety).
- We begin by understanding special extensions of types over models, which is easier, and then proceed to types over arbitrary sets, where the situation in stable theories is explained by M^{eq}.

Definition 3.17. A global type $p(x) \in S(\mathbb{M})$ is called *invariant* over C if it is invariant under all automorphisms of $\mathbb{M}$ fixing C. That is, for every $a \equiv_C b$ and $\phi(x,y) \in L$, $\phi(x,a) \in p \Leftrightarrow \phi(x,b) \in p$.

Applying Proposition 3.12 to $\{0,1\}$-measures, every global type invariant over A is non-forking over A.

Definition 3.18. Let $A \subseteq B$, $p \in S_x(A)$ and $q \in S_x(B)$ extending p be given (so $p = q \restriction_{\mathrm{Def}_x(A)}$, which we also denote as $p = q|_A$).

(1) We say that q is an *heir* of p (or "an heir over A") if for every formula $\phi(x,y) \in L(A)$, if $\phi(x,b) \in q$ for some $b \in B$, then $\phi(x,b') \in p$ for some $b' \in A$. Note that if q is an heir of p, then in fact A has to be a model of T.
(2) We say that q is a *coheir* of p ("coheir over A", "finitely satisfiable in A") if for any $\phi(x,b) \in q$ there is some $a \in A$ such that $\models \phi(a,b)$.

Exercise 3.19. Let $A \subseteq B$ be given.

(1) If a type $q \in S(B)$ is definable over A or is finitely satisfiable in A, then it *does not split* over A, i.e. for all $a \equiv_A a'$ from B and $\phi(x,y) \in L(A)$ we have that $\phi(x,a) \in q \iff \phi(x,a') \in q$. In particular, if $B = \mathbb{M}$ then q is A-invariant.
(2) If A is a model of T and $q \in S(B)$ is definable over A, then it is an heir over A.
(3) If $B = \mathbb{M}$ and $q \in S(B)$ is A-invariant then it doesn't fork over A (use 3.12).
(4) $\mathrm{tp}(a/Mc)$ is an heir of $\mathrm{tp}(a/M)$ if and only if $\mathrm{tp}(c/Ma)$ is a coheir of $\mathrm{tp}(b/M)$.

Example 3.20. Let $M = (\mathbb{Q}, <)$ and consider the type $p \in S_x(M)$ given by $p = \{a < x : a \in \mathbb{M}\}$. Now consider two global extensions $q_1, q_2 \in S_x(\mathbb{M})$ of p:

- $q_1(x) = \{a < x : a \in \mathbb{M}\}$,
- $q_2(x) = p(x) \cup \{x < b : M < b \in \mathbb{M}\}$.

It is easy to check that q_1 is M-definable, so it is an heir of p, but not a coheir of p. On the other hand, q_2 is a coheir of p, but it is not an heir over M.

Remark 3.21. Note that the space of A-invariant global types is a closed subset of $S(\mathbb{M})$ (as it equals $\bigcap_{\phi \in L, a \equiv_A b \in \mathbb{M}} \langle \phi(x,a) \leftrightarrow \phi(x,b) \rangle$), thus compact. Similarly, the space of types finitely satisfiable in A is a closed subset of A — it equals $\bigcap_{\phi(x,a) \in L(\mathbb{M}), \phi(A,a) = \emptyset} \langle \neg\phi(x,a) \rangle$. It can also be described as the closure of the set of types realized in A, i.e. of $\{\mathrm{tp}(a/\mathbb{M}) : a \in A\}$.

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Proposition 3.22. *Let $p \in S_x(M)$ be arbitrary, where $M \models T$ is a small model.*

(1) *There is a global coheir q of p.*
(2) *There is a global heir r of p.*

Proof. (1) Let $A \subseteq \mathbb{M}_x$ be small, and let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}(A)$. We can define a global type $q_{\mathcal{U}} \in S_x(\mathbb{M})$ in the following way. For a formula $\phi(x,b) \in L(\mathbb{M})$ we define $\phi(x,b) \in q_{\mathcal{U}} \iff \phi(A,b) \in \mathcal{U}$.

Then $q_{\mathcal{U}}$ is finitely satisfiable in A. Conversely, every global type q is finitely satisfiable in A is of the form $q_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on $\mathcal{P}(A)$. Namely, let $\mathcal{V} = \{\phi(A,b) : \phi(x,b) \in q\}$ — by finite satisfiability of q this is an ultrafilter on the Boolean algebra of all externally definable subsets of A. But then let $\mathcal{U} \supseteq \mathcal{V}$ be an arbitrary ultrafilter on $\mathcal{P}(A)$, it is easy to see that $q = q_{\mathcal{U}}$ (note that such a $\mathcal{U}$ need not be unique). Note that if $\mathcal{U}$ is a principal ultrafilter, then $q_{\mathcal{U}}$ is realized.

Now, any $p \in S_x(M)$ is finitely satisfiable in M (for any formula $\phi(x) \in p$ we have $\mathbb{M} \models \exists x \phi(x)$; as $\phi(x) \in L(M)$ and $M \prec \mathbb{M}$ we have $M \models \exists x \phi(x)$). It follows that $\{\phi(M) : \phi(x) \in p\}$ is a filter, so extends to some ultrafilter $\mathcal{U}$ on $\mathcal{P}(M)$. Then the global type $q_{\mathcal{U}}$ is a coheir of p.

(2) It is enough to show that the following set of formulas is consistent

$s(x) := p(x) \cup \{\phi(x,c) : c \in \mathbb{M}, \phi(x,y) \in L(M), \phi(x,m) \in p \text{ for all } m \in M\}.$

As then any complete type $r(x) \in S_x(\mathbb{M})$ with $r \supseteq s$ is an heir of p (indeed, the first condition guarantees that r extends p, and the second condition implies that for any $\phi(x,c) \in r$ there has to be some $c' \in M$ with $\phi(x,c') \in r|_M$).

Assume it is not consistent, then by compactness there are some formulas $\phi(x,c) \in p$ and $\phi_i(x,c_i), i < n$ from $s(x)$ such that their conjunction is inconsistent, i.e.

$\models \phi(x,c) \to \bigvee_{i<n} \neg\phi(x,c_i)$. As $\phi(x,c) \in L(M)$ and $M \prec \mathbb{M}$, it follows that there are some $m_i, i<n$ such that $M \models \phi(x,c) \to \bigvee_{i<n} \neg\phi(x,m_i)$. But by the definition of $s(x)$ we have $\phi(x,m_i) \in p$ for all $i<n$, as well as $\phi(x,c) \in p$ — thus their conjunction is consistent, a contradiction. $\square$

Proposition 3.23. *Let $p \in S_x(M)$ be a definable type. Then it has a unique global heir $q \supseteq p$, which is in fact definable over M.*

Proof. First we show that p has a global M-definable extension. As $p(x)$ is definable, it follows that for every $\phi(x,y) \in L$ there is some $d\phi(y) \in L(M)$ such that $\phi(x,a) \in p \iff \models d\phi(a)$, for all $a \in M$. Consider the following set of formulas

$$q(x) := \{\phi(x,a) : \phi(x,y) \in L, a \in \mathbb{M}_y, \models d\phi(a)\}.$$

We claim that it is consistent. Indeed, by compactness it is enough to show that for any $n \in \omega$ and any $\phi_i(x,y_i) \in L$,

$$\models \forall y_0 \dots y_{n-1} \exists x \left(\phi_i(x,y_i) \leftrightarrow d\phi_i(y_i)\right)$$

holds. But as $d\phi_i(y) \in L(M)$ and $M \prec \mathbb{M}$, this is equivalent to

$$M \models \forall y_0 \dots y_{n-1} \exists x \left(\phi_i(x,y_i) \leftrightarrow d\phi_i(y_i)\right)$$

Recalling that $\{d\phi_i(y_i)\}$ is in fact a definition schema for the type p, which is consistent, we see that this holds. Now it is easy to see that in fact $q(x)$ is a complete type extending $p(x)$.

Assume that q, r are two global types extending p which are both definable over M. This implies that for their corresponding defining schemas, $(d_q\phi(y))_{\phi(x,y)\in L}$ and $(d_r\phi(y))_{\phi(x,y)\in L}$ we must have $d_q\phi(M) = d_r\phi(M)$. But again, as $M \prec \mathbb{M}$, this implies that $d_q\phi(M) = d_r\phi(M)$, and so $q = r$.

By Exercise 3.19, $q(x)$ is an heir of $p(x)$. Now if $q' \neq q$ is another global type extending p, then for some $\phi(x,b) \in q'$ we have $\neg\phi(x,b) \in q$, so $\not\models d\phi(b)$, and so $(\phi(x,b) \wedge \neg d\phi(b)) \in q'$. But as there can be no $m \in M$ with $\models \phi(x,m) \wedge \neg d\phi(m)$, and as $\phi(x,y) \wedge \neg\phi(y) \in L(M)$, it follows that q' is not an heir of p. $\square$

Proposition 3.24. *Let $p \in S_x(\mathbb{M})$ be a global A-invariant type.*

1. *If p is definable, then in fact it is definable over A.*
2. *If p is finitely satisfiable in some small set, then in fact it is finitely satisfiable in any model $M \supseteq A$.*

Proof. (1) As p is definable, for any formula $\phi(x,y) \in L$ there is some $d\phi(y) \in L(\mathbb{M})$ such that for any $b \in \mathbb{M}$ we have $\phi(x,b) \in p \iff b \in d\phi(\mathbb{M})$. As p is A-invariant, the definable set $d\phi(\mathbb{M})$ is also (setwise) $\mathrm{Aut}(\mathbb{M}/A)$-invariant. But then the set $d\phi(\mathbb{M})$ is in fact A-definable by Lemma 1.12, which implies that p is definable over A.

(2) Let's say p is finitely satisfiable in some small model N. Let M be an arbitrary small model containing A. Let $\phi(x,b) \in p$ be arbitrary. Consider the type $\mathrm{tp}(N/M)$, so it is a type in $|N|$-many variables. By Proposition 3.22 this has a global coheir $r(\bar{x})$, let $N_1 \models r|_{Mb}$. Then by invariance p is finitely satisfiable in N_1, in particular $\phi(N_1,b) \neq \emptyset$. But as the type $\mathrm{tp}(N_1/Mb)$ is finitely satisfiable in M, it follows that $\phi(M,b) \neq \emptyset$. $\square$

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3.4. **Tensor product of invariant types and Morley sequences.**

Definition 3.25. Let $p \in S_x(\mathbb{M}), q \in S_y(\mathbb{M})$ be two global, A-invariant types. Then we define their tensor product $p \otimes q \in S_{xy}(\mathbb{M})$ as follows:

given a formula $\phi(x,y) \in L(B)$, $A \subseteq B \subseteq \mathbb{M}$, we set $\phi(x,y) \in p \otimes q \iff \phi(x,b) \in p$ for some (equivalently, any, by invariance of p) $b \in \mathbb{M}_y$ such that $b \models q|_B$.

Remark 3.26.

