INTRODUCTION TO GEOMETRIC STABILITY

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Lecture notes for the IMS Graduate Summer School in Logic, National University of Singapore, Jun 2017. The material is based on a number of sources including:

- Lecture notes on "Elements of Geometric Stability Theory" by Boris Zilber
- "Model Theory: An Introduction" by David Marker
- "A Course in Model Theory" by Katrin Tent and Martin Ziegler
- "Geometric stability theory" by Anand Pillay
- Lecture notes "Geometric Stability Theory" by Martin Bays

All comments and corrections are very welcome.

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1. Preliminaries

We recall briefly some basic facts and definitions in order to set up the notation for our discussion.

- $\mathcal{M} = (M, R_1, \ldots, f_1, \ldots, c_1, \ldots)$ denotes a first-order structure in a language $\mathcal{L}$.
- T denotes a first-order theory, $\mathcal{M} \models T$ if it satisfies all sentences in T.
- $\mathrm{Th}(\mathcal{M})$ denotes the complete theory of $\mathcal{M}$.
- Given $A \subseteq M$, $\phi(\bar{x})$ is an $\mathcal{L}(A)$-formula (formula over a set of parameters A) if it is of the form $\phi(\bar{x}) = \psi(\bar{x}, \bar{b})$ where $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{b}$ is tuple from A.
- We will abuse notation and write x, y, z, etc. to denote tuples of variables when there is no confusion.
- If $\Psi(x) \subseteq \mathcal{L}(A)$, a set of $\mathcal{L}(A)$ formulas and $a \in M$, we write $a \models \Psi(x) \iff a$ satisfies all formulas in Ψ.
- Given $B \subseteq M^{|x|}$, $\Psi(B) := \{b \in B : \mathcal{M} \models \Psi(a)\}$.
- $X \subseteq M^n$ is an *A-definable set* if $A = \psi(M)$ for some $\psi(x) \in \mathcal{L}(A)$.
- $\mathrm{Def}_n(A)$ is the boolean algebra of all A-definable subsets of M^n.
- Given $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$, write $\mathcal{M} \equiv \mathcal{N}$ to denote elementary equivalence ($\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$).
- Given a (partial) map $f : \mathcal{M} \to \mathcal{N}$, f is *elementary* if for all $a \in \mathrm{Dom}(f)$ and $\phi \in \mathcal{L}$, $\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(f(a))$.
- $\mathcal{M} \preceq \mathcal{N}$ is an *elementary substructure* if the embedding map is elementary.

Fact 1.1. *(Compactness theorem) Let $\mathcal{L}$ be any language and Ψ a set of $\mathcal{L}$-sentences (of any cardinality). If every finite $\Psi_0 \subseteq \Psi$ is consistent (i.e. there is some $\mathcal{L}$-structure $\mathcal{M} \models \Psi_0$), then Ψ is consistent.*

Fact 1.2. *(Löwenheim–Skolem theorem) Let $\mathcal{M} \models T$ be given, with $|\mathcal{M}| \geq \aleph_0$. Then for any cardinal $\kappa \geq |\mathcal{L}|$ there is some $\mathcal{N}$ with $|\mathcal{N}| = \kappa$ and such that:*

- *$\mathcal{M} \preceq \mathcal{N}$ if $\kappa > |\mathcal{M}|$,*
- $\mathcal{N} \preceq \mathcal{M}$ if $\kappa < |\mathcal{M}|$.

- For $A \subseteq M$, a *partial type* $\Phi(x)$ *over* A is *consistent* collection of $\mathcal{L}(A)$-formulas.
- $\Phi(x)$ is a *(complete) type* if $\phi(x) \in \Phi$ or $\neg\phi(x) \in \Phi$ for all $\phi \in \Phi$.
- For a tuple b in $\mathcal{M}$, $\mathrm{tp}(b/A) := \{\phi(x) : b \models \phi(x), \phi(x) \in \mathcal{L}(A)\}$.

Definition 1.3. Let κ be an infinite cardinal.

(1) $\mathcal{M}$ is *κ-saturated* if $\forall A \subseteq M$ with $|A| < \kappa$, every 1-type over A can be realized in $\mathcal{M}$.
(2) $\mathcal{M}$ is *κ-homogenous* if any partial elementary map from $\mathcal{M}$ to itself with a domain of size $< \kappa$ can be extended to an automorphism of $\mathcal{M}$.

Fact 1.4. *For any T and κ, there is a κ-saturated and κ-homogeneous model $\mathcal{M}$ of T.*

Definition 1.5. For $A \subseteq M$, we let $S_n(A)$ denote the space of all complete n-types over A.

- This is the *Stone dual* of $\mathrm{Def}_n(A)$, hence compact (=compactness theorem), Hausdorff space with a basis of clopens given by the sets $\langle\phi(x)\rangle = \{p \in S_n(A) : \phi(x) \in p\}$ for $\phi(x) \in \mathcal{L}(A)$.

- A type $p(x) \in S_n(A)$ is *isolated* if it is an isolated point of the top. space $S_n(A)$, i.e. $\exists \phi(x) \in p$ s.t. $\phi(x) \vdash p(x)$.

Example 1.6. In many natural structures it is possible to describe definable sets and types explicitly using quantifier elimination (i.e., every formula is equivalent to one not involving quantifiers, see Marker's book for the details).

(1) T_∞ — the theory of an infinite set, in the language of equality.
By QE, $\mathrm{Def}_1(M)$ consists of the finite and cofinite subsets.
$S_1(M) = \{\mathrm{tp}(a/M) : a \in M\} \cup \{p^*\}$, where $p^* = \{x \neq a : a \in M\}$ is the type of a "new element".

(2) We consider a vector space $\mathcal{M} = (M, 0, +, -, (r(x) : r \in K))$ where K is a field (finite or infinite), and $r(x)$ is a function $x \mapsto rx$.
Let $\mathrm{VS}_K := \mathrm{Th}(\mathcal{M})$.
It has QE, hence definable sets are given by the Boolean combinations of equations of the form $r_1x_1 + \ldots + r_nx_n = 0$.
In particular, every set in $\mathrm{Def}_1(M)$ is either finite or cofinite.
(Note that the field R is incorporated into the language, and is not a definable as a part of the structure.)

(3) ACF_p - the theory of an algebraically closed field of char p, where p is prime or 0 (e.g. $\mathrm{Th}(\mathbb{C}, +, \times, 0, 1) = \mathrm{ACF}_0$).
By Tarski's QE, definable sets are precisely the Boolean combinations of algebraic sets of the form $\{\bar{x} : p(\bar{x}) = 0\}$, where p is a polynomial.
Again, every set in $\mathrm{Def}_1(M)$ is either finite or cofinite.

(4) DLO=$\mathrm{Th}(\mathbb{Q}, <)$ — the theory of dense linear orders without end points.
By QE, $\mathrm{Def}_1(M)$ consists of finite unions of (bounded or unbounded) intervals (hence can be infinite-coinfinite).
$S_1(M) \leftrightarrow$ Dedekind cuts in $(M, <)$ (For a cut $C = (A, B)$, let $p_C := \{a < x < b : a \in A, b \in B\}$. This set of formulas is consistent by the density of the order, and defines a complete type by QE)

Exercise 1.7. Show:

- If $\mathcal{M}$ is κ-saturated and $A \subseteq M$, $|A| < \kappa$ then every n-type over A is realized in M.
- $(\mathbb{R}, +, \times, 0, 1)$ is not $\aleph_0$-saturated.
- (*) $(\mathbb{C}, +, \times, 0, 1)$ is $2^{\aleph_0}$-saturated (we will see later how it follows from the general theory).

2. Morley's categoricity theorem

In this section we prove Morley's categoricity theorem, using it as an opportunity to introduce some of the fundamental ideas of stability theory.

Fact 2.1. *(Morley, 1965) Let T be a countable complete first-order theory κ-categorical for some uncountable cardinal κ (i.e. admitting a unique model of cardinality κ, up to isomorphism). Then T is κ-categorical for every uncountable cardinal κ.*

Example 2.2. (1) T_∞ is uncountably categorical — indeed, every model of T_∞ is determined up to isomorphism by its size.

(2) Let K be a countable field, then VS_K is uncountably categorical — by the standard linear algebra, any two vector spaces of the same dimension

are isomorphic, and the dimension of any uncountable vector space over a countable field is equal to its size.
(3) Same holds for ACF_p with the transcendence degree over the prime field in the place of dimension.

Exercise 2.3. Show:
(1) Assume K is a countable field. How many countable models does VS_K have?
(2) DLO is not κ-categorical for any uncountable κ(but $\aleph_0$-categorical).
(3) RG, the theory of the countable random graph, has 2^κ models of size κ, $\forall\kappa$ uncountable (but $\aleph_0$-categorical).
(4) (*) In fact, DLO also has 2^κ models of size κ, $\forall\kappa$ uncountable.

2.1. Algebraic closure.

- If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(x)$ is an $\mathcal{L}(M)$-formula, then $\phi(M) := \{a \in M : a \models \phi(a)\}$.

Definition 2.4. Let $\mathcal{M}$ be a structure and A a subset of M.
(1) A formula $\phi(x) \in \mathcal{L}(A)$ is *algebraic* if $\phi(M)$ is finite.
(2) An element $a \in M$ is *algebraic over* A if it satisfies an algebraic $\mathcal{L}(A)$-formula. We say "*a is algebraic*" meaning "a is algebraic over $\emptyset$".
(3) The *algebraic closure of* A is the set $\mathrm{acl}(A) = \{a \in M : a \text{ is algebraic over } A\}$.
(4) A is *algebraically closed* if $\mathrm{acl}(A) = A$.

Remark 2.5. (1) Given $A \subseteq \mathcal{M} \preceq \mathcal{N}$, $\mathrm{acl}(A)$ calculated in $\mathcal{N}$ is equal to $\mathrm{acl}(A)$ calculated in $\mathcal{M}$ (an $\mathcal{L}(A)$-formula defining a finite set in $\mathcal{M}$ defines the same set in $\mathcal{N}$ by elementarity). In particular, elementary substructures are algebraically closed.
(2) $|\mathrm{acl}(A)| \leq \max(A, |T|)$.

Example 2.6. (1) If $\mathcal{M} \models T_\infty$, the $\mathrm{acl}(A) = A$ for any set A.
(2) If $\mathcal{M} \equiv (\mathbb{Z}, s)$, then $\mathrm{acl}(A)$ is the set of points reachable from A (Exercise).
(3) If $\mathcal{M} \models \mathrm{VS}_K$, then a vector $a \in \mathrm{acl}(A)$, if and only if a is in the span of A.
(4) If $\mathcal{M} \models \mathrm{ACF}_p$, then $a \in \mathrm{acl}(A) \iff a$ there is a polynomial with coefficients in the field generated by A such that a is its root).

Definition 2.7. A type $p(x) \in S(A)$ is *algebraic* if p contains an algebraic formula.

Remark 2.8. (1) Any such type is isolated by an algebraic formula $\phi(x) \in \mathcal{L}(A)$, namely any $\phi(x) \in p$ with the minimal number of solutions in $\mathcal{M}$ (Check!).
(2) Using it and compactness, $p \in S(A)$ is algebraic $\iff$ p has only finitely many realizations in any elementary extension of $\mathcal{M}$.

Exercise 2.9. (1) If $A \subseteq M$ and $\mathcal{M}$ is $|A|^+$-saturated, then $p \in S(A)$ is algebraic $\iff$ $p(M)$ is finite.
(2) Let A, B be subsets of $\mathcal{M}$ and $(c_0, \ldots, c_n)$ a sequence of elements non algebraic over A. If $\mathcal{M}$ is $|A \cup B|^+$-saturated, the type $\mathrm{tp}(c_0, \ldots, c_n/A)$ has a realization which is disjoint from B.
(3) Determine acl in DLO and RG.
(4) Given $A \subseteq \mathcal{N}$ such that $\mathcal{N}$ is $|A|^+$-saturated, show that $\mathrm{acl}(A) = \bigcap\{\mathcal{M} \preceq \mathcal{N} : A \subseteq \mathcal{M}\}$.

2.2. **Strongly minimal formulas and theories.** Let T be a complete theory with infinite models.

Definition 2.10. Let $\mathcal{M} \models T$, and $D(x) \in \mathcal{L}(M)$ a non-algebraic formula.

1. The set $D(M)$ is *minimal* in $\mathcal{M}$ if for all $\psi(x) \in \mathcal{L}(M)$, $D(M) \wedge \psi(M)$ is either finite or cofinite in $\phi(M)$.
2. The formula $D(x)$ is *strongly minimal* if $D(x)$ defines a minimal set in any elementary extensions $\mathcal{N} \succcurlyeq \mathcal{M}$.
3. T is *strongly minimal* if the formula "$x = x$" is strongly minimal.

Example 2.11. $T_\infty, \mathrm{VS}_K, \mathrm{ACF}_p$ are strongly minimal (by the discussion in Example 1.6).

Example 2.12. Minimal $\neq$ strongly minimal

Let $\mathcal{L} = \{E\}$ and let $\mathcal{M}$ be an $\mathcal{L}$ structure in which E is an equivalence relation with one class of size n for all $n \in \omega$ and no infinite classes. Then $x = x$ is a minimal formula. However, by compactness there is $\mathcal{N} \succ \mathcal{M}$ and $a \in N$ such that the equivalence class of a is infinite. Then $E(x,a)$ defines an infinite-coinfinite subset, hence $x = x$ is not strongly minimal.