(1) Note that $p \otimes q$ is indeed a complete type, as

$$p \otimes q = \bigcup_{A \subseteq B \subset_{\text{small}} \mathbb{M}} \{\text{tp}(ab/B) : a \models p|_{bB}, b \models q|_B\}.$$

(2) If both p, q are A-invariant, then $p \otimes q$ is also A-invariant (Exercise).
(3) The operation $\otimes$ is associative, i.e. $p \otimes (q \otimes r) = (p \otimes q) \otimes r$. The reason is that for any small set B, both products restricted to B are equal to $\text{tp}(abc/B)$ for $c \models r, b \models q|_{Bc}, a \models p|_{Bbc}$.
(4) However, $\otimes$ need not be commutative. Let T be DLO, and let $p = q$ be the type at $+\infty$, it is $\emptyset$-invariant. Then $p(x) \otimes q(y) \vdash x > y$, while $q(y) \otimes p(x) \vdash x < y$. (Check however that any two distinct types in DLO commute).
(5) In fact, in the definition of the tensor product, we have only used that p is invariant.

Definition 3.27. Let $p \in S_x(\mathbb{M})$ be a global A-invariant type. Then for any $n \in \omega$ we define by induction $p^{(1)}(x_0) := p(x_0)$ and $p^{(n+1)}(x_0, \ldots, x_n) := p(x_n) \otimes p^{(n)}(x_0, \ldots, x_{n-1})$. We also let $p^{(\omega)}(x_0, x_1, \ldots) := \bigcup_{n \in \omega} p^{(n)}(x_0, \ldots, x_{n-1})$. For any set $B \supseteq A$, a sequence $(a_i : i \in \omega) \models p^{(\omega)}|_B$ is called a *Morley sequence* of p over B (indexed by ω).

Remark 3.28.

(1) We can define $p^{(I)}$ for an arbitrary order type I in a natural way.
(2) Note that for any $(a_i : i < \omega), (b_i : i < \omega) \models p^{(\omega)}|B$ we have that $(a_i : i < \omega) \equiv_B (b_i : i < \omega)$. In particular, any Morley sequence of p over B is B-indiscernible, by the associativity of $\otimes$.

The following characterization of dividing will be very useful.

Lemma 3.29. *The following are equivalent.*

(1) $\text{tp}(a/Ab)$ *does not divide over* A.
(2) *For every infinite A-indiscernible sequence I such that $b \in I$, there is some $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.*
(3) *For every infinite A-indiscernible sequence I such that $b \in I$, there is some $J \equiv_{Ab} I$ such that J is Aa-indiscernible.*

Proof. The equivalence of (2) and (3) follows by taking an A-automorphism.

(3) implies (1) is clear since (3) cannot hold for any sequence I witnessing dividing, i.e. such that $I = (b_i)$ with $b_0 = b$, $\models \phi(a, b_0)$ and $\{\phi(x, b_i) : i \in \omega\}$ inconsistent.

We prove (1) implies (2). Without loss of generality $A = \emptyset, I = (b_i : i \in \omega)$ and $b_0 = b$. Let $p(x,b) = \text{tp}(a/b)$ and let $\Gamma(x, (x_i : i < \omega))$ be a set of formulas expressing that $(x_i : i \in \omega)$ is indiscernible over x. It is enough to show that $p(x, b_0) \cup \Gamma(x, (b_i : i \in \omega))$ is consistent, as then any a' realizing it satisfies the requirement. By (1) and compactness, the partial type $q(x) := \bigcup_{i\in\omega} p(x, b_i)$ is consistent. Let $c \models q(x)$, and let Γ_0 be an arbitrary finite subset of Γ, enough to show that Γ_0 is consistent. By Ramsey's theorem there is an order-preserving function $f : \omega \to \omega$ such that $\models \Gamma_0\left(c, \left(b_{f(i)} : i \in \omega\right)\right)$. By indiscernibility, $(b_i : i \in \omega) \equiv \left(b_{f(i)} : i \in \omega\right)$, and so by taking an automorphism we can find some c' such that $c'(b_i : i \in \omega) \equiv c\left(b_{f(i)} : i \in \omega\right)$. Then $c' \models q(x) \cup \Gamma_0(x, (b_i : i \in \omega))$, as wanted. □

Corollary 3.30. *If* $\mathrm{tp}(a/B)$ *does not divide over* $A \subseteq B$ *and* $\mathrm{tp}(b/Ba)$ *does not divide over* Aa*, then* $\mathrm{tp}(ab/B)$ *does not divide over* A*.*

Proof. Using the equivalence in Lemma 3.29. Let I be an arbitrary A-indiscernible sequence starting with B. Then by the first assumption we find $I' \equiv_B I$ such that I' is aA-indiscernible. By the second assumption, we find $I'' \equiv_{Ba} I'$ such that I'' is abB-indiscernible. As in particular $I'' \equiv_B I$, we conclude. □

Corollary 3.31. *If* $\phi(x, a)$ *k-divides over* A *and* $\mathrm{tp}(b/Aa)$ *does not divide over* A*, then* $\phi(x, a)$ *k-divides over* Ab*.*

Proof. Let $I = (a_i :\in \omega)$ be an infinite A-indiscernible sequence such that $a_0 = a$ and $\{\phi(x, a_i) : i \in \omega\}$ is k-inconsistent. By assumption and Lemma 3.29 there is some $J \equiv_{Aa} I$ which is Ab-indiscernible. Then J witnesses that $\phi(x, a)$ $k-$divides over Ab. □

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Proposition 3.32. *Let* $p \in S_x(\mathbb{M})$ *be a global type, and let* M *be a small model. TFAE:*

1. *If* p *is definable over* A*, then* p *does not divide over* A*.*
2. *If* T *is stable and* p *does not divide over* M*, then* p *is definable over* M*.*

Proof. We already know (1), and we prove (2).

Assume that T is stable and that p does not divide over M. We will show that p is an heir of $p|_M$, which is enough (as $p|_M$ is a definable type by stability and Theorem 2.23, which using Proposition 3.23 implies that p is definable over M). So let $\phi(x, y) \in L(M)$ be given, and assume that $\phi(x, b) \in p$. We want to show that $\phi(x, b') \in p$ for some $b' \in M$. Let $I = (b_i : i \in \omega)$ be a Morley sequence of a global coheir extension of $\mathrm{tp}(b/M)$ over M starting with $b_0 = b$ (exists by Proposition 3.22). Let $a \models p|_{Mb}$. Since $\mathrm{tp}(a/Mb)$ does not divide over M, by Lemma 3.29 we may assume that I is indiscernible over Ma. So we have $\models \phi(a, b_i)$ for all $i \in \omega$. Again by stability and Theorem 2.23, the type $q = \mathrm{tp}(a/MI)$ is definable. Let $n \in \omega$ be such that all of the parameters of $d\phi(y)$ are in $M \cup \{b_0, \ldots, b_{n-1}\}$. Since $\mathrm{tp}(b_n/b_{<n}M)$ is a coheir of $\mathrm{tp}(b/M)$ and $\models d\phi(b_n)$ (as $\models \phi(a, b_n)$), it follows that there is some $b' \in M$ with $\models d_q\phi(b')$. This implies that $\models \phi(a, b')$, and so $\phi(x, b') \in \mathrm{tp}(a/M) = p|_M$, as wanted. □

3.5. **Forking and dividing in simple theories.** Now we consider the question of the equality of forking and dividing (recall that in general they can be different, Example 3.9). We work in the larger context of simple theories.

Definition 3.33. A theory T is *simple* if every type $p \in S_x(A)$ does not divide over some subset $A_0 \subseteq A$ of size $|A| \leq |T|$.

Simple theories were introduced by Shelah [She80] and were extensively studied in the 90's by numerous authors starting with the work of Hrushovski in the finite rank case and Kim and Pillay in general. See e.g. [Har00, Cas11b, Cas07, Wag00].

Exercise 3.34. (1) Show that if T is stable then it is simple, and that if T is simple then it is NSOP.
(2) Show that the theory of a random graph is simple.

Example 3.35. (1) Let F be an arbitrary ultraproduct of finite fields. Then the theory $\mathrm{Th}(F)$ is simple (see [Cha97]).
(2) Any completion of the theory of an algebraically closed field with a generic automorphism (ACFA) is simple (see [Cha00]).

Note that, according to Definition 3.7, it is possible that a formula $\phi(x,a)$ divides over A, witnessed by a certain A-indiscernible sequence $I=(a_i)$ (i.e. $a_0=a$ and $\{\phi(x,a_i)\}$ is inconsistent), yet there is some other A-indiscernible sequence $J=(b_i)$ such that $b_0=a$ but $\{\phi(x,b_i)\}$ is consistent. However, we can isolate a class of indiscernible sequences which always witness that a formula divides.

Lemma 3.36. *[Kim's lemma for simple theories] Let T be simple. Assume that $\phi(x,a)$ divides over A and let $(a_i : i\in\omega)$ be an A-indiscernible sequence such that moreover $\mathrm{tp}(a_i/a_{<i}A)$ does not divide over A, for all i (such a sequence is also called a Morley sequence in the type $\mathrm{tp}(a/A)$). Then $\{\phi(x,a_i) : i\in\omega\}$ is inconsistent.*

Proof. Without loss of generality $A=\emptyset$. Assume that $\phi(x,a)$ divides over A, but for some Morley sequence (a_i) in $\mathrm{tp}(a/\emptyset)$ we have $\{\phi(x,a_i)\}$ is consistent. Let X be the linear order $\left(|T|^+\right)^*$, i.e. the reverse order for $|T|^+$. We may assume that in fact our sequence is $(a_i : i\in X)$ (by compactness, as dividing is $\mathrm{Aut}(\mathbb{M})$-invariant). Let $c\models\{\phi(x,a_i) : i\in X\}$. By simplicity there is some $Y\subseteq X$ with $|Y|\leq|T|$ such that $\mathrm{tp}(c/(a_i : i\in X))$ does not fork over $(a_i : i\in Y)$. By our choice of the order X there is some $i^*\in X$ with $i^*<Y$. Then $\mathrm{tp}((a_i : i\in Y)/a_{i^*})$ does not divide over $\emptyset$ (by assumption and applying Corollary 3.30 inductively). Since $\phi(x,a_{i^*})$ divides over $\emptyset$, it divides over $(a_i : i\in Y)$ by Corollary 3.31. But $\phi(x,a_{i^*})\in\mathrm{tp}(c/(a_i : i\in X))$, so $\mathrm{tp}(c/(a_i : i\in X))$ divides over $(a_i : i\in Y)$ — a contradiction. □

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But do we actually always have such Morley sequences?

Definition 3.37. We say that A is an *extension base* if every type over A does not fork over A. (Note that this is always true for non-dividing).

Proposition 3.38. *Let A be an extension base and $p\in S_x(A)$ be given. Then there is a Morley sequence in p.*

Proof. Since p doesn't fork over A and the set of all $L(\mathbb{M})$-formulas forking over A is an ideal, there is some global q extending p and non-forking over A. Then for any small cardinal κ we can find a sequence $\bar{a}=(a_i : i<\kappa)$ in $\mathbb{M}$ such that $a_i\models q|_{Aa_{<i}}$. Note that $\bar{a}$ need not be A-indiscernible. However, taking κ large enough compared to $|A|$ and $|a_i|$ and applying Fact 2.33, we find some A-indiscernible sequence $(a'_i : i\in\omega)$ which is A-indiscernible and such that for any $n\in\omega$ there are some $i_0<\ldots<i_{n-1}$ such that $\left(a'_j : j<n\right)\equiv_A\left(a_{i_j} : j<n\right)$. But by construction

of $\bar{a}$ and as dividing over A is $\mathrm{Aut}(A)$-invariant, it follows that $\mathrm{tp}\left(a'_i/a'_{<i}A\right)$ does not divide over A for all $i \in \omega$, so $(\bar{a}_i)$ is a Morley sequence in p. $\square$

Example 3.39.
(1) If T is arbitrary and $M \models T$ then M is an extension base (Exercise).
(2) If T is an arbitrary theory with Skolem functions, and A is an arbitrary set, then A is an extension base (Exercise). In particular, if $T = \mathrm{Th}(\mathbb{Q}_p, \times, +, 0, 1)$ then any set A is an extension base.
(3) If T is o-minimal and A is arbitrary, then A is an extension base.
(4) If $T = \mathrm{ACVF}$, i.e. the theory of algebraically closed valued fields. Then any set A is an extension base.

Lemma 3.40. *If $\phi(x, b)$ k-divides over A and $A \subseteq B$ then there is some $B' \equiv_A B$ such that $\phi(x, b)$ k-divides over B'.*

Proof. Let (b_i) be an A-indiscernible witnessing that $\phi(x, b)$ k-divides over A. Extract a B-indiscernible (b'_i) based on (b_i). Note that $b'_0 \equiv_A b_0$, so let $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ be such that $\sigma(b'_0) = b_0$. Then $(\sigma(b'_i) : i \in \omega)$ shows that $\phi(x, b)$ k-divides over $B' := \sigma(B)$. $\square$

Theorem 3.41. *Let T be simple, and let A be an arbitrary set. Then A is an extension base.*

Proof. Suppose $p \in S_x(A)$ forks over A, i.e. $p(x) \vdash \bigvee_{l<d} \phi_l(x, b)$ such that each of $\phi_l(x, b)$ k-divides over A. Let $\Delta = \{\phi_l(x, y) : l < d\}$. We show by induction on $n \in \omega$ that for any $n \in \omega$ there is a sequence $(\psi_i(x, a_i) : i < n)$ such that:

(1) $\psi_i(x, y) \in \Delta$,
(2) $\psi_i(x, a_i)$ k-divides over $A \cup \{a_j : j < i\}$,
(3) $p(x) \cup \{\psi_i(x, a_i) : i < n\}$ is consistent.

Assume we have found $\{\psi_i(x, a_i) : i < n\}$. There is some $b' \equiv_A b$ such that $\{\psi_i(x, a_i) : i < n\}$ satisfies (1)–(3) with Ab' instead of A (follows by a repeated application of Lemma 3.31). But now one of the formulas $\phi_l(x, b')$, say $\phi_0(x, b')$, has to be consistent with $p(x) \cup \{\psi_i(x, a_i) : i < n\}$. So the sequence

$$\phi_{l'}(x, b'), \psi_0(x, a_0), \ldots, \psi_n(x, a_{n-1})$$

satisfies (1)–(3) for $n + 1$.

But now, by pigeon-hole, there is in fact some $\phi(x, y) \in \Delta$ such that for any $n \in \omega$ we can find some $(a_i : i < n)$ such that $\{\phi(x, a_i) : i < n\}$ is consistent and $\phi(x, a_i)$ k-divides over $a_{<i}$. By compactness (and it's crucial here that we have k-dividing everywhere for a fixed k) we can find a sequence $\left(a_i : i < |T|^+\right)$ such that still $\left\{\phi(x, a_i) : i < |T|^+\right\}$ is consistent and $\phi(x, a_i)$ k-divides over $a_{<i}$ for all $i \in \kappa$. Let $B = \{a_i : i < \kappa\}$ and $q \in S(B)$ be any completion of $\{\phi(x, a_i)\}$. Then q divides over any subset of B of size $|T|$, contradicting simplicity. $\square$

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Theorem 3.42. *Let T be a simple theory.*

(1) *$\phi(x, a)$ forks over B if and only if $\phi(x, a)$ divides over B.*
(2) *Forking is symmetric: $\mathrm{tp}(a/bC)$ does not fork over C if and only if $\mathrm{tp}(b/aC)$ does not fork over C.*

Proof. (1) Assume $\phi(x,a)$ forks over B, i.e. $\phi(x,a) \vdash \bigvee_{i<k} \phi_i(x,a_i)$ for some $\phi(x,a_i)$ dividing over B. By Theorem 3.41 and Proposition 3.38, let $(a_j a_{0,j} \dots a_{k-1,j} : j \in \omega)$ be a Morley sequence in the type $\mathrm{tp}(aa_0 \dots a_{k-1}/A)$. Note that in particular each of $(a_{i,j} : j \in \omega)$ is a Morley sequence in $\mathrm{tp}(a_i/A)$, and so by Kim's lemma $\{\phi_i(x,a_{i,j}) : j \in \omega\}$ is inconsistent for each $i < k$. On the other hand, if $\phi(x,a)$ does not divide over A, then $\{\phi(x,a_j) : j \in \omega\}$ is consistent. But since $\phi(x,a_j) \vdash \bigvee_{i<k} \phi_i(x,a_{i,j})$ for all $j \in \omega$ by indiscernibility, it follows by pigeonhole that there is some $i < k$ and some infinite set $I \subseteq \omega$ such that $\{\phi_i(x,a_{i,j}) : j \in I\}$ is consistent. Again by indiscernibility this implies that $\{\phi_i(x,a_{i,j}) : j \in \omega\}$ is consistent, a contradiction.

(2) Assume that $\mathrm{tp}(a/bC)$ does not fork over C. By (1) it is enough to show $\mathrm{tp}(b/aC)$ does not divide over C. As $\mathrm{tp}(a/bC)$ does not fork over C, by the proof of Lemma 3.38 we can find a sequence $(a_i : i \in \omega)$ indiscernible over bC and such that $a_0 = a$ and $\mathrm{tp}(a_i/a_{<i}bC)$ does not divide over C. Let $p(x,y) = \mathrm{tp}(ab/C)$. Then $\bigcup_{i\in\omega} p(a_i,y)$ is consistent as it is realized by b. But then by Lemma 3.36 $p(a,y)$ does not divide over C. □

Remark 3.43. As with the other properties arising from Shelah's classification, there is a characterization of simplicity in terms of the combinatorics of the definable families of sets. Let us say that a formula $\phi(x,y)$ has the *tree property*, or TP, if there is a tree of parameters $(a_\eta : \eta \in \omega^{<\omega})$ and $k \in \omega$ such that:

(1) $\{\phi(x,a_{\eta i}) : i \in \omega\}$ is k-inconsistent for every $\eta \in \omega^{<\omega}$,
(2) $\{\phi(x,a_{\eta\restriction i}) : i \in \omega\}$ is consistent for every$\eta \in \omega^{\omega}$.

Then T is simple if and only if no formula has the tree property.

There are two extreme opposite ways in which a tree property can occur. Namely, the requirements (1) and (2) above do not specify what happens in general for a_η, a_ν with η and ν that are incomparable but are not siblings. Then the *tree property of the first kind*, or TP_1, essentially requires that all such pairs are inconsistent, while the tree property of the second kind, or TP_2, requires that all such pairs are consistent. This gives rise to two classes of theories NTP_1 (no formula has TP_1) and NTP_2 (no formula has TP_2), and a result of Shelah shows that $\mathrm{NTP}_1 \cap \mathrm{NTP}_2$ is exactly the class of simple theories.

Theory of forking over extension bases can be developed in a larger class of NTP_2 theories which contains both simple and NIP theories. See [CK12, BYC14, Che14] + my notes on forking in NTP_2 theories[4]. The case of NTP_1 theories is less understood so far (see e.g. [CR15]).

3.6. **Properties of forking in stable theories.**

Theorem 3.44. *Let T be stable, M a small model, $p \in S(M)$ and $A \supseteq M$ given. Then p has a unique extension $q \in S(A)$ with the following equivalent properties:*

(1) *q does not divide over M,*
(2) *q does not fork over M,*
(3) *q is definable over M,*
(4) *q is an heir of p,*
(5) *q is a coheir of p.*

Proof. As every stable theory is simple, forking equals dividing over arbitrary sets by Theorem 3.42.

[4]http://www.math.ucla.edu/~chernikov/teaching/ForkingLyon2012/ForkingLectures.pdf

If q does not fork over M, then it extends to a global type non-forking over M, which by Proposition 3.32 is definable over M, and so q itself is definable over M. Since p is definable over M, q is the unique heir of p by 3.23.

For the equivalence with (5), combine the equivalence of (2) and (4) with the symmetry of forking given by Theorem 3.42 (recalling that $\mathrm{tp}(a/Mb)$ is a coheir of $\mathrm{tp}(a/M)$ if and only if $\mathrm{tp}(b/Ma)$ is an heir of $\mathrm{tp}(b/M)$). $\square$

***

Now we would like to understand the situation over an arbitrary base set rather than over a model.

Definition 3.45. Let p be a global type. By a *canonical base* of p we mean a set of parameters A such that for any $f \in \mathrm{Aut}(\mathbb{M})$, $f(p) = p \iff f(A) = A$ (setwise).

Note that if A and B are canonical bases for p, then by 1.15 we have $\mathrm{dcl}(A) = \mathrm{dcl}(B)$. Thus if p has a canonical base, then there is a unique *definably closed* one, and we will denote it by $\mathrm{Cb}(p)$.

Proposition 3.46. *Assume that T has elimination of imaginaries, and let $p(x)$ be a definable global type. Then p has a canonical base, and in fact $\mathrm{Cb}(p)$ is the smallest definably closed set over which p is definable.*

Proof. As p is definable, for each formula $\phi(x,y) \in L$ we have a definition $d\phi(y) \in L(\mathbb{M})$. Let c_ϕ in $\mathbb{M}$ be a code for the definable set $d\phi(\mathbb{M})$, exists by the elimination of imaginaries. Let $A = \{c_\phi : \phi(x,y) \in L\}$, then for any $f \in \mathrm{Aut}(\mathbb{M})$ we have $f(p) = p \iff f(A) = A$ (pointwise). The second claim follows by Proposition 3.24(1) and the fact that a code a of a B-definable set is in $\mathrm{dcl}(B)$, for any a, B. $\square$

Remark 3.47. One can make sense of the notion of a canonical base in an arbitrary simple theory, however working with imaginaries is not enough and one has to work with *hyperimaginaries* — these are the classes of type-definable equivalence relations. See [Cas11b] for the details.

Definition 3.48. Let $p \in S(B)$ be a definable type, say defined by a definition schema $(d\phi(y) : \phi(x,y) \in L)$ with $d\phi(y) \in L(B)$. We say that this definition is *good* if it actually is a definition for some global type (or, equivalently, it is a definition for some type over some model M containing B).

Exercise 3.49. Find a theory T and a type p over $\emptyset$ such that p is definable, but no definition of p defines a global type.

Proposition 3.50. *Let T be stable. A type $p(x) \in S(B)$ does not fork over $A \subseteq B$ if and only if p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(A)$.*

Proof. If p does not fork over A, then it has a global extension $p' \in S(\mathbb{M})$ non-forking over A. Let M be an arbitrary model containing A. Then p' in particular does not fork over M, and by 3.44 p' is definable over M. By Proposition 3.46, $\mathrm{Cb}(p') \subseteq M^{\mathrm{eq}}$. By 1.17 it follows that $\mathrm{Cb}(p) \subseteq \bigcap_{M \supseteq A} M^{\mathrm{eq}} = \mathrm{acl}^{\mathrm{eq}}(A)$, which implies that p is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable.