Remark 2.13. The property "$\phi(x,a)$ is strongly minimal" depends only on $\mathrm{tp}(a)$. Namely, $\phi(x,a)$ is s.m. $\iff$ for all $\psi(x,z) \in \mathcal{L}$ the set

$$\Sigma_\psi(z,a) = \left\{\exists^{>k} x\,(\phi(x,a) \wedge \psi(x,z)) \wedge \exists^{>k} x\,(\phi(x,a) \wedge \neg\psi(x,z)) : k \in \mathbb{N}\right\}$$

cannot be realized in any elementary extension.

By compactness, this means that for all $\psi(x,z)$, there is a bound k_ψ such that

$$\mathcal{M} \models \forall z \left(\exists^{\leq k_\psi} x\,(\phi(x,a) \wedge \psi(x,z)) \wedge \exists^{\leq k_\psi} x\,(\phi(x,a) \wedge \neg\psi(x,z))\right).$$

— an *elementary property* of a, i.e. expressible by a first-order formula.

Hence, it makes sense to say "$\phi(x,b)$ is a s.m. formula" without specifying a model.

Corollary 2.14. *Assume $\mathcal{M} \models T$ is ω-saturated. Then any formula $\phi(x) \in \mathcal{L}(M)$ such that $\phi(M)$ is minimal, is strongly minimal.*

Proof. If $\mathcal{M}$ is ω-saturated and $\phi(x,a)$ is not s.m., then for some $\psi(x,z) \in \mathcal{L}$ the set $\Sigma_\psi(z,a)$ is realized in $\mathcal{M}$, so $\phi(M)$ is not minimal. □

Exercise 2.15. If $\mathcal{M}$ is strongly minimal, $A \subseteq M$ and satisfies $N := \mathrm{acl}(A)$ is infinite, then $N \preceq \mathcal{M}$.

2.3. **Algebraic closure in strongly minimal sets and pregeometries.**

Definition 2.16. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $D(x) \in \mathcal{L}(M)$ a strongly minimal formula. We consider the closure operator

$$\mathrm{cl} : 2^{D(M)} \to 2^{D(M)}$$

defined by

$$\mathrm{cl}(A) := \mathrm{acl}(A) \cap D(M).$$

Proposition 2.17. cl *satisfies the following properties for all $A \subseteq D(M)$ and $a, b \in D(M)$:*

1. *(Reflexivity)* $A \subseteq \mathrm{cl}(A)$,
2. *(Finite character)* $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A') : A' \subseteq A$ *finite*$\}$,

(3) *(Transitivity)* $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(4) *(Exchange)* $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A) \implies b \in \mathrm{cl}(Aa)$ *(abusing notation slightly, we write Ab to denote $A \cup \{b\}$).*

Proof. Properties (1)-(3) hold without any assumption on D. (1) and (2) are straightforward, and we check (3).

- Assume that $c \in \mathrm{cl}(\{b_1, \ldots, b_n\})$ and $b_i \in \mathrm{cl}(A)$.
- Let $\phi(x, b_1, \ldots, b_n) \in \mathcal{L}$ be an algebraic formula satisfied by c.
- Let $\phi_i(y) \in \mathcal{L}(A)$ be algebraic formulas satisfied by the b_i's.
- Then $\psi(x) :=$

$$\exists y_1 \ldots y_n \left(\phi_1(y_1) \wedge \ldots \phi_n(y_n) \wedge \exists^{\leq k} z \phi(z, y_1, \ldots, y_n) \wedge \phi(x, y_1, \ldots, y_n)\right)$$

  is an algebraic $\mathcal{L}(A)$-formula satisfied by c.

On the other hand, (4) is more subtle, and we show that it holds assuming D is s.m.

- Suppose $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$.
- Then $\mathcal{M} \models \phi(a, b)$, where $\phi(x, y) \in \mathcal{L}(A)$ and $|\phi(D, b)| = n$.
- Let $\psi(y) := \exists^{=n} x (D(x) \wedge \phi(x, y))$.
- If $|\psi(D)| < \infty$, then $b \in \mathrm{cl}(A)$, hence $a \in \mathrm{cl}(A)$ (as in the argument above) — a contradiction.
- Thus, $\psi(y)$ defines a cofinite subset of D.
- If $\{y \in D : \phi(a, y) \wedge \psi(y)\}$ is finite, we are done because then $b \in \mathrm{cl}(Aa)$.
- Towards contradiction, assume $|D \setminus \{y : \phi(a, y) \wedge \psi(y)\}| = l$ for some $l \in \omega$.
- Let $\chi(x) := \exists^{=l} y (D(y) \wedge \neg(\phi(x, y) \wedge \psi(y)))$.
- If $\chi(x)$ defines a finite subset of D, then $a \in \mathrm{cl}(A)$, a contradiction.
- Thus $\chi(x)$ defines a cofinite subset of D.
- Choose $a_1, \ldots, a_{n+1}$ such that $\models \chi(a_i)$.
- The set $B_i := \{y \in D : \phi(a_i, y) \wedge \psi(y)\}$ is cofinite for all $i = 1, \ldots, n+1$.
- Choose $b' \in \bigcap B_i$.
- Then $\models \phi(a_i, b')$ for each i,
- so $|\{x \in D : \models \phi(x, b')\}| \geq n+1$ — contradicting the fact that $\psi(b')$ holds.

□

Exercise 2.18. Give an example of a structure $\mathcal{M}$ in which acl doesn't satisfy the exchange.

2.4. **Dimension and independence in pregeometries.**

Definition 2.19. Any abstract closure operator $\mathrm{cl} : 2^X \to 2^X$ on subsets of an infinite set X is satisfying properties (1)-(4) in Proposition 2.17 is called a *pregeometry.*

- If the underlying set is finite, it is called a *matroid.* There is a rather rich theory of matroids, we will discuss this more.
- What is important for us here is that one can develop notions of independence, dimension, basis, etc. in any pregeometry, generalizing linear independence in vector spaces and algebraic independence in algebraically closed fields.

Definition 2.20. Let (D, cl) be a pregeometry. A subset $A \subseteq D$ is called:

(1) *independent (over C)* if $a \notin \operatorname{cl}(A \setminus \{a\})$ (resp., $a \notin \operatorname{cl}(C \cup (A \setminus \{a\}))$) for all $a \in A$;
(2) *a generating set* if $D = \operatorname{cl}(A)$;
(3) *a basis for* $Y \subseteq D$ if $A \subseteq Y$ is an independent generating set.

- Note: any maximal independent subset of Y is a basis for Y.
- Just as in vector spaces, we have:

Lemma 2.21. *(Replacement lemma) Let $A, B \subseteq D$ be independent with $A \subseteq \operatorname{cl}(B)$.*
(1) *Suppose $A_0 \subseteq A, B_0 \subseteq B$ and $A_0 \cup B_0$ is a basis for $\operatorname{acl}(B)$ and $a \in A \setminus A_0$. Then $\exists b \in B_0$ s.t. $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ is a basis for $\operatorname{acl}(B)$.*
(2) $|A| \leq |B|$.
(3) *If A, B are bases for $Y \subseteq D$, then $|A| = |B|$.*

Proof. (1)

- Let $C \subseteq B_0$ be of min. cardinality s.t. $a \in \operatorname{cl}(A_0 \cup C)$.
- As A is independent, $|C| \geq 1$. Let $b \in C$.
- By exchange, $b \in \operatorname{cl}(A_0 \cup \{a\} \cup (C \setminus \{b\}))$, thus
- $\operatorname{cl}(A_0 \cup \{a\} \cup (B \setminus b)) = \operatorname{cl}(B)$.
- If $a \in \operatorname{cl}(A_0 \cup (B_0 \setminus \{b\}))$, then $b \in \operatorname{cl}(A_0 \cup (B_0 \setminus \{b\}))$, contradicting $A_0 \cup B_0$ being a basis.
- Thus $A_0 \cup B_0$ is a basis.
- Thus $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ is independent.

(2)

- Supp. *B is finite*:
- say $|B| = n$ and $a_1, \ldots, a_{n+1} \in A$ are distinct.
- Let $A_0 = \emptyset$ and $B_0 = B$.
- Using (1) inductively, find $b_1, \ldots, b_n \in B$ distinct, s.t.
- $\{a_1, \ldots, a_i\} \cup (B \setminus \{b_1, \ldots, b_i\})$ is a basis for $\operatorname{cl}(B)$ for $i \leq n$.
- But then $\operatorname{cl}(a_1, \ldots, a_n) = \operatorname{cl}(B)$.
- As $a_{n+1} \in \operatorname{cl}(B)$, this contradicts the independence of A.
- *Supp. B is infinite:*
- then for any finite $B_0 \subset B$, $A \cap \operatorname{cl}(B_0)$ is finite and
- $A \subseteq \bigcup_{B_0 \subseteq B \text{ finite}} \operatorname{cl}(B_0)$.
- Thus $|A| \leq |B|$.

(3) is immediate from (2). □

Hence we can define:

Definition 2.22. If $Y \subseteq D$, then $\dim(Y)$, the *dimension* of Y, is the cardinality of a basis for Y.

2.5. **Uncountable categoricity of s.m. theories.** We go back to a matroid given by acl on a definable strongly minimal set D.

Lemma 2.23. *Supp. $\mathcal{M}, \mathcal{N} \models T$, $\phi(x) \in \mathcal{L}(A)$ is a s.m. formula, either $A = \emptyset$ or $A \subseteq M_0$ where $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$.*

If $a_1, \ldots, a_n \in \phi(M)$ are independent over A and $b_1, \ldots, b_n \in \phi(N)$ are independent over A, then $\operatorname{tp}^{\mathcal{M}}(\bar{a}/A) = \operatorname{tp}^{\mathcal{N}}(\bar{b}/A)$.

Proof. We assume $\phi(x) \in \mathcal{L}(A)$, $A \subseteq M_0$, $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$ (case $A = \emptyset$ is a straightforward modification of the argument).

- By induction on n.
- Assume $n = 1$, $a \in \phi(M) \setminus \mathrm{cl}(A)$ and $b \in \phi(N) \setminus \mathrm{cl}(A)$.
- Let $\psi(x) \in \mathcal{L}(A)$, and suppose $\mathcal{M} \models \psi(a)$.
- As $a \notin \mathrm{cl}(A)$, $\phi(M) \cap \psi(M)$ is cofinite in $\phi(M)$.
- Thus $\exists n$ s.t. $\mathcal{M} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = n$.
- Because $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$ and $b \notin \mathrm{cl}(A)$, also $\mathcal{N} \models \psi(b)$.
- As $\psi \in \mathcal{L}(A)$ was arbitrary, $\mathrm{tp}^{\mathcal{M}}(a/A) = \mathrm{tp}^{\mathcal{N}}(b/A)$.
- Assume the claim is true for n and $a_1, \ldots, a_{n+1} \in \phi(M), b_1, \ldots, b_{n+1} \in \phi(N)$ are independent sequence over A.
- Let $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n)$, by induction $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A) = \mathrm{tp}^{\mathcal{N}}\left(\bar{b}/B\right)$.
- Assume $\mathcal{M} \models \psi(\bar{a}, a_{n+1})$ for some $\psi(\bar{w}, v) \in \mathcal{L}(A)$.
- As $a_{n+1} \notin \mathrm{cl}(A\bar{a})$, $\phi(M) \setminus \psi(\bar{a}, M)$ is finite.
- Hence $\exists n$ s.t. $\mathcal{M} \models |\{v : \phi(v) \wedge \neg\psi(\bar{a}, v)\}| = n$.
- Because $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$ and $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A) = \mathrm{tp}^{\mathcal{N}}\left(\bar{b}/A\right)$, also $\mathcal{N} \models \left|\left\{v : \phi(v) \wedge \neg\psi\left(\bar{b}, v\right)\right\}\right| = n$.
- As $b_{n+1} \notin \mathrm{cl}\left(A\bar{b}\right)$, $\mathcal{N} \models \psi\left(\bar{b}, b_{n+1}\right)$, and we conclude.

□

Lemma 2.24. *If $\mathcal{M}, \mathcal{N}$ and ϕ are as above and* $\dim(\phi(M)) = \dim(\phi(N))$*, then there is a bijective partial elementary map $f : \phi(M) \to \phi(N)$.*

In particular, if T is s.m. and $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \cong \mathcal{N} \iff$ $\dim(M) = \dim(N)$.

Proof.
- Let B be a basis for $\phi(\mathcal{M})$ and C a basis for $\phi(\mathcal{N})$.
- As $|B| = |C|$ be assumption, let $f : B \to C$ be any bijection.
- By Lemma 2.23, f is elementary.
- Let $I := \{g : B' \to C' : B \subseteq B' \subseteq \phi(M), C \subseteq C' \subseteq \phi(N), f \subseteq g$ partial elementary$\}$.
- By Zorn's lemma, there is a maximal $g : B' \to C'$ in I.
- Supp. $b \in \phi(M) \setminus B'$.
- As $b \in \mathrm{cl}(B')$, there is a formula $\psi\left(v, \bar{d}\right)$ isolating $\mathrm{tp}^{\mathcal{M}}(b/B')$.
- As g is elementary, can find $c \in \phi(N)$ such that $\mathcal{N} \models \psi\left(c, g\left(\bar{d}\right)\right)$.
- Then $\mathrm{tp}^{\mathcal{M}}(b/B') = \mathrm{tp}^{\mathcal{N}}(c/C')$, and we can extend g by sending b to c.
- This contradicts the maximality of g!
- Thus $\phi(M) = B'$.
- An analogous argument shows that $C' = \phi(N)$.

□

Theorem 2.25. *If T is a countable strongly minimal theory, then T is κ-categorical for all uncountable κ.*

Moreover, it has at most countably many models of cardinality $\aleph_0$.