Conversely, if p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(A)$, then some global extension p' of p is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$, which implies by Proposition 3.32 that p' does not fork/divide over $\mathrm{acl}^{\mathrm{eq}}(A)$, and therefore does not fork over A. $\square$

Lemma 3.51. *Let T be stable, and let $p(x)$ and $q(y)$ be global types. Then for any formula $\phi(x,y) \in L(\mathbb{M})$ we have $d_p\phi(y) \in q(y) \iff d_q\phi(x) \in p(x)$ for any definitions such that $d_p\phi(b) \iff \phi(x,b) \in p(x)$ and $d_q\phi(a) \iff \phi(a,y) \in q(y)$ for all a, b in $\mathbb{M}$.*

Proof. Let A be a small set such that $p, q, \phi(x,y)$ are all definable over A. We define a sequence $(a_i, b_i : i \in \omega)$ recursively. Given $(a_i, b_i : i < n)$, let $b_n \models q|_{Aa_0\ldots a_{n-1}}$ and let $a_n \models p|_{Ab_0\ldots b_n}$. Then for $i < j$ we have

$$\models \phi(a_i, b_j) \iff \phi(a_i, y) \in q \iff \models d_q\phi(a_i) \iff d_q\phi(x) \in p,$$

and for $j \leq i$ we have

$$\models \phi(a_i, b_j) \iff \phi(x, b_j) \in p \iff \models d_p\phi(b_j) \iff d_p\phi(y) \in q.$$

Since $\phi(x,y)$ does not have the order property, the claim follows. □

Definition 3.52. A type p is *stationary* if it has a unique global non-forking extension.

Corollary 3.53. *Let T be stable. Then any type over $A = \operatorname{acl}^{\mathrm{eq}}(A)$ is stationary.*

Proof. Let $A = \operatorname{acl}^{\mathrm{eq}}(A)$, and let p', p'' be two global non-forking extensions of $p \in S(A)$. Let $\phi(x,b) \in L(\mathbb{M})$ be an arbitrary formula, and let q be a global non-forking extension of $\operatorname{tp}(b/A)$. By Proposition 3.50, p', p'' and q are definable over A. Applying Lemma 3.51 we have:

$$\phi(x,b) \in p' \iff \models d_{p'}\phi(b) \iff d_{p'}\phi(y) \in q \iff d_q\phi(x) \in p' \iff d_q\phi(x) \in p \iff$$

$$d_q\phi(x) \in p'' \iff d_{p''}\phi(y) \in q \iff \models d_{p''}\phi(b) \iff \phi(x,b) \in p''.$$

□

Corollary 3.54. *In a stable theory, all types over models are stationary.*

For the remainder of this section, let us assume that T eliminates imaginaries, that is we are working in T^{eq}.

Definition 3.55. In an arbitrary theory, we define a ternary notion of independence $\mathop{\smile\!\!\!\!\!\!|}$ on small subsets of the monster:
$a \mathop{\smile\!\!\!\!\!\!|}_C b \iff \operatorname{tp}(a/bC)$ does not fork over C.

Proposition 3.56. *Properties of $\mathop{\smile\!\!\!\!\!\!|}$ in arbitrary theories ("non-commutative forking calculus"):*

1. *Invariance under automorphisms: $a \mathop{\smile\!\!\!\!\!\!|}_C b$ if and only if $\sigma(a) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(b)$, for any $\sigma \in \operatorname{Aut}(\mathbb{M})$.*
2. *Finite character: $a \not\mathop{\smile\!\!\!\!\!\!|}_C b$ implies that $a' \not\mathop{\smile\!\!\!\!\!\!|}_C b'$ for some finite $a' \subseteq a, b' \subseteq b$.*
3. *Monotonicity: $aa' \mathop{\smile\!\!\!\!\!\!|}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!|}_C b$.*
4. *Base monotonicity: $a \mathop{\smile\!\!\!\!\!\!|}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!|}_{Cb'} b$.*
5. *Left transitivity: $a \mathop{\smile\!\!\!\!\!\!|}_C b$ and $a' \mathop{\smile\!\!\!\!\!\!|}_{aC} b$ implies $aa' \mathop{\smile\!\!\!\!\!\!|}_C b$.*
6. *Right extension: if $a \mathop{\smile\!\!\!\!\!\!|}_C b$, then for any d there is $d' \equiv_{bC} d$ such that $a \mathop{\smile\!\!\!\!\!\!|}_C bd'$.*

Proof. (1), (2), (3) are clear from the definition of forking.

(6) As non-forking types can be extended, there is some $|Cb|^+$-saturated model $M \supseteq Cb$ such that $a \mathop{\smile\!\!\!\!\!\mid}_C M$. But then for any d we can realize $\mathrm{tp}(d/bC)$ by some $d' \in M$ by saturation, and $a \mathop{\smile\!\!\!\!\!\mid}_C bd'$ by (3).

(5) First we know that it holds for $\mathop{\smile\!\!\!\!\!\mid}^d$, non-dividing, by Corollary 3.30. Now for forking. Let $M_1 \supseteq Cb$ be saturated enough with $a \mathop{\smile\!\!\!\!\!\mid}_C M_1$, and let $M_2 \equiv_{abC} M_1$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_{aC} M_2$ — exists by (6). It then follows that $a \mathop{\smile\!\!\!\!\!\mid}_C M_2$ by invariance, and together with $a' \mathop{\smile\!\!\!\!\!\mid}_{aC} M_2$ it implies $aa' \mathop{\smile\!\!\!\!\!\mid}^d_C M_2$. As M_2 is saturated enough it implies that $aa' \mathop{\smile\!\!\!\!\!\mid}_C M_2$ hence $aa' \mathop{\smile\!\!\!\!\!\mid}_C b$ by (3). $\square$

We will call any relation $\mathop{\smile\!\!\!\!\!\mid}'$ satisfying these axioms a *preindependence relation.*

Proposition 3.57. *Let T be simple. Then in addition $\mathop{\smile\!\!\!\!\!\mid}$ satisfies*

1. *Local character: For any a and B, there is some $C \subseteq B$ with $|C| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\mid}_C B$.*
2. *Symmetry: $a \mathop{\smile\!\!\!\!\!\mid}_C b \iff b \mathop{\smile\!\!\!\!\!\mid}_C a$.*

This is proved in Corollary 3.42. In fact, each of these properties of forking characterizes simplicity of the theory (see [Kim01]). Note that by symmetry we have right transitivity and left extension (i.e. Proposition 3.56(5) and (6) with the roles of the left and right sides interchanged). A symmetric preindependence relation is called an *independence relation.*

Finally, we specialize to the stable case.

Lemma 3.58. *Let $p', p'' \in S(\mathrm{acl}(A))$ be two extensions of $p \in S(A)$. Then p' and p'' are conjugate over A.*

Theorem 3.59. *Assume that T is stable. Then in addition forking satisfies the following.*

1. *(Conjugacy) Let A be a small set of parameters. Then all global non-forking extensions of $p \in S(A)$ are conjugate over A.*
2. *(Boundedness) Any $p \in S(A)$ has at most $2^{|T|}$ global non-forking extensions.*

Proof. (1) Let q_1, q_2 be two non-forking extensions of $p$to $\mathbb{M}$. Note that $\mathrm{acl}(A)$ is preserved by A-automorphisms, and that by homogeneity of $\mathbb{M}$ there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma|_{\mathrm{acl}(A)}$ takes $q_1|_{\mathrm{acl}(A)}$ to $q_2|_{\mathrm{acl}(A)}$. But since $q_i|_{\mathrm{acl}(A)}$ is stationary for $i = 1, 2$ by Corollary 3.53, and an A-conjugate of a type non-forking over A is still non-forking over A, it follows that $\sigma(q_1) = q_2$.

(2) Let $A_0 \subseteq A$ with $|A_0| \leq |T|$ be such that p does not fork over A_0 (exists as T is simple in particular). Then by (1) p has at most as many non-forking extensions as $p|_{A_0}$ has extensions to $\mathrm{acl}(A_0)$, of which there are $\leq 2^{|\mathrm{acl}(A_0)|} \leq 2^{|T|}$. $\square$

Corollary 3.60. *Let T be stable and $p \in S(A)$ be given. Then p is stationary if and only if it has a good definition over A.*

Proof. Let p be stationary, and let q be the global non-forking extension of p. So q is definable (apriori over some set larger than A). Since p is stationary and all of its non-forking extensions are conjugate over A by Theorem 3.59, it follows that q is actually A-invariant, and so definable over A by Lemma 3.24. It's definition gives a good definition for p.

Conversely, assume that p has a good definition over A. Then p has a global extension p' definable over A, so in particular p' is a non-forking extension. But again, since all global non-forking extensions of p are conjugates of p', and p' is fixed by automorphism over A, it is the unique non-forking extension of p. $\square$

Definition 3.61. Let $p \in S(A)$ be a stationary type.

(1) We define the canonical base of p, $\mathrm{Cb}(p)$, as $\mathrm{Cb}(q)$ for q the unique global non-forking extension of p.
(2) We say that p is *based* on B if p is *parallel* to some stationary type q defined over B, i.e. if p and q have the same global non-forking extension.

Lemma 3.62. *A stationary type $p \in S(A)$ is based on B if and only if $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(B)$. So p does not fork over $B \subseteq A$ if and only if $\mathrm{Cb}(p) \subseteq \mathrm{acl}(B)$.*

Proof. Let r be the global non-forking extension of p and let $q = r|_B$. Assume that p is based on B. Then q is stationary and r is the unique non-forking extension of q. By Corollary 3.60 q has a good definition over B, which also defines r. So $r$is definable over B, which implies $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(B)$.

Conversely, if r is definable over B, we know that r does not fork over B, and that q is stationary by Corollary 3.60.

For the last statement observe that p does not fork over B if and only if p is based on $\mathrm{acl}(B)$. $\square$

For $A \subseteq B$, let $N(B/A) \subseteq S(B)$ be the set of all types over B that do not fork over A. Note that $N(B/A)$ is a closed subset of $S(B)$ (as for every type that forks, there is some formula that forks). Let $\pi : S(B) \to S(A)$ be the restriction map $p \mapsto p|_A$, obviously π is continuous, and so closed (as a continuous map from a compact space to a Hausdorff space).

Theorem 3.63. *(Open mapping theorem) The restriction map $\pi : N(B/A) \to S(A)$ is open.*

Proof. First note that we may replace B by $\mathbb{M}$. Now if $\pi(q) = \pi(q')$ for some $q, q' \in N(\mathbb{M}/A)$, then q and q' are A-conjugate. So if O is an open subset of $N(\mathbb{M}/A)$ then $O' = \pi^{-1}(\pi(O)) = \bigcup \{\alpha(O) : \alpha \in \mathrm{Aut}(\mathbb{M}/A)\}$ is also open. Thus $S(A) \setminus \pi(O) = \pi(N(\mathbb{M}/A) \setminus O')$ is closed as it is the image of a closed set. $\square$

Exercise 3.64. (Herzog-Rothmaler) Show that a theory T is stable if and only if for any $M \models T$, the map $\pi : S(\mathbb{M}) \to M, p \mapsto p|_M$ admits a continuous section.

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Finally for this section, we demonstrate that in fact the existence of an independence relation $\mathop{\smile\!\!\!\!\!\!|}\,'$ on the monster model of T satisfying some natural axioms implies that T is stable, and that the relation $\mathop{\smile\!\!\!\!\!\!|}\,'$ is precisely the non-forking independence $\mathop{\smile\!\!\!\!\!\!|}$.

Theorem 3.65. *Let T be a complete theory and $n > 0$. Then T is stable if and only if there is a special class of extensions of n-types, denoted by $p \sqsubset q$, with the following properties.*

(1) *(Invariance) $\sqsubset$ is invariant under $\mathrm{Aut}(\mathbb{M})$.*
(2) *(Local character) There is a small cardinal κ such that for any $q \in S_n(C)$ there is some $C_0 \subseteq C$ of size at most κ and such that $q|_{C_0} \sqsubset q$.*

(3) *(Weak boundedness) For all A and $p \in S_n(A)$ there is a small cardinal μ such that p has at most μ special extensions $p \sqsubset q \in S_n(\mathbb{M})$.*

Moreover, if $\sqsubset$ satisfies in addition the following properties, then $\sqsubset$ coincides with non-forking.

(4) *(Existence) For all $p \in S_n(A)$ and $A \subseteq B$ there is some $q \in S_n(B)$ such that $p \sqsubset q$.*
(5) *(Transitivity) $p \sqsubset q \sqsubset r$ implies $p \sqsubset r$.*
(6) *(Weak monotonicity) If $p \sqsubset r$ and $p \subseteq q \subseteq r$ then $p \sqsubset q$.*

Proof. If T is stable, then non-forking extensions satisfy all of the listed properties by Proposition 3.56, Proposition 3.57 and Theorem 3.59.

Assume, conversely, that (1), (2) and (3) hold. Note that there are at most $2^{\kappa+|T|}$-types over $\emptyset$ in κ-many variables. For each such $r \in S_\kappa(\emptyset)$, let $A_r \models r$. Note that $|S_n(A_r)| \leq 2^\kappa$. Now, by (3) for each type $p \in S_n(A_r)$, there is some $\mu_{r,p}$ such that p has at most $\mu_{r,p}$ global $\sqsubset$-special extensions. Let $\mu' = \{\mu_{r,p} : r \in S_\kappa(\emptyset), p \in S_n(A_r)\}$ — still a small cardinal. By (1) it follows that for any set A of size $\leq \kappa$, for any $p \in S_n(A)$ there are at most μ' global special extensions of p.

Let A be an arbitrary set of parameters. As by (2) every type over A is a special extension of it's restriction to some subset of A of size $\leq \kappa$, it follows that the number of n-types over A is bounded by the (number of subsets A_0 of A of size at most κ)$\times$(the number of types p over A_0)$\times$(the number of special extensions of p to A). So we have

$$|S_n(A)| \leq |A|^\kappa \times 2^{\max\{\kappa,|T|\}} \times \mu',$$

which implies that $|S_n(A)| = |A|$ for all A of size λ with $\lambda = \lambda^\kappa$ and $\lambda \geq \max\{2^{|T|}, \mu'\}$. This implies stability as $|S_1(A)| \leq |S_n(A)|$ and it is enough to have such a bound for 1-types by 2.24

Assume now that (1)–(6) hold, and let $p \in S_n(A)$ and $q \in S_n(B)$ with $p \subseteq q$ be given.

Assume first that $p \sqsubset q$. Let μ be the cardinal given by (3) applied to p.

Claim. For any $r \in S_n(\mathbb{M})$, if r forks over A then r has more than μ conjugates over A.

Proof of the claim. If r forks over A, then some formula $\phi(x,b) \in r$ divides over A. Then there is some $k \in \omega$ such that by compactness for any small cardinal λ we can find an A-indiscernible sequence $(b_i : i \in \lambda)$ such that $\{\phi(x,b_i) : i \in \lambda\}$ is k-inconsistent. Note that by homogeneity of $\mathbb{M}$ we have that $\phi(x,b_i)$ belongs to some conjugate of r. If there were less than λ A-conjugates of r, it would belong to the same conjugate of r for infinitely many i's, contradicting k-inconsistency.

Now by (4) q has an extension $r \in S_n(\mathbb{M})$ with $q \sqsubset r$, and so by (5) we have $p \sqsubset r$. By (1) we have that $p \sqsubset r'$ holds for any A-conjugate r' of r. So r has no more than μ conjugates over A, which implies that r does not fork over A by the claim, and so in particular q is a non-forking extension of p.

Now assume that q is a non-forking extension of p. Let $r \in S_n(\mathbb{M})$ be a non-forking extension of q, and let $r' \in S_n(\mathbb{M})$ be such that $p \sqsubset r'$, exists by (4). By the above r' is a non-forking extension of p. Besides, we already know that T is stable. Then by Theorem 3.59 r and r' are conjugate over A. This implies that $p \sqsubset r$ by (1), and so $p \sqsubset q$ by (6). □

Remark 3.66. In a simple unstable theory, one can always find a type with unboundedly many global non-forking extensions. However, a version of Theorem 3.65 can be given to characterize simple theories and forking in simple theories. The key point is that boundedness (or stationarity) has to be replaced by an amalgamation statement (the so-called "independence" theorem):

Assume that we are given a type $p \in S(M)$, two sets of parameters $A, B \supseteq M$ with $A \downarrow_M B$ and $p_1 \in S(A), p_2 \in S(B)$ two non-forking extensions of p. Then there is some $q \in S(\mathbb{M})$ a global non-forking extension of p and such that $q_i \subseteq q$ for $i = 1, 2$.

3.7. **Forking and ranks in stable theories.** We have already considered the so-called "Shelah's 2-rank" in Section 2.3. The aim of this section is to demonstrate that in a stable (or simple) theory, if q is an extension of p that forks, then this is characterized by a drop of a certain "rank" or "dimension". This can be described via certain local and global ranks coming from the Cantor-Bendixson rank on the associated spaces of types. Recall:

Definition 3.67. Let X be a compact Hausdorff topological space.

(1) For a point $p \in X$, its *Cantor-Bendixson rank* $\mathrm{CB}(p)$ is defined by induction on an ordinal α:
   (a) $\mathrm{CB}(p) \geq 0$ for all $p \in X$,
   (b) $\mathrm{CB}(p) = \alpha$ iff p is isolated in the subspace $\{q \in X : \mathrm{CB}(q) \geq \alpha\}$.
(2) If $\mathrm{CB}(p) < \infty$ for all $p \in X$, then $\{\mathrm{CB}(p) : p \in X\}$ has the greatest element, say α, and the set $\{p \in X : \mathrm{CB}(p) = \alpha\}$ is finite, say of cardinality n (by compactness of X). We then say that α is the CB-rank if X, or $\mathrm{CB}(X) = \alpha$, and that n is the CB-multiplicity of X,$\mathrm{CB}-\mathrm{mult}(X) = n$.

First we consider *local ranks*. Let Δ be a **finite** collection of formulas of the form $\phi(x, y)$, x fixed and y may vary. We consider $S_\Delta(\mathbb{M})$, the space of global Δ-types.

Proposition 3.68. *Let all formulas in Δ be stable. Then* $\mathrm{CB}(S_\Delta(\mathbb{M})) < \omega$.

Proof. Similarly to the proof of Proposition 2.17: if the rank is infinite, we can construct a binary tree of formulas (witnessing the the type is not isolated) such that at each level the formulas corresponding to the two children split their parent into two disjoin parts; using that Δ is finite, we may assume that it is the same formula at each level, which produces too many types for it to be stable. See [Pil96, Lemma 3.1] for the details. □

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Remark 3.69. It also follows that if Y is an arbitrary compact subspace of $S_\Delta(\mathbb{M})$, then $\mathrm{CB}(Y)$ is finite as well.

Definition 3.70. Let $\Phi(x)$ be a set of formulas over a small set of parameters. By the Δ-rank of $\Phi(x)$, or $R_\Delta(\Phi(x))$, we mean the CB-rank of the subspace $Y = \{q \in S_\Delta(\mathbb{M}) : q(x) \cup \Phi(x) \text{ is consistent}\}$. And by the Δ-multiplicity of $\Phi(x)$, or $\mathrm{mult}_\Delta(\Phi)$, we mean the CB-multiplicity of Y.

The following list of properties are more or less immediate from the basic properties of the CB-rank.

Lemma 3.71. (1) *If $\Psi(x) \vdash \Phi(x)$ then $R_\Delta(\Psi) \leq R_\Delta(\Phi)$.*
(2) $R_\Delta(\Phi(x)) = \min\{R_\Delta(\Phi'(x)) : \Phi' \subseteq \Phi$ *finite*$\}$.

(3) $R_\Delta(\phi(x) \vee \psi(x)) = \max\{R_\Delta(\phi(x)), R_\Delta(\psi(x))\}$.
(4) *If $\Phi(x)$ is a set of formulas over a small set A, then there is some $p \in S_\Delta(A)$ such that $R_\Delta(\Phi(x) \cup p(x)) = R_\Delta(\Phi(x))$.*
(5) *If $\Phi(x)$ is a set of Δ-formulas, then $R_\Delta(\Phi(x)) = \max\{\mathrm{CB}(p) : p \in S_\Delta(\mathbb{M}), \Phi(x) \subseteq p\}$.*
(6) *R_Δ is $\mathrm{Aut}(\mathbb{M})$-invariant.*
(7) *If $\phi(x)$ is a Δ-formula, then $R_\Delta(\Phi) \geq n+1$ if there is an infinite set $\{\phi_i(x) : i < \omega\}$ of pairwise contradictory Δ-formulas each implying $\phi(x)$ and such that $R_\Delta(\phi_i) \geq n$ for all i.*
(8) *Let $\phi(x)$ be a Δ-formula with $R_\Delta(\phi) = n$. Then $mult_\Delta(\phi)$ is the maximal $k \in \omega$ for which there are pairwise contradictory Δ- formulas $\phi_1, \ldots, \phi_k$ with $\phi_i \vdash \phi$ and $R_\Delta(\phi_i) = n$ for all $i = 1, \ldots, k$.*

Proposition 3.72. *Let $A \subseteq B$, $q(x) \in S_\Delta(B)$ and $p(x) = q|_A \in S_\Delta(A)$. Then q does not fork over A if and only if $R_\Delta(q) = R_\Delta(p)$.*

Proof. Suppose q does not fork over A. Let $\phi(x) \in q$ by such that $R_\Delta(\phi) = R_\Delta(q)$, exists by Lemma 3.71(2). By inspecting carefully the proof of the open mapping theorem, there is some positive Boolean combination $\psi(x)$ of A-conjugates of $\phi(x)$ such that $\psi(x) \in p$. By Lemma 3.71 (1) and (3) we have $R_\Delta(\psi(x)) \leq R_\Delta(\phi(x))$. Thus $R_\Delta(p) \leq R_\Delta(q)$ by Lemma 3.71(2), and we get the equality by (1).

Conversely, suppose that q forks over A. Let $q' \in S_\Delta(\mathbb{M})$ be a non-forking extension of q (i.e. q' does not fork over B). By the first part of the proof $R_\Delta(q) = R_\Delta(q')$. Also q' forks over A, thus $q'(x)$ has infinitely many conjugates under A-automorphisms, say $\{q_i : i \in \omega\}$ (by the claim in the proof of Theorem 3.65). Then we can find an infinite set $S \subseteq \omega$ and formulas $\phi_i(x) \in q_i$ for $i \in S$ such that $\{\phi_i : i \in S\}$ are pairwise contradictory. It follows, using Lemma 3.71(6) and (7), that $R_\Delta(q') < R_\Delta(p)$ and thus $R_\Delta(q) < R_\Delta(p)$. □

Corollary 3.73. *Let $A \subseteq B$, $q(x) \in S(B)$, $p(x) = q|_A$. Then q does not fork over A if and only if $R_\Delta(p|_\Delta) = R_\Delta(q|_\Delta)$ for all finite sets $\Delta(x)$ of L-formulas iff $R_\Delta(p) = R_\Delta(q)$ for all finite sets of formulas.*

Remark 3.74. Let $p(x) \in S_\Delta(A)$ be given. Then p is stationary if and only if $\mathrm{mult}_\Delta(p) = 1$.