Proof. If T is countable, then $|\mathrm{acl}(A)| \leq |A| + \aleph_0$, hence any basis for $\mathcal{M}$ with $|\mathcal{M}| = \kappa > \aleph_0$ has cardinality κ.

If $|M| \leq \aleph_0$, then $\dim(M) \leq \aleph_0$. □

2.6. ω-stability.

- We aim to show that if T is κ-categorical for some uncountable κ, then it is a "fibration" over a definable strongly minimal set.
- For this, we isolate some consequences of κ-categoricity.

Definition 2.26. T is ω-stable if for every $\mathcal{M} \models T$ and every $A \subseteq M$, $|A| \leq \aleph_0$ $\implies$ $|S_1(A)| \leq \aleph_0$.

Exercise 2.27.
(1) Show that under ω-stability, also $|S_n(A)| \leq \aleph_0$ for all countable A and $n \in \omega$.
(2) Show that a countable T is ω-stable $\iff$ $|S_1(M)| = |M|$ for all infinite $M \models T$.

- By Stone duality, the space of types is small $\iff$ definable sets are not so complicated.

Example 2.28. It T is strongly minimal, then T is ω-stable.

Indeed, $\forall\ A \subseteq \mathcal{M} \models T$, the types over A are the algebraic types (all realized in $\operatorname{acl}(A)$, $|\operatorname{acl}(A)| \leq |A| + \aleph_0$) and the unique non-algebraic type.

Theorem 2.29. *Let T be ω-stable. If $\mathcal{M} \models T$, then there is a minimal formula in $\mathcal{M}$.*

Proof. Suppose no formula is minimal. Then by induction we can build a tree of $\mathcal{L}(M)$-formulas $(\phi_\eta(x) : \eta \in 2^{<\omega})$ such that:

- $\phi_\emptyset$ is "$x = x$",
- $\eta \subseteq \nu \in 2^{<\omega} \implies \phi_\nu \vdash \phi_\eta$,
- $\phi_{\eta\frown i}(x) \vdash \phi_{\eta\frown(1-i)}(x)$,
- $\phi_\eta(M)$ is infinite for all $\eta \in 2^{<\omega}$.

Let A be the set of all parameters of all formulas ϕ_η, $|A| \leq \aleph_0$.

For each $\eta \in 2^\omega$, the set $\{\phi_{\eta\restriction n}(x) : n \in \omega\}$ is consistent, hence extends to some $p_\eta \in S_1(A)$.

And if η, ν are incomparable, then $p_\eta \neq p_\nu$. Hence $|S_1(A)| = 2^{\aleph_0}$, contradicting ω-stability. □

Definition 2.30. Given $A \subseteq \mathcal{M} \models T$, we say that $\mathcal{M}$ *is prime over* A if $\forall \mathcal{N} \models T$ and $f : A \to N$ partial elementary, there is an elementary $f^* : \mathcal{M} \to \mathcal{N}$ extending f.

Lemma 2.31. *Let a countable T be ω-stable, $\mathcal{M} \models T$, $A \subseteq M$. The isolated types in $S_n(M)$ are dense.*

Proof. If not, can build a binary tree of formulas as in Theorem 2.29. □

Theorem 2.32. *Let a countable T be ω-stable, $\mathcal{M} \models T$ and $A \subseteq M$. There is $\mathcal{M}_0 \preceq \mathcal{M}$, a prime model over A.*

Proof.
- We find an ordinal δ and subsets $(A_\alpha : \alpha \leq \delta)$ of M s.t.
  (1) $A_0 = A$.
  (2) α limit $\implies$ $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.
  (3) if no element of $M \setminus A_\alpha$ realizes an isolated type over A_α, stop and let $\delta = \alpha$;
  (4) otherwise, pick a_α with $\operatorname{tp}(a_\alpha/A_\alpha)$ isolated and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.
- Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with $M_0 := A_\delta$.
- **Claim 1.** $\mathcal{M}_0 \preceq \mathcal{M}$.
- By Tarski-Vaught test: suppose $\mathcal{M} \models \phi(v)$ with $\phi(x) \in \mathcal{L}(A_\delta)$.
- By the lemma, $\exists b \in \mathcal{M}$ s.t. $\operatorname{tp}(b/A_\delta)$ is isolated and $\mathcal{M} \models \phi(b)$.
- By choice of δ, $b \in A_\delta$.

- **Claim 2.** $\mathcal{M}_0$ is a prime model extension of A.
- Let $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ partial elementary.
- We find by induction $f = f_0 \subseteq \ldots \subseteq f_\alpha \subseteq f_\delta$ with $f_\alpha : A_\alpha \to \mathcal{N}$ elementary.
- α limit: let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.
- Given $f_\alpha : A_\alpha \to \mathcal{N}$ partial elementary, let $\phi(v, a)$ isolate $\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)$.
- Then $\phi(v, f_\alpha(a))$ isolates $f_\alpha\left(\mathrm{tp}^{\mathcal{M}_0}(a_\alpha/A_\alpha)\right)$ in $S_1^{\mathcal{N}}(f_\alpha(A))$.
- As f_α is elementary, $\exists b \in N$ s.t. $\mathcal{N} \models \phi(b, f_\alpha(a))$.
- Hence $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ is elementary.

- Hence $f_\delta : \mathcal{M}_0 \to \mathcal{N}$ is elementary, as wanted.

□

Theorem 2.33. *If T is κ-categorical for some uncountable κ, then T is ω-stable.*

- Before we can prove it, we need an auxilliary construction.

2.7. **Ehrenfeucht–Mostowski Models.**

Definition 2.34. Let I be a linear ordering. A sequence $(a_i : i \in I)$ of tuples in an $\mathcal{L}$-structure $\mathcal{M}$ is *indiscernible over $B \subseteq \mathcal{M}$* if $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/B) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/B)$ $\forall n \in \omega$, $\bar{i}, \bar{j}$ strictly increasing tuples from I.

Example 2.35. We saw that if T is strongly minimal, and $(a_i : i \in I)$ is a sequence of acl-independent elements, then it is indiscernible.

Fact 2.36. *For any T in language $\mathcal{L}$ and I there is some $\mathcal{M} \models T$ and $(a_i : i \in I)$ a non-constant indiscernible sequence in $\mathcal{M}$.*

Proof.
- Recall Ramsey theorem: for any $n, k \in \omega$ and $f : \omega^m \to k$, there is an infinite subset $S \subseteq \omega$ s.t. f is constant on all increasing m-tuples from S.
- Consider the partial type $\Sigma(x_i : i \in I)$ containing formulas
  1. T,
  2. $x_i \neq x_j$, $\forall i \neq j$,
  3. $\phi(x_{i_1}, \ldots, x_{i_n}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_n})$, $\forall \phi \in \mathcal{L}$ and $\bar{i}, \bar{j}$ increasing tuples from I.
- Any finite $\Sigma_0 \subseteq \Sigma$ is consistent:
  - let $(a_i : i \in \omega)$ be any infinite sequence in some $\mathcal{M} \models T$, and Δ a finite set of formulas.
  - By Ramsey, $(a_i : i \in \omega)$ contains an infinite subsequence satisfying condition (3) (and the other conditions hold).
- Conclude by compactness.

□

Theorem 2.37. *Let $\mathcal{L}$ be countable, T an $\mathcal{L}$-theory. For all $\kappa \geq \aleph_0$ there is $\mathcal{M} \models T$ with $|M| = \kappa$ such that if $A \subseteq M$, then $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types over A.*

Proof.
- Let T^* be an expansion of T in a language $\mathcal{L}^* \supseteq \mathcal{L}$ with Skolem functions (i.e. for each formula $\phi(x, y) \in \mathcal{L}^*$ there is a function symbol $f_\phi(y)$ such that $T^* \vdash \forall y (\exists x \phi(x, y) \to \phi(f_\phi(y), y))$. Exists by taking a model of T and expanding until catching the tail.)
- Let $\mathcal{M}^* \models T^*$ and contain I, an $\mathcal{L}^*$-indiscernible sequence of order type $(\kappa, <)$.

- Let $\mathcal{M}$, an $\mathcal{L}^*$-substructure generated by I.
- In particular, the $\mathcal{L}$-reduct of $\mathcal{M}$ is a model of T, and $|\mathcal{M}| = \kappa$.
- Let $A \subseteq M$. For each $a \in A$, $a = t_a(\bar{x}_a)$ for some term t_a and $\bar{x}_a$ from I.
- Let $X := \{x \in I : x \text{ occurs in } \bar{x}_a \text{ for some } a \in A\}$.
- Then $|X| \leq |A| + \aleph_0$.
- By $\mathcal{L}^*$-indiscernibility: if $\bar{y}, \bar{z}$ are tuples from κ with the same order type over X (denoted by $\bar{y} \sim_X \bar{z}$) and t is an $\mathcal{L}^*$-term, then $t(\bar{y}), t(\bar{z})$ have the same type over A.
- Hence enough to show: for any $n \in \omega$, $|I^n / \sim_X| \leq |A| + \aleph_0$.
- For $y \in I \setminus X$, let $C_y := \{x \in X : x < y\}$.
- Then $\bar{z} \sim_X \bar{y} \iff \forall 1 \leq i \leq n$:
  1. if $y_i \in X$, then $y_i = z_i$,
  2. if $y_i \notin X$, then $z_i \notin X$ and $C_{y_i} = C_{z_i}$.
- As I is well-ordered, $C_y = C_z \iff C_y = C_z = \emptyset$ or $\inf\{i \in I : i > C_y\} = \inf\{i \in I : i > C_z\}$.
- In particular, there are at most $|X| + 1$ possible cuts C_y.
- Hence $|I^n / \sim_X| \leq |A| + \aleph_0$, as wanted.

□

Proof. (of Theorem 2.33).

Let κ be any uncountable cardinal.

Assume T is not ω-stable, then $\exists \mathcal{M} \models T$ and $A \subseteq M$ countable with $|S_1(A)| > \aleph_0$. We can find $\mathcal{N}_0 \succeq \mathcal{M}$ with $|\mathcal{N}_0|$ realizing uncountably many types from $S_1(A)$. By Theorem 2.37, $\exists \mathcal{N}_1 \models T$, $|N_1| = \kappa$, and $\forall B \subseteq N_1$, $|B| = \aleph_0$, $\mathcal{N}_1$ realizes at most $\aleph_0$ types over B.

Then $\mathcal{N}_0 \ncong \mathcal{N}_1$, contradicting κ-categoricity. □

2.8. Two-cardinal models and elimination of $\exists^\infty$.

Definition 2.38. Let $\kappa > \lambda \geq \aleph_0$ be cardinals. Then $\mathcal{M} \models T$ is a (κ, λ)-model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

Proposition 2.39. *A countable T is κ-categorical $\implies$ T has no (κ, λ)-models for any $\lambda < \kappa$.*

Proof. By compactness and downwards LS, any countable T has a model $\mathcal{M}$ of size κ s.t. every infinite $\emptyset$-definable subset of $\mathcal{M}$ also has cardinality κ. □

Fact 2.40. *(Vaught's two-cardinal theorem) If T has a (κ, λ)-model with $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.*

- Almost any other implication of this form is false or independent from ZFC.

Definition 2.41. T has a *Vaught pair* if there are $\mathcal{M} \preceq \mathcal{N} \models T$ and $\phi(x) \in \mathcal{L}(M)$ such that:

1. $M \neq N$,
2. $\phi(M)$ is infinite,
3. $\phi(M) = \phi(N)$.

Lemma 2.42. *If T has a (κ, λ)-model, then it has a Vaught pair.*

Proof.
- Let $\mathcal{N}$ be a (κ, λ)-model.
- Supp. $|\phi(N)| = \lambda$.
- By the LS theorem, $\exists \mathcal{M} \prec \mathcal{N}$ s.t. $\phi(N) \subseteq M$ and $|M| = \lambda$.

- As $\phi(N) = \phi(M)$, this is a Vaught pair.

∎

- Vaught's theorem really proves the following:

Lemma 2.43. *If T has a Vaught pair, then it has a $(\aleph_1, \aleph_0)$-model.*

Corollary 2.44. *If T is $\aleph_1$-categorical, then T has no Vaught pairs.*

- Assuming ω-stability, we have the following converse.

Theorem 2.45. *Suppose T is ω-stable and T has an $(\aleph_1, \aleph_0)$-model. If $\kappa > \aleph_1$, then T has a $(\kappa, \aleph_0)$-model.*

Proposition 2.46. *If T has no Vaught pairs, then T eliminates $\exists^\infty$. That is, for each $\phi(\bar{v}, \bar{w}) \in \mathcal{L}$ $\exists n_\phi \in \omega$ s.t. if $\mathcal{M} \models T$, $\bar{a} \in M$ and $|\phi(M, \bar{a})| > n$, then $\phi(\mathcal{M}, \bar{a})$ is infinite.*

Proof.
- Let $\mathcal{L}^* := \mathcal{L} \cup \{P(x), c_1, \ldots, c_n\}$.
- Let T^* be the theory of all $\mathcal{L}^*$-structures $(M, P(M), c_1, \ldots, c_n)$ where:
  - M is a model of T,
  - $P(M) \prec_{\mathcal{L}} M$,
  - $P(c_i)$ holds $\forall i$,
  - $\phi(M, \bar{a}) \subseteq P(M)$.
- Suppose no bound $n_\phi \in \omega$ exists.
- Then $\forall n \exists \mathcal{N} \models T$ and $\bar{a} \in N$ s.t. $\phi(N, \bar{a})$ is finite, but has more than n elements.
- Let $\mathcal{M}$ be a proper elem. ext. of $\mathcal{N}$.
- Then $\phi(M, \bar{a}) = \phi(N, \bar{a})$ and $(M, N, \bar{a}) \models T^*$.
- Hence $T^* \cup \{\exists^{>n} x \phi(x, \bar{c}) : n \in \omega\}$ is finitely satisfiable.
- By compactness it has a model, which gives a Vaught pair for T.