Proof. By Proposition 3.72 and Lemma 3.71(8). □

Remark 3.75. Note that this characterization of forking via the drop of the local ranks only works in stable theories. A version of it holds in simple theories as well, however in general, e.g. in NIP, an appropriate rank does not seem to exist.

The global analogue of R_Δ is called Morley rank, and historically this was the first CB-type rank introduced in model theory. This time we consider the space $S_x(\mathbb{M})$ of *complete* global types.

Definition 3.76. (1) T is called *totally transcendental*, or t.t., if for every variable x, $\mathrm{CB}(S_x(\mathbb{M})) < \infty$ (i.e. bounded by some ordinal).
(2) Let $\Phi(x)$ be a set of formulas over a small set. Let $Y = \{p \in S_x(\mathbb{M}) : \Phi \subseteq p\}$, a closed subspace of $S_x(\mathbb{M})$. By the *Morley rank* of Φ, $\mathrm{RM}(\Phi)$, we mean $\mathrm{CB}(Y)$. If $\mathrm{CB}(Y) < \infty$, then the Morley degree of Φ, $\mathrm{dM}(\Phi)$, is defined to be the $\mathrm{CB}-\mathrm{mult}(Y)$.

So, essentially, $\mathrm{RM}(-)$ is $R_\Delta(-)$ where Δ is the set of *all* L-formulas. We have an analogue of Lemma 3.71.

Lemma 3.77.
(1) $\mathrm{RM}(\phi(x)) \geq \alpha+1$ *if and only if there is an infinite set* $\{\phi_i(x) : i \in \omega\}$ *of pairwise contradictory formulas such that* $\phi_i(x) \vdash \phi(x)$ *and* $\mathrm{RM}(\phi_i) \geq \alpha$ *for all* $i \in \omega$.
(2) $\mathrm{RM}(\Phi(x)) = \max\{\mathrm{CB}(p) : p \in S_x(\mathbb{M}), \Phi \subseteq p\}$.
(3) $\mathrm{RM}(\Phi(x)) = \min\{\mathrm{RM}(\Phi'(x)) : \Phi' \subseteq \Phi$ *finite* $\}$. *If* $\mathrm{RM}(\Phi) < \infty$, *then* $dM(\Phi)$ *is the minimum of* $dM(\phi)$ *where* ϕ *is a finite conjunction of formulas in* Φ *and* $\mathrm{RM}(\phi) = \mathrm{RM}(\Phi)$.
(4) $\mathrm{RM}(\phi(x)) = 0$ *iff* $\phi(x)$ *is algebraic.*
(5) $\mathrm{RM}(\phi(x) \vee \psi(x)) = \max\{\mathrm{RM}(\phi), \mathrm{RM}(\psi)\}$.
(6) *For any set* $\Phi(x)$ *of formulas over* A, *there is a complete type* $p \in S_x(A)$ *such that* $\mathrm{RM}(\Phi) = \mathrm{RM}(p)$.
(7) *If* $\mathrm{RM}(\phi) = \alpha$ *then* $dM(\phi)$ *is the greatest* $k \in \omega$ *such that there are pairwise contradictory* $\phi_1, \ldots, \phi_k$ *with* $\phi_i(x) \vdash \phi(x)$ *and* $\mathrm{RM}(\phi_i) \geq \alpha$.
(8) *Suppose* $p \in S_x(A), q \in S_x(B), A \subseteq B$ *and* $p(x) \subseteq q(x)$. *Suppose that at least one of* $\mathrm{RM}(p), \mathrm{RM}(q)$ *is* $< \infty$. *Then* q *does not fork over* A *if and only if* $\mathrm{RM}(p) = \mathrm{RM}(q)$.

Exercise 3.78. Prove this lemma.

***

Finally, we discuss one more global rank.

Definition 3.79. Let T be a stable theory. For a type p, we define $\mathrm{SU}(p) \geq \alpha$ by recursion on α:

(1) $\mathrm{SU}(p) \geq 0$ for all types p,
(2) $\mathrm{SU}(p) \geq \beta+1$ if p has a forking extension q with $\mathrm{SU}(q) \geq \beta$,
(3) $\mathrm{SU}(p) \geq \lambda$ for a limit ordinal λ if $\mathrm{SU}(p) \geq \beta$ for all $\beta < \lambda$,

and the SU-rank $\mathrm{SU}(p)$ is the maximal α such that $\mathrm{SU}(p) \geq \alpha$. If there is no maximum, we set $\mathrm{SU}(p) = \infty$.

Exercise 3.80. ("Diamond lemma") Let T be simple and $p \in S_x(A)$. Let q be a non-forking extension of p and r any extension of p (all types over small sets). Then there is an A-conjugate r' of r and a non-forking extension s of r' such that s also extends q.

Lemma 3.81. *Let* T *be simple. Let* p *have ordinal-valued* SU*-rank and let* q *be an extension of* p. *Then* q *is a non-forking extension of* p *if and only if* q *has the same* SU*-rank as* p. *If* p *has* SU*-rank* ∞, *then so does any non-forking extension of it.*

Proof. Clearly the SU-rank of an extension cannot increase. So it is enough to show for all α that $\mathrm{SU}(p) \geq \alpha$ implies $\mathrm{SU}(q) \geq \alpha$ whenever q is a non-forking extension of $p \in S(A)$. The case of a limit α is obvious, so assume $\alpha = \beta + 1$. Then p has a forking extension r with $\mathrm{SU}(r) \geq \beta$.

By the "Diamond lemma" there is an A-conjugate r' of r and s a non-forking extension of r, such that s also extends q. By the inductive assumption $\mathrm{SU}(s) \geq \beta$, but s is a forking extension of q, so $\mathrm{SU}(q) \geq \beta$. □

Remark 3.82. Since every type does not fork over some set of size $\leq |T|$, there are at most $2^{|T|}$ different SU-ranks. Since they form an initial segment of the ordinals, it follows that all ordinal ranks are $\leq \left(2^{|T|}\right)^+$.

Exercise 3.83. Show that actually they are smaller than $|T|^+$.

Definition 3.84. A theory T is supersimple if every type does not fork over some *finite* subset of its domain. A stable, supersimple theory is called *superstable*.

Exercise 3.85. Any totally transcendental theory is superstable.

Lemma 3.86. *T is supersimple if and only if every type has* SU*-rank* $< \infty$.

Proof. If $\mathrm{SU}(p) = \infty$, there is an infinite sequence $p = p_0 \subseteq p_1 \subseteq \dots$ of forking extensions of SU-rank ∞. Then the union $\bigcup_{i \in \omega} p_i$ forks over every finite subset of its domain.

If $p \in S(A)$ forks over every finite subset of A, we can choose an infinite sequence $A_0 \subseteq A_1 \subseteq \dots$ of finite subsets of A such that $p|_{A_{i+1}}$ forks over A_i. This shows that $p|_{\emptyset}$ has SU-rank ∞. $\square$

We give the proof of the stable cases in Fact 2.5.

Theorem 3.87. *Let T be a countable complete theory. Then one of the following cases occurs.*

1. *T is totally transcendental. Then $f_T(\kappa) = \kappa$ for all $\kappa \geq \aleph_0$.*
2. *T is superstable, but not totally transcendental. Then $f_T(\kappa) = 2^{\aleph_0} + \kappa$.*
3. *T is stable, but not superstable. Then $f_T(\kappa) = \kappa^{\aleph_0}$.*

Proof. (1) This follows from the well-known fact that a countable theory T is totally transcendental then it is $\aleph_0$-stable, i.e. there can be only κ many complete types over a set A of size κ. (Idea: assume not, then we can find infinitely many disjoint clopen sets each of which contains more than κ types, then each of those clopens can be split again into disjoint clopens each of which contains $> \kappa$ types, etc - this produces infinite Morley rank. See e.g. [TZ12, Chapter 6 + Theorem 5.2.6] for the details).

(2) Let T be superstable and $|A| = \kappa$. Since every type doesn't fork over a finite subset of A, we have: $|S(A)| \leq$(the number of finite subsets E of A)$\times$(the number of types $p \in S(E)$)$\times$(the number of non-forking extensions of p to A)$\leq \kappa \times 2^{\aleph_0} \times 2^{\aleph_0} = \max\left\{2^{\aleph_0}, \kappa\right\}$ (using Theorem 3.59). If T is not totally transcendental, then by the usual construction of a splitting tree we can produce $2^{\aleph_0}$ complete types over a countable set.

(3) We know that if T is stable, then $|S(A)| \leq |A|^{|T|} = |A|^{\aleph_0}$ by Theorem 2.24. If T is not superstable, it follows from the proof of Lemma 3.86 that there is a type $p \in S(\emptyset)$ with $\mathrm{SU}(p) = \infty$ with a forking extension p' of infinite SU-rank. Let q be a non-forking global extension of p' and let $\kappa \geq \aleph_0$. By the Claim in the proof of Theorem 3.65, q has κ many different A-conjugates $\{q_\alpha : \alpha < \kappa\}$ (since p' does). Let A_0 of size κ be such that all of $p_\alpha := q_\alpha|_{A_0}$ are different. By Lemma 3.81 we have $\mathrm{SU}(p_\alpha) = \infty$. Continuing in this manner we get a sequence $A_0 \subseteq A_1 \subseteq \dots$ of parameter sets and a tree of types $p_{\alpha_0 \dots \alpha_n} \in S(A_{n+1})$, $n < \omega, \alpha_i < \kappa$. We may assume that all A_i have size κ, and each path through this tree defines a type over $A = \bigcup_{n<\omega} A_n$. This implies that $|S(A)| \geq \kappa^{\aleph_0}$. $\square$

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4. Stable groups

We proceed to study stable groups. A group G is *stable* if it is definable in a stable theory, i.e. the underlying set G is a definable subsets of $\mathbb{M}^n$ for some $n \in \omega$ and the group operation $\cdot : G(\mathbb{M}^n) \times G(\mathbb{M}^n) \to G(\mathbb{M}^n)$ is a definable function.

For simplicity of notation we assume that $n = 1, G(\mathbb{M}) = \mathbb{M}$ and that $G, \cdot$ are $\emptyset$-definable.

Recall from Example 2.48:

Example 4.1. The following groups are stable.

(1) Affine algebraic groups over algebraically closed fields (e.g. $\mathrm{GL}_n(\mathbb{C})$, etc.). These are definable in a strongly minimal theory of algebraically closed fields, so they are of finite Morley rank. A long standing open conjecture of Cherlin-Zilber is that every simple group of finite Morley rank is algebraic over an algebraically closed field (we know that every superstable field is algebraically closed).
(2) Algebraic matrix groups over any stable ring, abelian varieties (equipped with their induced structure from the underlying field).
(3) Abelian groups (as pure groups).
(4) Free group on n generators, $n \in \omega$ (as a pure group). More generally, torsion-free hyperbolic groups are stable (both are deep results of Z. Sela).
(5) In general, given an arbitrary stable theory T in a finite relational language, Mekler's construction produces a pure group G (in fact nilpotent, class 2) such that G is stable and interprets T. This shows in a sense that one can't really hope to obtain any strong classification result for general stable groups. However, Mekler's construction doesn't preserve $\aleph_0$-categoricity (i.e. the property of having a unique countable model, up to isomorphism). It is known that every $\aleph_0$-categorical stable group is nilpotent-by-finite (i.e. it contains a normal nilpotent subgroup of finite index), and that every $\aleph_0$-categorical ω-stable group is abelian-by-finite (Felgner'78, Baur, Cherlin, Macintyre'79).
Open problem: Is every $\aleph_0$-categorical stable group abelian-by-finite?
By a result of Lachlan, every $\aleph_0$-categorical superstable theory is ω-stable, but Hrushovski has constructed $\aleph_0$-categorical stable theories that are not superstable.