∎

Corollary 2.47. *If T has no Vaught pairs, then any min. formula is s.m.*

Proof. If $\phi(x, \bar{a})$ is minimal in $\mathcal{M}$ and T eliminates $\exists^\infty$, then for each $\psi(x, z) \in \mathcal{L}$,

$$\neg\left(\exists^\infty x\left(\phi(x, \bar{a}) \wedge \psi(x, \bar{z})\right) \wedge \exists^\infty x\left(\phi(x, \bar{a}) \wedge \neg\psi(x, \bar{z})\right)\right)$$

is an elementary property of $\bar{z}$ and holds in $\mathcal{M}$, hence holds in every model of T. ∎

Exercise 2.48. Show that RG has a Vaught pair.

2.9. **Morley's theorem, proof by Baldwin-Lachlan.**

- Now we put everything together.

Theorem 2.49. *(Baldwin-Lachlan) Let κ be an uncountable cardinal. A countable T is κ-categorical $\iff$ T is ω-stable and has no Vaughtian pairs.*

- As the RHS doesn't depend on the specific uncountable κ, Morley's categoricity theorem follows.

Proof. $(\Rightarrow)$ We saw that T is κ-cat $\implies$ T is ω-stable and has no Vaught pairs.
$(\Leftarrow)$
- As T is ω-stable, it has a prime model $\mathcal{M}_0$ (over $\emptyset$).
- By Theorem 2.29 and Corollary 2.47, $\exists$ an s.m. formula $\phi(v) \in \mathcal{L}(M_0)$.
- Let $\mathcal{M}, \mathcal{N} \models T$, of card. $\kappa \geq \aleph_1$.

- WLOG $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$.
- Then $|\phi(M)| = |\phi(M)| = \kappa$ (as T has no (κ, λ)-models)
- Hence $\dim(\phi(M)) = \dim(\phi(N)) = \kappa$.
- By Lemma 2.24, there is $f : \phi(M) \to \phi(N)$ a partial elementary bijection.
- **Claim.** $\mathcal{M}$ is prime over $\phi(M)$.
  - No proper $\mathcal{K} \prec \mathcal{M}$ conains $\phi(M)$, as otherwise $\phi(K) = \phi(M)$ and $\phi(M)$ is infinite, hence $(\mathcal{M}, \mathcal{K})$ would be a Vaught pair.
  - As T is ω-stable, by Theorem 2.32, $\exists \mathcal{K} \preceq \mathcal{M}$ a prime model over $\phi(M)$.
  - By the above, $\mathcal{K} = \mathcal{M}$.
- Hence, by def. of primality, f extends to an elementary $f' : M \to N$.
- But as in the claim, $\mathcal{N}$ has no proper elem. submodels containing $\phi(N)$.
- Hence f' is surjective, so an isomorphism.

$\square$

- It follows from the proof that if T is uncountably categorical, then it has at most countably many countable models.
- This can be refined to:

Fact 2.50. *(Baldwin-Lachlan) If T is uncountably categorical but not $\aleph_0$-categorical, then it has exactly $\aleph_0$ countable models.*

2.10. **Further work on counting models.**

Definition 2.51. For a theory T, consider the function $I_T(\kappa)$ = the number of models of T of size κ, up to isomorphism.

- Note that $1 \leq I_T(\kappa) \leq 2^\kappa$ for all infinite κ bigger than the cardinality of the language.
- Morley's theorem says: let T be a countable theory. If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1$ for all uncountable κ.

Stability theory developed historically in Shelah's work as a chunk of machinery intended to generalize Morley's theorem to a computation of the possible "spectra" of complete first order theories, in particular to prove the following conjectures of Morley.

Theorem 2.52. *[Shelah]*

(1) *The function $I_T(\kappa)$ is non-decreasing on uncountable cardinals.*
(2) *For theory T of any cardinality, there is some cardinal $\kappa(T)$ such that if $I_T(\lambda) = 1$ for some $\lambda > \kappa(T)$, then $I_T(\lambda) = 1$ for all $\lambda > \kappa(T)$.*

- This project, at least for countable theories, was essentially completed by Shelah in the early 80's with the "Main gap theorem" [2].
- Shelah isolated a bunch of "dividing lines" (in the form of T being able to encode in a definable manner a certain finitary combinatorical configuration) on the space of first-order theories, showing that all theories on the non-structure side of the dividing line have as many models as possible, and on the structure side developing some kind of dimension theory and showing that the isomorphism type of a model can be described by some "small" invariants implying e.g. that there are few models (so in the case of a vector space we only need one cardinal invariant, but if we have an equivalence relation then we need to know the size of each of its classes).

- A complete classification of the possible functions $I_T(\kappa)$ for countable theories was given by Hart, Hrushovski, Laskowski [1] (required some descriptive set-theoretic ideas in order to prove a "continuum hypothesis" for a certain notion of dimension).
- Later on, motivated by the aim to understand totally categorical theories, other perspectives developed, in the work of Macintyre, Zilber, Cherlin, Poizat, Hrushovski, Pillay and many others, in which stability theory is seen rather as a way of classifying definable sets in a structure and describing the interaction between definable sets. Eventually this theory started to be seen as having a "geometric meaning". It turned out that the study of the definability of certain algebraic objects such as groups and fields is essential even for purely logical questions of categoricity.
- Moreover, more recently it was realized that a lot of techniques from stability can still be developed in larger contexts, and the so-called "generalized stability" is currently an active area of research.

Exercise 2.53. Show that if T is ω-stable, then for any formula $\phi(x, y) \in \mathcal{L}$ and any model $\mathcal{M} \models T$, we cannot find a sequence $(a_i, b_i : i \in \omega)$ in M such that $\mathcal{M} \models \phi(a_i, b_j) \iff i \leq j$.

Any such formula is called *stable,* and large part of the theory developed in this course can be generalized to theories in which every formula is stable.

3. Morley rank and forking independence in ω-stable theories

- We fix a theory T and a κ-saturated and κ-homogeneous "monster model" $\mathbb{M} \models T$, with $\kappa = \kappa(\mathbb{M})$ sufficiently large.
- All sets, tuples and models we consider will be contained in $\mathbb{M}$.
- We say that a set $A \subseteq \mathbb{M}$ is *small*, or *bounded*, if $|A| < \kappa$.

3.1. M^{eq} and canonical parameters.

- All the notions above generalize in an obvious way to *multi-sorted structures*, i.e. the underlying set of our model is now partitioned into several sorts, and for each of the variables for each relation and function symbols we specify which sort they live in. Then formulas and other notions are defined and evaluated accordingly.
- We recall Shelah's construction of imaginaries, which allows us to view quotients of definable sets by definable equivalence relations as definable objects themselves.
- Given $\mathcal{M} \models T$, we construct a structure $\mathcal{M}^{\mathrm{eq}}$ in a language $\mathcal{L}^{\mathrm{eq}}$ extending $\mathcal{L}$ as follows:
  - add a new sort M^n/E, $\forall\ n \in \omega$ and $E(x, y) \in \mathcal{L}$ defining an equivalence relation on M^n.
  - $\mathcal{L}^{\mathrm{eq}}$ contains an n-ary function symbol π_E for the projection map $\pi_E : M^n \to M^n/E$, $\pi_E(a) = [a]_E$, the E-equivalence class of $a \in M^n$.
  - Identify M with $M/=$.
  - The sort $M/=$ is called the main sort, and the elements from the other sorts are called *imaginaries.*
- Then $T^{\mathrm{eq}} := \mathrm{Th}_{\mathcal{L}^{\mathrm{eq}}}(\mathcal{M}^{\mathrm{eq}})$.

Exercise 3.1. • T^{eq} doesn't depend on the choice of the initial model $\mathcal{M} \models T$.

- Every automorphism of $\mathcal{M}$ extends uniquely to an automorphism of $\mathcal{M}^{\mathrm{eq}}$.
- If $\mathcal{M} \models T$ is κ-saturated (-homogeneous), then $\mathcal{M}^{\mathrm{eq}}$ is also κ-saturated (resp., κ-homogeneous).
- For every $\psi(x, x_1, \ldots, x_n) \in \mathcal{L}^{\mathrm{eq}}$ with x in the main sort and x_i is of sort E_i, $\exists \phi(x, y_1, \ldots, y_n) \in \mathcal{L}$ with $|y_i| = \operatorname{arity}(E_i)$ s.t. for all tuples $a, a_1, \ldots, a_n \in M$:
$$\mathcal{M}^{\mathrm{eq}} \models \psi(a, \pi_{E_1}(a_1), \ldots, \pi_{E_n}(a_n)) \iff \mathcal{M} \models \psi(a, a_1, \ldots, a_n).$$
- If T is t.t., then T^{eq} is also t.t.
- In particular, $\mathbb{M}^{\mathrm{eq}}$ is a monster model for T^{eq} and T^{eq} doesn't define any new subsets of the main sort.

Definition 3.2. Given $\psi(x, a) \in \mathcal{L}(\mathbb{M})$, consider the equiv. rel. $E_\psi(y, z) \iff \forall x (\psi(x, y) \leftrightarrow \psi(x, z))$, we define the *canonical parameter* $\ulcorner \psi(x, a) \urcorner$ *of* $\psi(x, a)$ as the imaginary $[a]_{E_\psi}$.

- Note: $E_\psi(y, z)$ holds $\iff \psi(\mathbb{M}, y) = \psi(\mathbb{M}, z)$.

Definition 3.3.
- An element $a \in \mathbb{M}$ is definable over $A \subseteq \mathbb{M}$ if $\exists \phi(x) \in \mathcal{L}(A)$ s.t. $\models \phi(a)$ and $|\phi(\mathbb{M})| = 1$.
- $\operatorname{dcl}(A) := \{a \in \mathbb{M} : a \text{ is definable over } A\}$

- Note: $\operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$.
- We will write $\operatorname{acl}^{\mathrm{eq}}, \operatorname{dcl}^{\mathrm{eq}}$ to denote acl and dcl in the structure $\mathbb{M}^{\mathrm{eq}}$.

Exercise 3.4. Assume that $\phi(\mathbb{M}, a) = \psi(\mathbb{M}, b)$ for any $\phi(x, a), \psi(x, b) \in \mathcal{L}(\mathbb{M})$. Then $\ulcorner \phi(x, a) \urcorner$ and $\ulcorner \psi(x, b) \urcorner$ are *interdefinable*, i.e. $\ulcorner \phi(x, a) \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner \psi(x, b) \urcorner)$ and $\ulcorner \psi(x, b) \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner \phi(x, a) \urcorner)$. Hence given $X \subseteq \mathbb{M}^n$, we can talk about the canonical parameter $\ulcorner X \urcorner$ of X (well-defined up to interdefinability).

- Canonical parameters help dealing with automorphisms and give a Galois-theoretic characterization of definability:

Lemma 3.5. *Let $X \subseteq \mathbb{M}^n$ be definable and $A \subseteq \mathbb{M}$ a small set (i.e. $|A| < \kappa(\mathbb{M})$). TFAE:*

1. *X is A-definable.*
2. *X is A-invariant, i.e. $\sigma(X) = X$ (setwise) for all $\sigma \in \operatorname{Aut}(\mathbb{M}/A) = \operatorname{Aut}(\mathbb{M}^{\mathrm{eq}}/A)$ (i.e. σ is an automorphism of $\mathbb{M}$ fixing A pointwise).*
3. *$\ulcorner X \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(A)$.*

Proof. (1) $\implies$ (2):

- Supp. $X = \psi(\mathbb{M}, b)$, $b \in A$.
- For any $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$ we have $\sigma(b) = b$, hence

$$a \in X \iff \models \phi(a, b) \iff \models \phi(\sigma(a), \sigma(b)) \iff \models \phi(\sigma(a), b) \iff \sigma(a) \in X.$$

(2) $\implies$ (1):

- Supp. $X = \phi(\mathbb{M}, b)$ with $b \in \mathbb{M}$ and X is A-invariant.
- Let $p(y) := \operatorname{tp}(b/A)$.
- By A-invariance of X, $p(x) \vdash \forall x (\phi(x, y) \leftrightarrow \phi(x, b))$.
- By compactness $\theta(y) \vdash \forall x (\phi(x, y) \leftrightarrow \phi(x, b))$ for some $\theta(y) \in p(y)$.
- Let $\chi(x) := \exists z (\theta(z) \wedge \phi(x, z))$, then $\chi(x) \in \mathcal{L}(A)$.
- Check: $\chi(\mathbb{M}) = X$.

(2) $\iff$ (3):

- Let $\phi(x,y) \in \mathcal{L}, c \in \mathbb{M}$ s.t. $X = \phi(\mathbb{M}, c)$.
- Let $E(y,z)$ be the definable e.r. $\phi(\mathbb{M}, y) = \phi(\mathbb{M}, z)$.
- Thus $[c]_E = \ulcorner \phi(x,c) \urcorner$.
- By (2) $\iff$ (1) applied to the definable set $\{[c]_E\}$, we see that $[c]_E \in \mathrm{dcl}^{\mathrm{eq}}(A) \iff \sigma([c]_E) = [c]_E$ for all $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$.
- For any $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ we have:

$$\sigma([c]_E) = [c]_E \iff E(c, \sigma(c)) \iff \phi(\mathbb{M}, c) = \phi(\mathbb{M}, \sigma(c)) \iff \sigma(X) = X,$$

- and $[c]_E \in \mathrm{dcl}^{\mathrm{eq}}(A) \iff$ it is fixed by all automorphisms.