Large part of the basic results and terminology around stable groups, especially in the finite Morley rank case, is motivated by the study of algebraic groups over algebraically closed fields (see [Poi01]). More recently it was realized that it also follows from the general theory of WAP dynamical systems ([Gla03]).

4.1. **Chain conditions.** By a *uniformly definable family* of subgroups of G we mean a family of subgroups $(H_i : i \in I)$ of G such that for some $\phi(x, y) \in L$ we have $H_i = \phi(\mathbb{M}, a_i)$ for some parameter a_i, for all $i \in I$.

Lemma 4.2. *Let G be an* NSOP *group. For every formula $\phi(x, y)$ there is some $n = n(\phi) \in \omega$ such that every chain $H_1 \subseteq H_2 \subseteq \dots$ of subgroups of G uniformly defined by ϕ has length at most n.*

Proof. Immediate from the definition of NSOP (see Definition 2.39). □

Lemma 4.3. *Let G be an* NIP *group. For every formula $\phi(x, y)$ there is some number $m = m(\phi) \in \omega$ such that if I is finite and $(H_i : i \in I)$ is a uniformly definable family of subgroups of G of the form $H_i = \phi(\mathbb{M}, a_i)$ for some parameters a_i, then $\bigcap_{i \in I} H_i = \bigcap_{i \in I_0} H_i$ for some $I_0 \subseteq I$ with $|I_0| \leq m$.*

Proof. Otherwise for each $m \in \omega$ there are some subgroups $(H_i : i \leq m)$ such that $H_i = \phi(\mathbb{M}, a_i)$ and $\bigcap_{i \leq m} H_i \subsetneq \bigcap_{i \leq m, i \neq j} H_i$ for every $j \leq m$. Let b_j be an element from the set on the right hand side and not in the set on the left hand side. Now, if $I \subseteq \{0, 1, \ldots, m\}$ is arbitrary, define $b_I := \prod_{j \in I} b_j$. It follows that $\models \phi(b_I, a_i) \iff i \notin I$. This implies that $\phi(x, y)$ is not NIP. $\square$

Combining, we get:

Theorem 4.4. *(Baldwin-Saxl) Let G be stable. Then for any formula $\phi(x, y)$ there is some $k = k(\phi) \in \omega$ such that any descending chain of intersections of ϕ-definable subgroups has length at most k.*

Proof. By Lemma 4.3, every element of such a chain is an intersection of at most m_ϕ ϕ-definable subgroups, and so we may assume that the elements of the chain are themselves uniformly definable. But then by Lemma 4.2 such a chain can only have length at most n_ψ where $\psi(x, \bar{y}) = \bigwedge_{i < m_\phi} \phi(x, y_i)$. $\square$

Corollary 4.5. *It follows that if G is stable and $A \subset G$ then there is some finite $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$, where $C_G(A) = \{g \in G : g \cdot a = a \cdot g \text{ for all } a \in A\}$ is the centralizer of A in G.*

Proof. Apply Theorem 4.4 to the formula $\phi(x, y)$ given by $x \cdot y = y \cdot x$. $\square$

Corollary 4.6. *Let G be a stable group, and let $A \subseteq G$ be an abelian subgroup (not necessarily definable). Then there is a* ***definable*** *abelian subgroup $A' \supseteq A$ of G. The same statement is true for nilpotent and solvable subgroups.*

Proof. Let A' be the center of the centralizer of A. It is an abelian subgroup of G, and by Corollary 4.5 it is definable. $\square$

If G is moreover ω-stable, then the same chain condition holds with respect to all families of definable subgroups, not only the uniformly definable ones.

Proposition 4.7. *If G is ω-stable then G has no infinite decreasing chains of definable subgroups.*

Proof. If $H_{i+1} \subsetneq H_i$ then there are at least two disjoint cosets of H_{i+1} in H_i, and so either $\mathrm{RM}(H_{i+1}) < \mathrm{RM}(H_i)$ or $\mathrm{RM}(H_{i+1}) = \mathrm{RM}(H_i)$, in which case the Morley degree of H_{i+1} has to be smaller than the Morley degree of H_i (see Lemma 3.77). As Morley rank is an ordinal and Morley degree is a natural number, this decrease can happen only finitely many times. $\square$

4.2. **Connected components.**

Definition 4.8.
1. Given a definable group G (recall, we are always assuming it is $\emptyset$-definable) and a set of parameters A, we let $S_G(A)$ denote the set of all types in $S(A)$ which concentrate on G, i.e. $S_G(A) = \{p \in S(A) : G(x) \in p\}$.
2. For any set of parameters A, we consider the action of $G(A)$ on $S_G(A)$ given by: $g \cdot p := \mathrm{tp}(g \cdot b/A)$, where b is some/any element realizing p. Exercise: check that this is a well-defined continuous action of $G(A)$ on $S_G(A)$ by homeomorphisms.

We would like to understand better this action in the case of a stable group G. First we discuss several model-theoretic connected components associated to a definable group G.

Definition 4.9. Let G be a stable group.

(1) For $\phi(x,y) \in L$, let $G^0_\phi := \bigcap \{H \leq G : H = \phi(\mathbb{M}, a) \text{ for some } a \text{ and } [G:H] < \infty\}$. By Theorem 4.4 G^0_ϕ is in fact a definable subgroup. Besides, it is clear from the definition that G^0_ϕ is $\mathrm{Aut}(\mathbb{M}/\emptyset)$-invariant, which implies that G^0_ϕ is $\emptyset$-definable.

(2) Let $G^0 := \bigcap_{\phi \in L} G^0_\phi$. This is a normal subgroup of G (as the set of all definable subgroups of finite index is closed under conjugation) of *bounded* index (i.e. the index is small, compared to the saturation of the monster). In fact, $\left[G : G^0\right] < 2^{|T|}$.

(3) Similarly, one can define:

$$G^{00} := \bigcap \{H \leq G : [G:H] \text{ is bounded and } H \text{ is type-definable}\}.$$

(4) In a stable theory, $G^0 = G^{00}$, as any type-definable group is an intersection of definable groups.

Exercise 4.10. Stability is necessary in the last claim. Give an example of a type-definable group which is not an intersection of definable groups.

Remark 4.11. The term "connected component" comes from algebraic geometry. Namely, if G is an algebraic group, then G^0 is precisely the connected component in the sense of Zariski geometry. If G is ω-stable then G^0 is $\emptyset$-definable.

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The quotient G/G^{00} can be equipped with a natural topology.

Definition 4.12. Let E be a type-definable (over $\emptyset$) equivalence relation on $\mathbb{M}$. We say that E is *bounded* if $\mathbb{M}/E$ has small cardinality.

Exercise 4.13. E is bounded if and only for some (every) small model M, $a \equiv_M b$ implies $E(a,b)$.

Now fix a type-definable bounded equivalence relation, and let $\pi : \mathbb{M} \to M/E$ be the quotient map. We define *logic topology* on $\mathbb{M}/E$: a set $S \subseteq \mathbb{M}/E$ is closed if $\pi^{-1}(S)$ is type-definable over some (any) small model M.

Proposition 4.14. *The space $\mathbb{M}/E$ equipped with the logic topology is a compact Hausdorff space.*

Proof. Given a small model M, we have a map $f : S(M) \to \mathbb{M}/E$ given by $p \mapsto a/E$ for some/any $a \models p$. This is well-defined by Exercise 4.13, and f is clearly continuous by the definition of the logic topology. Thus $\mathbb{M}/E$ is compact, as an image of a compact set.

Let now $a, b \in \mathbb{M}$ be arbitrary such that $\neg E(a,b)$. Then for any $x.y \in \mathbb{M}$ we have $E(x,a) \wedge E(y,b) \to \neg E(x,y)$. As E is type-definable, by compactness there is a formula $\phi(x,y)$ such that $E(x,y) \vdash \phi(x,y)$ and $\models \phi(x,a) \wedge \phi(y,b) \to \neg E(x,y)$. Let $N_a := \left\{x \in \mathbb{M}/E : \pi^{-1}(x) \subseteq \phi(\mathbb{M}, a)\right\}, N_b := \left\{y \in \mathbb{M}/E : \pi^{-1}(y) \subseteq \phi(\mathbb{M}, b)\right\}$. Then N_a and N_b are two disjoint open neighborhoods of $\pi(a)$ and $\pi(b)$, respectively. $\square$

Now if G is definable (or just type-definable), the equivalence relation $E(x,y)$ given by $xG^{00} = yG^{00}$ is a bounded equivalence relation type-definable over $\emptyset$ (as explain in 4.9). Thus we can equip G/G^{00} with the logic topology, and in fact the groups operation is compatible with it (exercise).

Proposition 4.15. *The group G/G^{00} equipped with the logic topology is a compact Hausdorff group.*

Example 4.16. (1) If $G = G^{00}$, for example if G is a stable group, then G/G^{00} is a profinite group — it is the inverse image of the groups G/H with H ranging over all definable subgroups of finite index.
(2) Say, consider $G = (\mathbb{Z}, +)$. Then $G^0 = G^{00}$ is the set of elements divisible by all $n \in \mathbb{N}$ (of course, it has no elements in the standard model $\mathbb{Z}$, so needs to be calculated in the monster). Then G/G^{00} is isomorphic as a topological group to $\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z}$.
(3) Consider an unstable example. Let $G = S_1$ be the circle group defined in a saturated real closed field R. Then G^{00} is the set of infinitesimal elements of G, and G/G^{00} is isomorphic to the Lie group of the *standard* circle $S_1(\mathbb{R})$.

Remark 4.17. One can consider an even smaller connected component, G^{000} (also denoted as G^{∞}) given by the intersection of all $\mathrm{Aut}(\mathbb{M}/\emptyset)$-invariant subgroups of G of bounded index. By a result of Shelah and Gismatullin, assuming that T is NIP, the groups G^{00} and G^{∞} are themselves type-definable (invariant) subgroups of G of bounded index. In a general NIP theory, each of the inclusions $G^{\infty} \subseteq G^{00} \subseteq G^0$ can be strict (see [CP12]). In a stable theory we have $G^0 = G^{00} = G^{\infty}$.

4.3. Generics.

Definition 4.18. (1) A definable subset A of G is called *left generic* if G is covered by finitely many left translates of A, i.e. $G = g_1 \cdot A \cup \ldots \cup g_n \cdot A$ for some $g_1, \ldots, g_n \in G$.
(2) Similarly, A is *right generic* (*bi-generic*) if $G = \bigcup_{i<n} A \cdot g_i$ for some $g_i \in G$ (resp., $G = \bigcup_{i<n} g_i \cdot A \cdot h_i$ for some $g_i, h_i \in G$).
(3) A (partial) type $\pi(x)$ is (left-, right-, bi-) generic if it only contains (left-, right-, bi-) generic formulas.

In topological dynamics generic sets are called *syndetic*. We will show that all three notions coincide in stable groups, but **for now by generic we will mean bi-generic**. Note that a (two-sided) translate of a generic set is generic and both left generic sets and right generic sets are bi-generic. Also if p is a generic type, then p^{-1} is generic as well. Also note that any automorphism in $\mathrm{Aut}(\mathbb{M})$ preserves genericity.

Lemma 4.19. *Let G be stable, and let A be a definable subset of G. Then either A is left-generic, or its complement $\neg A$ is right-generic.*

Proof. Suppose not. Then for any $a_1, \ldots, a_n \in G$ we can find some $d \in G$ such that $d \notin \bigcup_{1 \leq i \leq n} (\neg A) \cdot a_i^{-1}$. Hence $da_i \in A$ for $1 \leq i \leq n$. Analogously, there is some e such that $a_i e \in \neg A$ for $1 \leq i \leq n$.

Using this we can choose inductively $(b_i, c_i : i \in \omega)$ in G such that $c_{n+1}b_1, \ldots, c_{n+1}b_n \in A$ and $c_1 b_{n+1}, \ldots, c_n b_{n+1} \in \neg A$. Then $i < j \implies c_i \cdot b_j \notin A$, and $j < i \implies c_i b_j \in A$. This shows that the formula $\phi(x, y) := (x \cdot y \in A)$ has the order property. ☐

Corollary 4.20. *If G is a stable group, then the family of all non-generic sets is an ideal (recall that "generic" refers to "bi-generic" for now).*

Proof. We show that if $A \cup B$ is generic, then either A or B is generic. If $G = \bigcup_{i<n} a_i \cdot (A \cup B) \cdot b_i$, then $G = \left(\bigcup_{i<n} a_i \cdot A \cdot b_i\right) \cup \left(\bigcup_{i<n} a_i \cdot B \cdot b_i\right)$. By Lemma 4.19 either the first or the second union is generic. Hence either A or B is generic. □

Corollary 4.21. *If G is stable then it admits a global generic type.*

Proof. By Corollary 4.20 and compactness. □

Theorem 4.22. *Let G be stable and $p \in S_G(\mathbb{M})$ generic. Then p does not fork over $\emptyset$.*

Proof. We are using the local ranks (see Lemma 3.71). Recall that if q is an extension of p which forks, witnessed by a formula $\phi(x, a)$, then $R_{\phi(x,y)}(p) > R_{\phi(x,y)}(q)$. The formula $\phi(x, a)$ can also be written as $\phi(1 \cdot x \cdot 1, a)$ where 1 is the identity element of the group, so we also have $R_{\phi(u \cdot x \cdot v, y)}(p) > R_{\phi(u \cdot x \cdot v, y)}(q)$ where u, v, y are viewed as the parameter variables.

Now let $p \in S_G(\mathbb{M})$ be generic, and let $q \in S_G(\mathbb{M})$ be a type of maximal $R_{\phi(u \cdot x \cdot v, y)}$-rank (exists by Lemma 3.71). For every definable subset A of G, if $A \in p$ then $a \cdot A \cdot b \in q$ for some $a, b \in G$ by genericity of p, and so $R_{\phi(u \cdot x \cdot v, y)}(A) = R_{\phi(u \cdot x \cdot v, y)}(a \cdot A \cdot b)$. Therefore p has maximal local rank with respect to all of these formulas simultaneously, and thus does not divide over $\emptyset$. □

Lemma 4.23. *For every formula $\phi(x, y)$ there is a number $n = n(\phi) \in \omega$ such that for any parameter c, if $\phi(x, c)$ is generic then $G(\mathbb{M})$ can be covered by n two-sided translates of $\phi(\mathbb{M}, c)$.*

Proof. Let $p \in S_G(\mathbb{M})$ be generic, it exists by Corollary 4.20. Note that p is definable by stability. We have that $\phi(x, c)$ is generic if and only if there are some $a, b \in G(\mathbb{M})$ with $a \cdot \phi(x, c) \cdot b \in p$. As p is definable, we have $\phi(x, c)$ is generic iff $\models \exists u, v\, d_\phi(u, v, c)$, where $d_\phi(u, v, y)$ is the definition for $\phi(u \cdot x \cdot v, y) \in p$. The lemma follows from compactness. □

Thus, for every formula $\phi(x, y)$ there is some formula $\psi_\phi(y)$ such that $\models \psi_\phi(c) \iff \phi(x, c)$ is generic.

Corollary 4.24. *If $p \in S_G(M)$ is generic, $B \supseteq M$ and $q \in S_G(B)$ is a non-forking extension of p then q is generic.*

Proof. By Theorem 3.44, q is an heir of p. Assume it is not generic, then $\phi(x, c) \wedge \neg\psi_\phi(c) \in q$ for some ϕ and c. But as q is an heir of p, $\phi(x, c') \wedge \neg\psi_\phi(c') \in p$ for some $c' \in M$ — which is impossible as p is generic. □

Exercise 4.25. Show that the same statement is true for $p \in S_G(A)$ where A is an arbitrary set (reduce to the case of a model).

Proposition 4.26.
(1) *If a and b are generic and independent over A, then ab is generic. Furthermore, a and ab are independent over $\emptyset$, as well as b and ab.*
(2) *Any $g \in G$ is a product of two generics.*

Proof. If $\operatorname{tp}(a/A)$ is generic and $a \mathop{\smile\!\!\!\!\!\mid}_A b$, then the $\operatorname{tp}(a/Ab)$ is also generic (by Exercise 4.25). In particular, $\operatorname{tp}(a \cdot b/Ab)$ only contains generic formulas (if $a \cdot b \models \phi(x) \in L(Ab)$ then $a \models \phi(x)\, b^{-1} \in L(Ab)$ and genericity is preserved by translation). In particular, $a \cdot b \mathop{\smile\!\!\!\!\!\mid} b$ by Theorem 4.22. Similarly for $a \cdot b \mathop{\smile\!\!\!\!\!\mid} a$.

(2) Assume we are given $g \in G$. Let $h \mathop{\smile\!\!\!\!\!\!\;|}\, g$ be such that $\operatorname{tp}(h/g)$ is generic. Note that $\operatorname{tp}\left(h^{-1}/g\right)$ is also generic, as well as $\operatorname{tp}\left(h^{-1}g/g\right)$ (as in (1)). Then $g = h \cdot \left(h^{-1} \cdot g\right)$ is a product of two generics (which need not be independent in general). □

Corollary 4.27. *A formula $\phi(x)$ is left-generic if and only if it is right-generic if and only if it is bi-generic.*

Proof. By symmetry it is enough to show that a bi-generic formula $\phi(x)$ is left-generic. Let M be a model containing the parameters of $\phi(x)$. By compactness it is enough to show that every type $p \in S_G(M)$ is in some left $G(M)$-translate of $\phi(x)$. Take $p = \operatorname{tp}(b/M)$ arbitrary, and let $\operatorname{tp}(a/Mb)$ be a bi-generic containing $\phi(x)$ (exists by Corollary 4.21). If $c := a \cdot b^{-1}$ then $\operatorname{tp}(c/Mb)$ is also generic, $c \mathop{\smile\!\!\!\!\!\!\;|}_M b$ and $c \models \phi(x \cdot b)$. By stability $\operatorname{tp}(c/bM)$ is finitely satisfiable in M, and so exists some $c' \in M$ with $\phi(c' \cdot b)$, i.e. $(c')^{-1}\phi(\cdot x) \in p$. □

Recall that $G^0 = \bigcap_{\phi \in L} G^0_\phi$ and G^0_ϕ is $\emptyset$-definable.

Definition 4.28. Let $p \in S_G(\mathbb{M})$ and let $\phi(x, y)$ be a formula. We define stabilizers $\operatorname{Stab}_\phi(p) := \{g \in G : \forall y\,(\phi(x, y) \in p \iff \phi(g \cdot x, y) \in p)\} = \{g \in G : gp|_\phi = p|_\phi\}$ and $\operatorname{Stab}_\phi(p) := \bigcap_{\phi \in L} \operatorname{Stab}_\phi(p) = \{g \in G : gp = p\}$. Both are subgroups of G.

By the definability of the type p it follows that $\operatorname{Stab}_\phi(p)$ is a definable group (as $\{g \in G : \forall y\,(d_p\phi(x, y) \leftrightarrow d_p\phi(g \cdot x, y))\}$), and so $\operatorname{Stab}(p)$ is type-definable.

Consider a formula $\phi(x; y, u) = \phi'(u \cdot x, y)$. Then any translate of an instance of ϕ is again an instance of ϕ, and any definable set is defined by an instance of such formula.

Proposition 4.29.
(1) *For any formula ϕ, the set $\{p|_\phi : p$ is generic$\}$ is finite.*
(2) $\operatorname{Stab}_\phi(p) \subseteq G^0_\phi$ *and* $\operatorname{Stab}(p) \subseteq G^0$ *for any* $p \in S_G(\mathbb{M})$.
(3) *If p is generic then* $\operatorname{Stab}_\phi(p)$ *has finite index in G, and* $\operatorname{Stab}(p) = G^0$.

Proof. (1) The type p contains the information about its coset modulo $G^0(\phi(ux, y))$. Let $\psi(x)$ define G^0_ϕ. There is some $b \in G$ with $\psi\left(b^{-1}x\right) \in p$ (as p has to be in some coset of G^0_ϕ). If $g \in \operatorname{Stab}_\phi(p)$ then $\psi\left(b^{-1}gx\right) \in p$. Hence $b^{-1}gb$ and $g \in G^0_\phi$.

(2) As generic types do not fork over $\emptyset$, their number is bounded by 3.59. Hence there are only finitely many generic ϕ-types, as otherwise could produce unboundedly many by compactness.

(3) Follows from (2) as a translate of a generic is generic. □

Theorem 4.30. *("Fundamental theorem of stable group theory")*
(1) *There is a unique generic in every coset of G^0, G/G^0 is a profinite group acting transitively on its generics, p is generic iff* $\operatorname{Stab}(p) = G^0$ *(and generics form the unique minimal flow).*
(2) *The following are equivalent for a definable set $\phi(x)$:*
   (a) *$\phi(x)$ is generic, in the sense of the definition above.*
   (b) *no translate of $\phi(x)$ forks over $\emptyset$.*
   (c) *$\phi(x)$ has positive measure with respect to all G-invariant Keisler measures.*

Proof. Let M be a small model containing representatives of every coset of G^0, possible since there are boundedly many of them.

Note that $G^0(x)$ is a generic partial type (as each G^0_ϕ is of finite index, so finitely many translates cover G). Hence there is some generic type p in G^0 — we call it *principal generic*. By translation, in every coset of G^0 there is a generic. By translation, enough to show that there is only one generic in G^0.

Choose realizations a and b of two generic types concentrated on G^0 and independent over M. Since $\mathrm{Stab} = G^0$, b and ab realize the same type over Ma, and so over M. Similarly, a and ab have the same type over M. Hence a and b have the same type over M. Works for any model M, so determines the complete type.

If p is generic, we already know $\mathrm{Stab}(p) = G^0$. Assume $\mathrm{Stab}(p) = G^0$. Then this is true for every heir of p, by definability. Let a realize p and b realize the principal generic over Ma. Then $a \downarrow_M b$ and $a, b \cdot a$ have the same type over Mb. Furthermore, $b \cdot a$ is generic over Ma. Therefore a is generic over M. □

Remark 4.31. The theory of generics, in the sense of forking, can be generalized to groups definable in simple theories (see e.g. [Wag00]). Similarly, there is a generalization to the case of NIP groups, where the dynamical counterpart becomes more subtle and which is currently an active research area [CS15a].

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