$\square$

- Similarly for $\mathrm{acl}^{\mathrm{eq}}$ (Exercise, use the proof of the previous lemma and compactness):

Lemma 3.6. *Let $X \subseteq \mathbb{M}^n$ be definable and A small. TFAE:*

(1) *X has a finite* $\mathrm{Aut}(\mathbb{M}/A)$*-orbit.*
(2) *X has a bounded* $\mathrm{Aut}(\mathbb{M}/A)$*-orbit (bounded = smaller than $\kappa(\mathbb{M})$).*
(3) $\ulcorner X \urcorner \in \mathrm{acl}^{\mathrm{eq}}(A)$.

3.2. Morley rank.

- In this section we develop an (ordinal-valued) notion of "dimension" for definable sets and types in ω-stable theories.
- It can be defined topologically using the Cantor-Bendixson rank on the associated spaces of types. Recall:

Definition 3.7. Let X be a compact Hausdorff topological space.

(1) For a point $p \in X$, its *Cantor-Bendixson rank* $\mathrm{CB}(p)$ is defined by induction on an ordinal α:
    (a) $\mathrm{CB}(p) \geq 0$ for all $p \in X$,
    (b) $\mathrm{CB}(p) = \alpha$ iff p is isolated in the subspace $\{q \in X : \mathrm{CB}(q) \geq \alpha\}$.
    (c) $\mathrm{CB}(p) = \infty$ if (b) doesn't hold for any α.
(2) If $\mathrm{CB}(p) < \infty$ for all $p \in X$, then $\{\mathrm{CB}(p) : p \in X\}$ has the greatest element, say α, and the set $\{p \in X : \mathrm{CB}(p) = \alpha\}$ is finite, say of cardinality n (by compactness of X). We then say that α is the CB-rank if X, or $\mathrm{CB}(X) = \alpha$, and that n is the CB-multiplicity of X,$\mathrm{CB}-\mathrm{mult}(X) = n$.

Definition 3.8. (1) T is called *totally transcendental*, or t.t., if for every $n \in \omega$, $\mathrm{CB}(S_n(\mathbb{M})) < \infty$ (i.e. bounded by some ordinal).
(2) Let $\Phi(x)$ be a set of formulas over a small set. Let $Y = \{p \in S_n(\mathbb{M}) : \Phi \subseteq p\}$, a closed subspace of $S_n(\mathbb{M})$. By the *Morley rank* of Φ, $\mathrm{RM}(\Phi)$, we mean $\mathrm{CB}(Y)$. If $\mathrm{CB}(Y) < \infty$, then the Morley degree of Φ, $\mathrm{dM}(\Phi)$, is defined to be the $\mathrm{CB}-\mathrm{mult}(Y)$.

Exercise 3.9. Show that if $\mathrm{CB}(S_1(\mathbb{M})) < \infty$, then T is t.t.

- We write $\mathrm{RM}(\bar{a}/A)$ for $\mathrm{RM}(\mathrm{tp}(\bar{a}/A))$.

Lemma 3.10. (1) $\mathrm{RM}(\phi(x)) \geq \alpha + 1$ *if and only if there is an infinite set* $\{\phi_i(x) : i \in \omega\}$ *of pairwise contradictory formulas such that* $\phi_i(x) \vdash \phi(x)$ *and* $\mathrm{RM}(\phi_i) \geq \alpha$ *for all* $i \in \omega$.

(2) $\mathrm{RM}(\Phi(x)) = \max\{\mathrm{CB}(p) : p \in S_x(\mathbb{M}), \Phi \subseteq p\}$.
(3) $\mathrm{RM}(\Phi(x)) = \min\{\mathrm{RM}(\Phi'(x)) : \Phi' \subseteq \Phi$ *finite* $\}$. *If* $\mathrm{RM}(\Phi) < \infty$, *then* $dM(\Phi)$ *is the minimum of* $dM(\phi)$ *where* ϕ *is a finite conjunction of formulas in* Φ *and* $\mathrm{RM}(\phi) = \mathrm{RM}(\Phi)$.
(4) $\mathrm{RM}(\phi(x)) = 0$ *iff* $\phi(x)$ *is algebraic.*
(5) $\mathrm{RM}(\phi(x) \vee \psi(x)) = \max\{\mathrm{RM}(\phi), \mathrm{RM}(\psi)\}$.
(6) *For any set* $\Phi(x)$ *of formulas over* A, *there is a complete type* $p(x) \in S_{|x|}(A)$ *such that* $\mathrm{RM}(\Phi) = \mathrm{RM}(p)$.
(7) *If* $\mathrm{RM}(\phi) = \alpha$ *then* $dM(\phi)$ *is the greatest* $k \in \omega$ *such that there are pairwise contradictory* $\phi_1, \ldots, \phi_k$ *with* $\phi_i(x) \vdash \phi(x)$ *and* $\mathrm{RM}(\phi_i) \geq \alpha$.
(8) *Suppose* $A \subseteq \mathbb{M}, \bar{a}, b \in \mathbb{M}$ *and* $b \in \mathrm{acl}(A\bar{a})$. *Then* $\mathrm{RM}(\bar{a}b/A) = \mathrm{RM}(\bar{a}/A)$.
(9) *If* $X \subseteq \mathbb{M}^n, Y \subseteq \mathbb{M}^m$ *are definable and* $f : X \to Y$ *is a definable function s.t.* $\mathrm{RM}\left(f^{-1}(a)\right) = k$ *for all* $a \in Y$, *then* $\mathrm{RM}(X) = \mathrm{RM}(Y) + k$.
(10) *In particular, definable finite-to-one maps preserve* RM.

Exercise 3.11. Prove this lemma.

Lemma 3.12. *It T is countable, then T is ω-stable $\iff$ T is t.t.*

Proof. ($\implies$): ω-stability $\implies$ isolated types are dense, hence every type has an ordinal-valued CB-rank.

($\Leftarrow$): Exercise. □

Fact 3.13. (1) *A formula* $\phi(x)$ *is s.m.* $\iff$ $\mathrm{RM}(\phi) = dM(\phi) = 1$.
(2) *Supp.* T *is s.m. If* $A \subseteq \mathbb{M}$ *and* $\bar{a} \in \mathbb{M}$, *then* $\mathrm{RM}(\bar{a}/A) = \dim(\bar{a}/A)$.

3.3. **Definability of types.**

Definition 3.14. $p \in S_n(A)$ is *definable* over B if $\forall \phi(x,y) \in \mathcal{L}$ there is $d_p\phi(y) \in \mathcal{L}(B)$ s.t. $\phi(x,a) \in p \iff \models d_p\phi(a)$ for all $a \in A$.

Theorem 3.15. *If T is ω-stable and $\mathcal{M} \prec \mathbb{M}$ and $p \in S_n(M)$, then p is definable.*

- Recall that a formula $\delta(x,y)$ is *stable* if
  $(*)$ there is some $n = n(\phi) \in \omega$ such that in no model $\mathcal{M} \models T$ we can find $(a_i, b_i : i \in n+1)$ such that $\models \delta(a_i, b_j) \iff i \leq j$.

Exercise 3.16. Assume that $\delta(x, y_1), \gamma(x, y_2)$ are stable. Then:
- $\delta'(y_1, x) := \delta(x, y_1)$ is stable,
- $\psi(x, y_1 y_2) := \delta(x, y_1) \vee \delta(x, y_2)$ and $\psi'(x, y_1) := \neg\delta(x, y_1)$ are stable (hence any Boolean combination of stable formulas is stable).

- Theorem 3.15 follows from the following fundamental lemma using Exercise 2.53 and compactness. Namely, given $p \in S_n(M)$, let $a \in \mathbb{M}$ s.t. $a \models p(x)$. Then for any $\delta(x,y) \in \mathcal{L}$, we can take $d_p\phi(y) := \bigvee_j \left(\bigwedge_i \delta\left(a_i^j, y\right)\right) \in \mathcal{L}(M)$ from the lemma.

Lemma 3.17. *Let $\delta(x,y)$ be stable, $\mathcal{M} \preceq \mathbb{M}$ and $a \in \mathbb{M}$. Then there is a finite set $\left\{a_i^j : i,j < N\right\}$ of tuples in M s.t. for any $b \in M$, $\models \delta(a,b) \iff \models \bigvee_j \left(\bigwedge_i \delta\left(a_i^j, b\right)\right)$.*

Proof.
- Let N_1 be as given by $(*)$ for $\delta(x,y)$.

- **Claim.** For any $\psi(x) \in \operatorname{tp}(a/M)$, $\exists a_1, \dots, a_n \in M$ for some $n \leq N_1$ s.t. $\forall b \in M$, $\bigwedge_{i=1}^{n} \delta(a_i, b) \to \delta(a, b)$.
  - We choose inductively $a_i \in M, a_i \models \psi(x)$ and $b_i \in M$ s.t. $\models \neg\delta(a, b_i)$ and $\models \delta(a_i, b_j) \iff i \leq j$.
  - Suppose we have already found such $a_1, \dots, a_n, b_1, \dots, b_n$.
  - Then $a \models \psi(x) \wedge \bigwedge_{i=1}^{n} \neg\delta(x, b_i)$, and this is an $\mathcal{L}(M)$-formula.
  - As $a \in \mathbb{M}$ and $\mathcal{M} \preceq \mathbb{M}$, there is also some $a_{n+1} \in M$ satisfying it.
  - If $(a_1, \dots, a_{n+1})$ satisfies the claim, we stop.
  - Otherwise there is some $b_{n+1} \in M$ such that $\models \bigwedge_{i=1}^{n+1} \delta(a_i, b)$, but $\models \neg\delta(a, b)$.
  - This process must stop after $\leq N_1$ steps by the choice of N_1.

- Let $\chi(x_1, \dots, x_{N_1}; y) := \delta(x_1, y) \wedge \ldots \wedge \delta(x_{N_1}, y)$. By Exercise 3.16, χ and $\neg\chi$ are stable.
- Let N_2 be as given by $(*)$ for $\neg\chi$.
- We choose inductively $\bar{a}^1, \bar{a}^2, \dots$ in M satisfying the claim above, and $b_i \in M$ s.t.
  (1) $\models \chi(\bar{a}^i, b_j) \iff i > j$,
  (2) $\models \delta(a, b_i)$ for all i.
- Suppose we have already chosen such $\bar{a}^i, b_i$ for $i = 1, \dots, n$.
- By the claim, $\exists \bar{a}^{n+1} \in M$ s.t. $\models \chi(\bar{a}^{n+1}, b_i)$ for $i = 1, \dots, n$ and s.t. for any $b \in M$, $\chi(\bar{a}^{n+1}, b) \to \delta(a, b)$.
- So if $\bigvee_{i=1}^{n+1} \chi(\bar{a}^i, y)$ doesn't satisfy the requirements of the lemma, then $\exists b_{n+1} \in M$ s.t. $\models \delta(a, b_{n+1})$ but $\models \neg\chi(\bar{a}^i, b_{n+1})$ for $i = 1, \dots, n+1$ and the construction can be continued.
- But it must stop at some $n \leq N_2$.

$\square$

3.4. **Forking independence.**

- We assume T is ω-stable.
- Using this notion of rank, we can define a "free" extension of a type to a larger set of parameters that doesn't add any new restrictions.

Definition 3.18. Supp. $A \subseteq B$, $p \in S_n(A), q \in S_n(B)$ and $p \subseteq q$. We say that q is a *non-forking* extension of p if $\mathrm{RM}(q) = \mathrm{RM}(p)$.

Theorem 3.19. *(Properties of non-forking extensions) Suppose $p \in S_n(A)$ and $A \subseteq B$.*

(1) *$\exists q \in S_n(B)$, a non-forking extension of p.*
(2) *There are at most $\deg_M(p)$ n.f. extension of p in $S_n(B)$, and exactly $\deg_M(p)$ in $S_n(\mathbb{M})$.*
(3) *There is at most one $q \in S_n(B)$, a n.f. extension of p with $\deg(p) = \deg(q)$.*
In part, $\deg(p) = 1 \implies p$ has a unique n.f. extension in $S_n(B)$.
(4) *If $M \preceq \mathbb{M}$,$p \in S_n(M)$, $M \subseteq B \subseteq \mathbb{M}$ and $q \in S_n(B)$ extends p, then $q \in S_n(B)$ is a non-forking extension $\iff$ q is definable over M.*
Namely, $\forall \phi(x, y) \in L, b \in B$ we have $\phi(x, b) \in q \iff \models d_p\phi(b)$.

- Using it, we can define a notion of independence between subsets of $\mathbb{M}$.

Definition 3.20. Given A, B, C small subsets of $\mathbb{M}$, we write $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if $\operatorname{tp}(A/BC)$ doesn't fork over C.

Fact 3.21. *(Forking calculus) If T is ω-stable (or just stable), $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the folowing conditions.*

1. *Invariance: $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $\sigma \in \operatorname{Aut}(\mathbb{M}) \implies \sigma(A) \mathop{\smile\!\!\!\!\!\mid}_{\sigma(C)} \sigma(B)$.*
2. *Finite character: $A \mathop{\smile\!\!\!\!\!\mid}_C B \iff A_0 \mathop{\smile\!\!\!\!\!\mid}_C B_0$ for any finite $A_0 \subseteq A$ and $B_0 \subseteq B$.*
3. *Extension: If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then $\forall D \subseteq \mathbb{M}\, \exists \sigma \in \operatorname{Aut}(\mathbb{M}/CB)$ s.t. $\sigma(A) \mathop{\smile\!\!\!\!\!\mid}_C BD$.*
4. *Local character: if A is finite and B any set, then there is some $C \subseteq B$ with $|C| \leq |T|$ and s.t. $A \mathop{\smile\!\!\!\!\!\mid}_C B$.*
5. *Transitivity: $A \mathop{\smile\!\!\!\!\!\mid}_C B \wedge A \mathop{\smile\!\!\!\!\!\mid}_{BC} D \iff A \mathop{\smile\!\!\!\!\!\mid}_C BD$.*
6. *Symmetry: $A \mathop{\smile\!\!\!\!\!\mid}_C B \iff B \mathop{\smile\!\!\!\!\!\mid}_C A$.*
7. *Stationarity over models: if $M \prec \mathbb{M}$, $\operatorname{tp}(A/M) = \operatorname{tp}(B/M)$ and $A \mathop{\smile\!\!\!\!\!\mid}_M C, B \mathop{\smile\!\!\!\!\!\mid}_M C$, then $\operatorname{tp}(A/CM) = \operatorname{tp}(B/CM)$.*

- Moreover, we have:

Fact 3.22. *Let T be any theory, $\mathbb{M} \models T$. Assume that there is some relation $\mathop{\smile\!\!\!\!\!\mid}^*$ on small subsets of $\mathbb{M}$ satisfying (1)-(7) in Fact 3.21. Then T is stable, and $\mathop{\smile\!\!\!\!\!\mid}^* = \mathop{\smile\!\!\!\!\!\mid}$.*

Example 3.23. If T is a strongly minimal theory, $\mathop{\smile\!\!\!\!\!\mid}$-independence coincides with acl-independence.

3.5. **Canonical bases.**

Definition 3.24. A type $p \in S(A)$ is *stationary* if it has a unique non-forking extension to a *global type* $q \in S(\mathbb{M})$.

- We already saw that any typ $p \in S(M)$ over a model is stationary.
- A more careful analysis yields the following:

Lemma 3.25. *Let T be stable, $A = \operatorname{acl}^{\mathrm{eq}}(A)$. Then any $p \in S(A)$ is stationary.*

- We call the $\operatorname{tp}(a/\operatorname{acl}^{\mathrm{eq}}(A))$ the *strong type of a over A, denoted* $\operatorname{stp}(A)$.

Definition 3.26. Let $p \in S(\mathbb{M})$ be a definable global type (i.e. definable over some small $A \subseteq \mathbb{M}$) in a stable theory.

The *canonical base* of p, $\operatorname{Cb}(p)$, is $:= \operatorname{dcl}^{\mathrm{eq}}(\{\ulcorner d_p\phi(y) \urcorner : \phi(x,y) \in \mathcal{L}\})$.

- For any $\sigma \in \operatorname{Aut}(\mathbb{M})$, σ fixes the canonical parameter $d_p\phi(y) \iff$ it permutes the set $p \restriction_{\phi(x,y)}$. Hence using Lemma 3.5 we have:

Remark 3.27. Let $p \in S(\mathbb{M})$, and A a small set. Then p is A-invariant $\iff$ $\operatorname{Cb}(p) \subseteq \operatorname{dcl}^{\mathrm{eq}}(A)$.

In particular, $\operatorname{Cb}(p)$ is the smallest definably closed set in $\mathbb{M}^{\mathrm{eq}}$ over which p is invariant.

Proposition 3.28. *Let T be stable and $p \in S(\mathbb{M})$.*

1. *p doesn't fork over $A \iff \operatorname{Cb}(p) \subseteq \operatorname{acl}^{\mathrm{eq}}(A)$.*
2. *p doesn't fork over A and $p \restriction_A$ is stationary $\iff \operatorname{Cb}(p) \subseteq \operatorname{dcl}^{\mathrm{eq}}(A)$.*

Definition 3.29. Let T be stable and $A \subseteq \mathbb{M}$ be small. Let $p \in S(A)$ be a stationary type. We define $\operatorname{Cb}(p) := \operatorname{Cb}(p^*)$, where p^* is the unique global non-forking extension of p.

Proposition 3.30. *Let T be stable.*

(1) *TFAE:*
  (a) $a \mathop{\smile\!\!\!\!\!\!\;|}_A B$,
  (b) $\mathrm{Cb}(a/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$,
  (c) $\mathrm{Cb}(a/A) = \mathrm{Cb}(a/B)$.

(2) *Let $p \in S(A)$ be stationary type and let $(a_i : i < \omega)$ be a sequence in $\mathbb{M}$ such that $a_i \models p|_{Aa_{<i}}$ for all $i \in \omega$ (exists by saturation of $\mathbb{M}$). Then* $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}((a_i : i \in \omega))$.

Fact 3.31. *Moreover, if T is s.m. then for any stationary type p,* $\mathrm{Cb}(p)$ *is a* ***finite*** *set, hence an element in $\mathbb{M}^{\mathrm{eq}}$.*

- See my notes on stability at `http://www.math.ucla.edu/~chernikov/teaching/StabilityTheory285D/StabilityNotes.pdf` for more details.

4. A Closer look at pregeometries and Zilber's trichotomy principle

- Recall: every uncountably categorical model is prime over a s.m. set, hence understanding what s.m. sets can look like is crucial.
- We return to pregeometries and have a closer look at their properties.

Remark 4.1. (1) Equivalently, a pregeometry is a set X with a collection $\mathcal{C} \subseteq 2^X$ of *closed* subsets such that:
  - $\mathcal{C}$ is closed under intersections: $\mathcal{B} \subseteq \mathcal{C} \implies (\bigcap \mathcal{B}) \in \mathcal{C}$.
  - (Exchange) If $C \in \mathcal{C}$ and $a \in X \setminus C$, there is an i*mmediate closed extension* C' of C containing a, i.e.
    $\exists C' \in \mathcal{C}$ s.t. $C' \supseteq Ca$ and $\nexists C'' \in \mathcal{C}$ with $C \subsetneq C'' \subsetneq C'$.
  - (Finite charactrer) If $\mathcal{B} \subseteq \mathcal{C}$, then $\bigvee \mathcal{B} = \bigcup_{\mathcal{B}_0 \subseteq \mathcal{B} \text{ finite}} (\bigvee \mathcal{B}_0)$, where $\bigvee \mathcal{B} := \bigcap \{C \in \mathcal{C} | \bigcup \mathcal{B} \subseteq C\}$ is the smallest closed set containing all $B \in \mathcal{B}$.

(2) Setting $\bigwedge \mathcal{B} := \bigcap \mathcal{B}$ and $A \leq B \iff A \subseteq B$, $\mathcal{C}$ forms a complete lattice.
(3) Given a closure operator cl on X, define $\mathcal{C} := \{\mathrm{cl}(A) : A \subseteq X\}$.
(4) Conversely, given a set $\mathcal{C}$ of closed subsets, define $\mathrm{cl}(A) := \bigcap \{C \in \mathcal{C} : A \subseteq C\}$.

- For $A, B \subseteq X$, $\dim(A/B)$ is the cardinality of a *basis* for A over B (i.e. a subset $A' \subseteq A$ of minimal size such that $\mathrm{cl}(A'B) = \mathrm{cl}(AB)$).

Exercise 4.2. Equivalently, $\dim(A/B)$ is the length d of any chain of immediate extensions of closed sets $\mathrm{cl}(B) = C_0 \subsetneq C_1 \subsetneq \ldots \subsetneq C_d = \mathrm{cl}(AB)$.

- The *localization of X at A* is $X_A := (X, \mathrm{cl}_A)$ where $\mathrm{cl}_A(B) := \mathrm{cl}(AB)$.
- Equivalently, the closed sets of X_A are the closed sets of X which contain A.

Example 4.3. In the case of a s.m. set D with the acl closure operator, localizing at $A \subseteq D$ corresponds to adding the constant for all elements of A to the language.

- A *geometry* is a pregeometry (X, cl) s.t. $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(\{a\}) = \{a\}$ for any $a \in X$.
- If (X, cl) is any pregeometry, there is a natural associated geometry:
  Let $X_0 := X \setminus \mathrm{cl}(\emptyset)$, and consider the relation $\sim$ on X given by $a \sim b \iff \mathrm{cl}(\{a\}) = \mathrm{cl}(\{b\})$.
  By exchange, $\sim$ is an equivalence relation.

Let $\hat{X} := X_0/\sim$, and define $\hat{\mathrm{cl}}$ on $\hat{X}$ by $\hat{\mathrm{cl}}\,(A/\sim) = \{b/\sim : b \in \mathrm{cl}\,(A)\}$. Then $\left(\hat{X}, \hat{\mathrm{cl}}\right)$ is a geometry.

Exercise 4.4. Equivalently, $\left(\hat{X}, \hat{\mathrm{cl}}\right)$ is the set of dimension 1 closed sets of X with $\hat{\mathrm{cl}}$ defined by $A \in \hat{\mathrm{cl}}\,(B) \iff A \subseteq \mathrm{cl}\,(\bigcup B)$.

Definition 4.5. A pregeometry is *trivial*, or *disintegrated*, if $\mathrm{cl}\,(A) = \bigcup_{a \in A} \mathrm{cl}\,(A)$.

- Equivalently, $\bigvee = \bigcup$ in the lattice of closed sets.

Example 4.6. (1) $\mathbb{M} \models T_\infty$ is a strongly minimal set with $(\mathbb{M}, \mathrm{acl})$ a trivial pregeometry as all subsets are closed, $\mathrm{acl}\,(B) = B$.

(2) An action of a group G on an infinite set D without fixed points in the language $\left(D, (g\,(x))_{g \in G}\right)$ where $g\,(x) = g \cdot x$ is a s.m. structure with trivial pregeometry.
Closed sets are unions of G-orbits, $\mathrm{acl}\,(B) = \{g \cdot b : g \in G, b \in B\}$.

Exercise 4.7. Any pregeometry satisfies the *submodular law*: $\dim\,(A \cup B) \leq \dim\,(A) + \dim\,(B) - \dim\,(A \cap B)$.

Definition 4.8. A pregeometry (X, cl) is *modular* if for any finite-dimensional closed $A, B \subseteq X$,

$$\dim\,(A \cup B) = \dim\,(A) + \dim\,(B) - \dim\,(A \cap B)\,.$$

- In other words, $\dim\,(A/B) = \dim\,(A/A \cap B)$.

Example 4.9. Let V be a vector space over a division ring K, with closed sets the vector subspaces. For V finite dimensional, this is the acl-pregeometry of a s.m. structure $\mathcal{M} \models \mathrm{VS}_K$. By standard linear algebra, it is modular. This is not a geometry as $\mathrm{cl}\,(\emptyset) = \{0\}$ and for any $a \in V \setminus \{0\}$, $\mathrm{cl}\,(a)$ is the line through a and 0.

The corresponding *geometry* is obtained by projectivising — deleting 0 and quotiening by the action of scalar multiplication — the projective space of $\dim\,(V) - 1$.

- A form of converse holds:

Fact 4.10. *Any non-disintegrated modular geometry of* $\dim \geq 4$ *is isomorphic to a projective geometry over a division ring.*

Exercise 4.11. (1) Prove that disintegrated pregeometries are modular.

(2) Prove that for $\mathcal{M} \models \mathrm{VS}_K$, (M, acl) is not disintegrated.

- Localization of a modular pregeometry is also modular (Exercise). However, the converse is false.

Example 4.12. Let V be an infinite dimensional vector space over a division ring K, with closed sets the **affine** spaces, i.e. translates of subspaces, and $\mathrm{cl}\,(\emptyset) = \emptyset$.

This is the acl-pregeometry of the s.m. structure $(V, (\{z = \lambda x + (1 - \lambda)\, y\} : \lambda \in K))$. Here $\mathrm{cl}\,(\{a\}) = \{a\}$, hence it is a geometry.

It is not modular: parallel lines within a common plane are dependent, but have trivial intersection.

Localizing at 0 yields the projective geometry of the previous example.

Definition 4.13. (X, cl) is *locally modular* if the localization at any $e \in X \setminus \mathrm{cl}\,(\emptyset)$ is modular.

Example 4.14. Let $K \models \mathrm{ACF}$ be a saturated model (hence, of infinite transcendence degree). Then (K, acl) is not locally modular.

- Let $k \prec K$ be a subfield of transcendence degree n.
- We show that even localizing at k the geometry is not modular.
- Let a, b, x be acl-independent over k.
- Let $y = ax + b$.
- Then $\dim(k(x, y, a, b)/k) = 3$ while $\dim(k(x, y)/k) = \dim(k(a, b)/k) = 2$.
- We show $\mathrm{acl}(k(x, y)) \cap \mathrm{acl}(k(a, b)) = k$ contradicting modularity.
- To see this, suppose $d \in \mathrm{acl}(k(a, b))$ and $y \in \mathrm{acl}(k(d, x))$. Let $k_1 = \mathrm{acl}(k(d))$.
- Then $\exists p(X, Y) \in k_1[X, Y]$ an irreducible polynomial s.t. $p(x, y) = 0$.
- By model completeness of ACF, $p(X, Y)$ is still irreducible over $\mathrm{acl}(k(a, b))$.
- Thus $p(X, Y)$ is $\alpha(Y - ax - b)$ for some $\alpha \in \mathrm{acl}(k(a, b))$ which is impossible as then $\alpha \in k_1$ and $a, b \in k_1$.

- ACF's are the only known natural examples of non locally modular s.m. sets.
- Zilber's philosophy: categorical objects are not random, and are canonical. Trichotomy principle: every s.m. set should be essentially T_∞, VS_K or ACF.
- **Zilber's conjecture:** Every non locally modular s.m. set is essentially an ACF (e.g., interprets an ACF. Conversely, a theorem of Macintyre says that every infinite ω-stable field is algebraically closed).
- It was refuted by Hrushovski via a combinatorial construction: there are non locally modular s.m. sets which do not even interpret any infinite groups!
- [Hrushovski, Zilber] Holds for "Zariski geometries".

4.1. **Characterization of local modularity.**

Example 4.15. If V is a vector space, $a_1, \ldots, a_n$ is a basis for a subspace $A \subseteq V$, $b_1, \ldots, b_m$ is a basis for B subspace of V. Then any $x \in \mathrm{Span}(A \cup B)$ is the sum of some $a \in A$ and $b \in B$.

- An analogous property characterizes modularity:

Lemma 4.16. *Let (X, cl) be a pregeometry. TFAE:*

1. *(X, cl) is modular,*
2. *If $A \subseteq X$ is closed, non-empty, $b \in X$ and $x \in \mathrm{cl}(A, b)$, then $\exists a \in A$ s.t. $x \in \mathrm{cl}(a, b)$.*
3. *If $A, B \subseteq X$ are closed, non-empty, and $x \in \mathrm{cl}(A, B)$, then $\exists a \in A, b \in B$ s.t. $x \in \mathrm{cl}(a, b)$.*

Proof. (1) $\Rightarrow$ (2):

- By fin. char., WMA $\dim A$ is finite.
- If $x \in \mathrm{cl}(b)$, we are done.
- So WMA $x \notin \mathrm{cl}(b)$.
- By modularity, $\dim(Abx) = \dim A + \dim(bx) - \dim(A \cap \mathrm{cl}(bx))$ and
- $\dim(Abx) = \dim(Ab) = \dim(A) + \dim b - \dim(A \cap \mathrm{cl}(b))$.
- Because $\dim(bx) = \dim(b) + 1$, $\exists a \in A$ s.t. $a \in \mathrm{cl}(bx) \setminus \mathrm{cl}(b)$.

- By exchange, $x \in \mathrm{cl}(ba)$.

$(2) \Rightarrow (3)$

- WMA A, B are finite dim.
- Proof by induction on $\dim A$.
- If $\dim A = 0$, then (3) holds.
- Supp. $A = \mathrm{cl}(A_0 a)$, where $\dim A_0 = \dim A - 1$.
- Then $x \in \mathrm{cl}(A_0 B a)$.
- By (2) $\exists c \in \mathrm{cl}(A_0 B)$ s.t. $x \in \mathrm{cl}(c, a)$.
- By induction, $\exists a_0 \in A_0$ and $b \in B$ s.t. $c \in \mathrm{cl}(a_0, b)$.
- By (2) again, $\exists a^* \in \mathrm{cl}(a_0 a) \subseteq A$ s.t. $x \in \mathrm{cl}(a^* b)$.

$(3) \Rightarrow (1)$.

- Supp. $A, B \subseteq X$ are fin. dim. and closed.
- We prove (1) by induction on $\dim A$.
- If $\dim A = 0$, done.
- Supp. $A = \mathrm{cl}(A_0 a)$, where $\dim A_0 = \dim A - 1$ and assume inductively $\dim(A_0, B) = \dim A_0 + \dim B - \dim(A_0 \cap B)$.
- Assume $a \in \mathrm{cl}(A_0 B)$.
- Then $\dim(A_0 B) = \dim(AB)$ and, as $a \notin A_0$, $\dim A = \dim A_0 + 1$.
- As $a \in \mathrm{cl}(A_0 B)$, by (3) $\exists a_0 \in A_0$ and $b \in B$ s.t. $a \in \mathrm{cl}(a_0 b)$.
- As $a \notin \mathrm{cl}(a_0)$, by exchange $b \in \mathrm{cl}(a a_0)$. Thus $b \in A$.
- But $b \notin A_0$, as otherwise $a \in A_0$.
- Hence $\dim(A \cap B) = \dim(A_0 \cap B) + 1$ as desired.
- Next, supp. $a \notin \mathrm{cl}(A_0 B)$. Then need to show $A \cap B = A_0 \cap B$.
- Supp. $b \in B$ and $b \in \mathrm{cl}(A_0 a) \setminus \mathrm{cl}(A_0)$.
- Then by exchange $a \in \mathrm{cl}(A_0 b)$, a contradiction.

□

4.2. **Families of plane curves.**

- The following definition aims to capture the idea that any parametrized family of "curves" on the "plane" D^2 is at most 1-dimensional.

Definition 4.17. Supp. $D = \phi(\mathbb{M}) \subseteq \mathbb{M}^n$ is s.m. Then D is *linear* if $\forall p \in S_2(D)$, if $\phi(v_1) \wedge \phi(v_2) \in p$ and $\mathrm{RM}(p) = 1$, then $\mathrm{RM}(\mathrm{Cb}(p)) \leq 1$.

Theorem 4.18. *Let $D \subseteq \mathbb{M}^n$ be a s.m. set. TFAE:*

1. *for some small $B \subseteq D$, the pregeometry D_B is modular,*
2. *D is linear,*
3. *for any $b \in D \setminus \mathrm{acl}(\emptyset)$, D_b is modular,*
4. *D is locally modular.*

Proof. We will assume that $D = \mathbb{M}$ (can always be achieved without loss of generality using definability of types and working inside the induced structure on D).

$(1) \Rightarrow (2)$

- **Claim.** D is non-linear $\implies$ D_B is non-linear.
- Supp. $p \in S_2(D)$, $\mathrm{RM}(p) = 1$, $\alpha = \mathrm{Cb}(p) \in \mathbb{M}^{\mathrm{eq}}$, $\mathrm{RM}(\alpha) \geq 2$.
- If α' realizes a n.f. ext. of $\mathrm{tp}(\alpha)$ to B, then α' is a canonical base for a rank 1 type and D_B is non-linear.

- Thus, adding B to the language, WMA $B = \emptyset$ and D is modular.

- Let $p \in S_2(D)$ with $\mathrm{RM}(p) = 1$.
- Let $\phi(v_1, v_2, \bar{a})$ ne a s.m. formula in p.
- Let $(b_1, b_2) \models p(v_1, v_2)$.
- Let $X := \mathrm{acl}(\bar{a}) \cap \mathrm{acl}(b_1, b_2)$.
- By modularity, $\dim(X) = \dim(\bar{a}) + \dim(b_1, b_2) - \dim(\bar{a}, b_1, b_2)$.
- As $\dim(\bar{a}, b_1, b_2) = \dim(\bar{a}) + 1$ and $1 \leq \dim(b_1, b_2) \leq 2$, $\dim X \leq 1$.
- Thus, $\dim(b_1 b_2/\bar{a}) = \dim(b_1, b_2, \bar{a}) = \dim(\bar{a}) = \dim(b_1, b_2) - \dim(X) = \dim(b_1, b_2/X)$.
- Thus $\dim(b_1 b_2/X) = 1$ and p d.n.f. over X.
- Hence $\mathrm{Cb}(p) \subseteq \mathrm{acl}^{\mathrm{eq}}(X)$ by the above theorem, so $\mathrm{RM}(\alpha) \leq 1$.

(2) $\Rightarrow$ (3)

- Let $b \in D \setminus \mathrm{acl}(\emptyset)$. We will use Lemma 4.16.
- Supp. B is fin.dim. closed set.
- Supp. $a_1 \in \mathrm{acl}(a_2 B b)$.
- We must find: $d \in \mathrm{acl}(Bb)$ s.t. $a_1 \in \mathrm{acl}(a_2 d b)$.
- WMA: $a_1 \notin \mathrm{acl}(Bb)$, $a_2 \notin \mathrm{acl}(Bb)$ and $a_1 \notin \mathrm{acl}(a_2 b)$ (otherwise we are done).
- Thus, $\dim(a_1 a_2/b) = 2$ and $\dim(a_1 a_2/Bb) = 1$.
- Let $\alpha \in \mathrm{acl}^{\mathrm{eq}}(Bb)$ be a canonical base for the type of a_1, a_2 over $\mathrm{acl}^{\mathrm{eq}}(Bb)$.
-
- **Claim.** $\alpha \in \mathrm{acl}^{\mathrm{eq}}(a_1 a_2)$.
- As α is a canonical base, $\mathrm{RM}(a_1 a_2 \alpha) = 1$.
- As $\mathrm{RM}(a_1 a_2/\alpha) < \mathrm{RM}(a_1 a_2)$, $\mathrm{RM}(\alpha/a_1 a_2) < \mathrm{RM}(\alpha)$.
- By (2), $\mathrm{RM}(\alpha) = 1$. Thus $\alpha \in \mathrm{acl}^{\mathrm{eq}}(a_1 a_2)$.
-
- As $\mathrm{RM}(a_1 a_2/b) = 2$ and $\mathrm{RM}(a_1 a_2/\alpha) = 1$, $\alpha \notin \mathrm{acl}(b)$.
- Thus $b \notin \mathrm{acl}(\alpha)$ (as $b \notin \mathrm{acl}(\emptyset)$).
- Hence a_1, b have the same type over α, hence by saturation of $\mathbb{M}$, $\exists d \in D$ s.t. $\mathrm{tp}(a_1 a_2/\alpha) = \mathrm{tp}(bd/\alpha)$.
- then $d \in \mathrm{acl}(b\alpha) \subseteq (\mathrm{acl}(a_1 a_2 b) \cap \mathrm{acl}(Bb))$and $d \notin \mathrm{acl}\, b$.
- We have $d \notin \mathrm{acl}(a_2 b)$ (if $d \in \mathrm{acl}(a_2 b)$, then as $d \in \mathrm{acl}\, b$, we have $a_2 \in \mathrm{acl}(db) \subseteq \mathrm{acl}(Bb)$ — a contradiction).
- Thus, as $d \in \mathrm{acl}(a_1 a_2 b) \setminus \mathrm{acl}(a_2 b)$, $a_1 \in \mathrm{acl}(a_2 b d)$ — as wanted.

The other implications are easy. $\square$

- We mention one more characterization of local modularity.

Definition 4.19. Supp. T is an ω-stable theory, $\mathbb{M} \models T$. T is 1-*based* if whenever $A, B \subseteq \mathbb{M}^{\mathrm{eq}}, A = \mathrm{acl}^{\mathrm{eq}}(A), B = \mathrm{acl}^{\mathrm{eq}}(B)$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} B$.

- The following is fundamental fact:

Theorem 4.20. *Suppose that* $\mathbb{M}$ *is a s.m., non-trivial, locally modular. Then there is an infinite group definable in* $\mathbb{M}^{\mathrm{eq}}$.

- Returning to our background question, [Zilber], [Hrushovski] at a larger level of generality, proved the following:

Fact 4.21. *If D is a s.m. set in an ω-categorical theory, then D is* 1-*based (so locally modular).*

5. Group configuration theorem

5.1. Germs of definable functions.

Definition 5.1. Let $\bar{p}, \bar{q} \in S(\mathbb{M})$.

(1) If f_1, f_2 are definable partial functions defined at $\bar{p}$ (meaning $\bar{p}(x) \vdash x \in \mathrm{dom}(f_i)$), then we say that they have the same germ at $\bar{p}$ if $\bar{p}(x) \vdash f_1(x) = f_2(x)$.
(2) The *germ of f at $\bar{p}$* is the equivalence class $\tilde{f}$ of f under this equivalence relation.
(3) We write $\tilde{f} : \bar{p} \to \bar{q}$ if $\bar{p}(x) \vdash \bar{q}(f(x))$ for some (any) representative f.
(4) If $\tilde{f} : \bar{p} \to \bar{q}$ has a representative f defined over b and $a \models \bar{p}|_b$, let $\tilde{f}(a) := f(a)$.
(5) This is well-defined and $\tilde{f}(a) \models \bar{q}|_b$ (Check!).
(6) Given $\tilde{f} : \bar{p} \to \bar{q}$ and $\sigma \in \mathrm{Aut}(\mathbb{M})$, we have a well-defined germ $\sigma\left(\tilde{f}\right) : \sigma(p) \to \sigma(q)$.
(7) $b \in \mathbb{M}^{\mathrm{eq}}$ is a *code* for $\tilde{f}$ if $\forall \sigma \in \mathrm{Aut}(\mathbb{M})$, $b = \sigma(b) \iff \tilde{f} = \sigma\left(\tilde{f}\right)$.
(8) If b is a code for $\tilde{f}$, we define $\left\lceil \tilde{f} \right\rceil := \mathrm{dcl}^{\mathrm{eq}}(b)$.
(9) If p, q are stationary types and $\bar{p}, \bar{q}$ are their global non-forking extensions, a *germ at p* is defined as a germ at $\bar{p}$, and we write $\tilde{f} : p \to q$ to denote $\tilde{f} : \bar{p} \to \bar{q}$.
(10) For a stationary p, we write $p|_A = \bar{p} \restriction_A$ — the unique non-forking extension of p to A.

Remark 5.2.
- Composition of germs on global types (hence also on stationary types) is well-defined.
- Hence we have a category with objects=stationary types and morphisms=germs.
- Write $\tilde{f} : \bar{p} \tilde{\to} \bar{q}$ if $\tilde{f}$ is invertible in this category.
- Equivalently, $\tilde{f}$ is injective on $\bar{p}$ (clearly necessary, and if holds then by compactness f is already injective on some $\phi(x) \in \bar{p}$, hence f has a well-defined definable inverse).
- Note: $\left\lceil \tilde{f} \circ \tilde{g} \right\rceil \subseteq \mathrm{dcl}^{\mathrm{eq}}\left(\left\lceil \tilde{f} \right\rceil, \lceil \tilde{g} \rceil\right)$, and $\left\lceil \tilde{f}^{-1} \right\rceil = \left\lceil \tilde{f} \right\rceil$.

Remark 5.3.
- As T is stable, $\bar{p}$ and $\bar{q}$ are definable.
- Hence given an $\emptyset$-definable family f_z of partial functions (i.e. some formula $F(x, y; z)$ s.t. for any z, $f_z(x, y)$ is a partial function), equality of germs of f_b, f_c at $\bar{p}$ is a definable equivalence relation $E(b, c)$:
  consider the formula $\phi(x; b, c) := x \in \mathrm{dom}(f_b) \cap \mathrm{dom}(f_c) \wedge f_b(x) = f_c(x)$.
  Then:
  $$\tilde{f}_b = \tilde{f}_c \iff \phi(x; b, c) \in \bar{p} \iff \models d_p\phi(b, c),$$
  hence taking $E(b, c) := d_p\phi(b, c)$, $\models E(b, c) \iff \tilde{f}_b = \tilde{f}_c$.
- Then, since $\sigma\left(\tilde{f}_b\right) = \tilde{f}_{\sigma(b)}$, we have $\left\lceil \tilde{f}_b \right\rceil = \mathrm{dcl}^{\mathrm{eq}}(b/E)$.

Definition 5.4.
- If $p, q, s \in S(\emptyset)$ are stationary, *a family $\tilde{f}_s$ of germs $p \to q$* is the family $\tilde{f}_s := \left(\tilde{f}_b : b \models s\right)$ of germs at p of an $\emptyset$-definable family f_z of partial functions, which is such that $\tilde{f}_b : p \to q$ whenever $b \models s$.
- The family is

– *canonical* if b is a code for $\tilde{f}_b$ for all $b \models s$.
– *generically transitive* if $\tilde{f}_b(x) \mathop{\smile\!\!\!\!\!\!|}\, x$ for some (any) b and x such that $b \models s$ and $x \models p|_b$.

Exercise 5.5. $\tilde{f}_s$ is generically transitive $\iff$ $\forall x \models p, y \models q|_x$, there is $b \models s$ s.t. $\tilde{f}_b(x) = y$.

Proposition 5.6. *(Finding families of germs)*

Suppose $p, q, s \in S(\emptyset)$ are stationary, and $\tilde{f}_s$ is a family of germs $p \to q$. Let $b \models s$, $x \models p|_b$ and $y = \tilde{f}_b(x)$. Then

$$x \mathop{\smile\!\!\!\!\!\!|}\, b, y \mathop{\smile\!\!\!\!\!\!|}\, b, y \in \operatorname{dcl}^{\mathrm{eq}}(bx) \qquad (*)$$

Conversely, if any (b, x, y) satisfy $(*)$, let $f_b(x) = y$ be a formula witnessing $y \in \operatorname{dcl}^{\mathrm{eq}}(bx)$ (where f_b is a partial definable function), let $s := \operatorname{tp}(b), p := \operatorname{tp}(x), q := \operatorname{tp}(y)$. Then $\tilde{f}_s$ is a family of germs $p \to q$.

Moreover, we have:

(1) *$\tilde{f}_b$ is invertible $\iff$ also $x \in \operatorname{dcl}^{\mathrm{eq}}(by)$;*
(2) *the family $\tilde{f}_s$ is generically transitive $\iff$ $x \mathop{\smile\!\!\!\!\!\!|}\, y$;*
(3) *$\left\lceil \tilde{f}_b \right\rceil = \operatorname{Cb}(xy/b)$ (so $\tilde{f}_s$ is canonical $\iff$ $\operatorname{Cb}(xy/b) = b$).*

Proof. $(1), (2)$ — Exercise.

(3)

- Let $\sigma \in \operatorname{Aut}(\mathbb{M})$.
- Let $\bar{p}$ be the global non-forking extension of p, hence $\sigma(\bar{p}) = \bar{p}$.
- Let $\bar{r}$ be the global non-forking extension of $\operatorname{stp}(xy/b)$.
- Then $\bar{r}(x, y)$ is equivalent to $\bar{p}(x) \cup \{y = f_b(x)\}$.
- We have $\sigma(\operatorname{Cb}(xy/b)) = \operatorname{Cb}(xy/b) \iff \sigma(\operatorname{Cb}(\bar{r})) = \operatorname{Cb}(\bar{r}) \iff \sigma(\bar{r}) = \bar{r} \iff \bar{p}(x) \cup \left\{y = f_{\sigma(b)}(x)\right\} \equiv \bar{p}(x) \cup \{y = f_b(x)\} \iff \bar{p}(x) \vdash f_{\sigma(b)}(x) = f_b(x) \iff \sigma\left(\tilde{f}_b\right) = \tilde{f}_b$.

□

5.2. **Hrushovski-Weil.**

- This is a slightly more general version of the "group chunk theorem" of Hrushovski-Weil.

Definition 5.7. (1) A $\bigwedge$-definable homogeneous space is given by a $\bigwedge$-definable set S and a $\bigwedge$-definable group G (with the graph of the operation $*$ of the group and the action $G \times S \to S$ are relatively definable), with G acting transitively on X.
(2) S is *connected* if $\exists$ a stationary type s extending S s.t. if $g \in G$ and $b \models s|_g$, then $g * b \models s|_g$ (i.e. $\operatorname{Stab}(s) = G$). Then s is called the *generic type* of G.
(3) Generics/connectedness of a group G are defined by considering the action of G on itself.

Exercise 5.8. If (G, S) is definable of finite Morley rank, S is connected $\iff$ $\deg(S) = 1$, and $\operatorname{tp}(a)$ for $a \in S$ is generic $\iff$ $\operatorname{RM}(a) = \operatorname{RM}(S)$.

Theorem 5.9. *(Hrushovski-Weil) Let $p, s \in S(\emptyset)$ be stationary, and supp. $\tilde{f}_s$ is a generically transitive canonical family of invertible germs $p \to p$. Supp. $\tilde{f}_s$ is closed* *under inverse and generic composition, meaning that* $\forall\, b \models s$ *there exists* $b' \models s$ *s.t.* $\tilde{f}_b^{-1} = \tilde{f}_{b'}$ *and for* $b_1, b_2 \models s, b_1 \perp b_2$*, there exists* $b_3 \models s$ *s.t.* $\tilde{f}_{b_1} \circ \tilde{f}_{b_2} = \tilde{f}_{b_3}$ *and* $b_i \perp b_3$ *for* $i = 1, 2$.

Then in $\mathbb{M}^{\mathrm{eq}}$ *there is a* $\bigwedge$*-definable over* $\emptyset$ *homogeneous space* (G, S)*, a definable embedding of* s *into* G *as its unique generic type, and a definable embedding of* p *into* S *as its unique generic type, s.t. the generic action of* s *on* p *agrees with that of* G *on* S*, i.e.* $\tilde{f}_b(a) = b * a$ *for* $b \models s$ *and* $a \models p|_b$.

Proof.

- Let G be the group of germs generated by $\tilde{f}_s$.

Claim 5.10. Any element of G is a composition of two generators.

Proof.
- The family is closed under inverses $\implies$ the identity is the composition of two generators.
- s is a complete type + generic composition and automorphism $\implies$ any generator is the composition of two generators.
- So: suffices to show any composition of three generators $\tilde{f}_{b_1} \circ \tilde{f}_{b_2} \circ \tilde{f}_{b_3}$ is the composition of two.
- Let $b' \models s|_{b_1 b_2 b_3}$.
- Then $\tilde{f}_{b_1} \circ \tilde{f}_{b_2} \circ \tilde{f}_{b_3} = \tilde{f}_{b_1} \circ \tilde{f}_{b'} \circ \tilde{f}_{b'}^{-1} \circ \tilde{f}_{b_2} \circ \tilde{f}_{b_3}$.
- Now $b' \perp b_2$, so $\tilde{f}_{b'}^{-1} \circ \tilde{f}_{b_2} = \tilde{f}_{b''}$ for some $b'' \models s, b'' \perp b', b'' \perp b_2$.
- Now $b' \perp_{b_2} b_3$, so $b'' \perp_{b_2} b_3$ since $b'' \in \mathrm{dcl}^{\mathrm{eq}}(b' b_2)$.
- Since $b'' \perp b_2$, we have $b'' \perp b_3$ by transitivity of $\perp$.
- Also $b' \perp b_1$.
- So the germs $\tilde{f}_{b_1} \circ \tilde{f}_{b'}$ and $\tilde{f}_{b'}^{-1} \circ \tilde{f}_{b_2} \circ \tilde{f}_{b_3}$ appear in the family $\tilde{f}_s$ by generic composition.

□

- Now G is $\bigwedge$-definable as pairs of realizations of s, modulo equality of the corresponding composition of germs, and the group operation is defined by composition of germs.

Claim. G is connected with the generic type s: If $g \in G$, then for some (hence any) $b \models s|_g$, we have $g * b \models s|_g$.

Proof.
- This holds for $g \models s$ by assumption.
- Let $g \in G$, say $g = g_1 * g_2$ with $g_1, g_2 \models s$.
- Let $b \models s|_{g_1 g_2}$.
- Then $g_2 * b \models s|_{g_1 g_2}$, so $g_1 * g_2 * b \models s|_{g_1 g_2}$.
- Now $g = g_1 * g_2 \in \mathrm{dcl}^{\mathrm{eq}}(g_1 g_2)$. so $b \models s|_g$ and $g * b \models s|_g$.

□

- G acts generically on p by application of germs.

Claim 5.11. This action is transitive: if $a, a' \models p$, then $a' \in G * a$

Proof. Indeed, let $c \models p|_{aa'}$.
Then by generic transitivity of $\tilde{f}_s$, $\exists g, g' \models s$ s.t. $g * a = c$ and $g' * c = a'$.
Then $(g' * g) * a = a'$. □

- Define $S := (G \times p)/E$ where $(g, a)\, E\, (g', a') \iff (h * g) * a = (h * g') * a'$ for $h \models s|_{aa'gg'}$.

- E is definable by definability of the type s.
- Define the action of G by $h*((g,a)/E) := (h*g,a)/E$.
- This is well-defined:
  - if $(g,a)\,E\,(g',a')$ and $h\in G$, then if $h'\models s|_{gg'aa'h}$, then also $h'*h\models s_{gg'aa'h}$ by genericity.
  - Hence $h'*h*g*a = h'*h*g'*a'$, so $(h*g,a)\,E\,(h*g',a')$.
  - p embeds via $a\mapsto (1,a)/E$.
  - $G*p=S$, so the action is transitive.
- Finally, p is the generic type: if $g\in G$ and $a\models p|_g$, then $g*a\models p|_g$ as this is the action of a germ.
- For the group configuration theorem, we need to generalize to families of bijections between two types.

Lemma 5.12. *(Hrushovski-Weil for bijections) Supp.* $\mathrm{acl}^{\mathrm{eq}}(\emptyset)=\mathrm{dcl}^{\mathrm{eq}}(\emptyset)$*, let* $p,q,r\in S(\emptyset)$*, and supp.* $\tilde{f}_r$ *is a gen. trans. canonical family of invertible germs* $p\to q$*. Let* $b_1,b_2\models r$ *be independent and* $\tilde{f}_{b_1}^{-1}\circ\tilde{f}_{b_2}=\tilde{g}_c$ *with* $\tilde{g}_s$ *a canonical family of invertible germs* $p\to p$ *and* $s=\mathrm{tp}(c)$*, and suppose* $c\mathop{\smile\!\!\!\!\!\!\,|}\, b_i$ *for* $i=1,2$*. Then* $\tilde{g}_s$ *satisfies the assumptions, hence the conclusions, of the Hrushovski-Weil theorem.*

Proof.

- Let $c'\models s|_c$. Let $b\models r|_{c,c'}$.
- Then, as $c\mathop{\smile\!\!\!\!\!\!\,|}\, b_i$, $bc\equiv b_2c$, so say $b_1'bc\equiv b_1b_2c$.
- Similarly $bc'\equiv b_1c$, so say $bb_2'c'\equiv b_1b_2c$.
- Then $\tilde{g}_c\circ\tilde{g}_{c'}=\tilde{f}_{b_1'}^{-1}\circ\tilde{f}_b\circ\tilde{f}_b^{-1}\circ\tilde{f}_{b_2'}=\tilde{f}_{b_1'}^{-1}\circ\tilde{f}_{b_2'}$.
- Now $b_i'\mathop{\smile\!\!\!\!\!\!\,|}\, b$ by choice of b_i', since $b_1\mathop{\smile\!\!\!\!\!\!\,|}\, b_2$.
- Also $b_1'\mathop{\smile\!\!\!\!\!\!\,|}_b b_2'$, since $c\mathop{\smile\!\!\!\!\!\!\,|}_b c'$, since $c\mathop{\smile\!\!\!\!\!\!\,|}\, c'$ and $b\mathop{\smile\!\!\!\!\!\!\,|}\, cc'$.
- Hence $b_1'\mathop{\smile\!\!\!\!\!\!\,|}\, bb_2'$, so $b_1'\mathop{\smile\!\!\!\!\!\!\,|}\, b_2'$ and $b\mathop{\smile\!\!\!\!\!\!\,|}\, b_1'b_2'$.
- Hence $\tilde{f}_{b_1'}^{-1}\circ\tilde{f}_{b_2'}=\tilde{g}_{c''}$ for some $c''\models s|_{b_i'}$.
- Then as $b\mathop{\smile\!\!\!\!\!\!\,|}\, b_1'b_2'$, we have $b\mathop{\smile\!\!\!\!\!\!\,|}\, c''b_1'$, hence $c''\mathop{\smile\!\!\!\!\!\!\,|}\, b_1'b$, so $c''\mathop{\smile\!\!\!\!\!\!\,|}\, c$.
- Similarly $c''\mathop{\smile\!\!\!\!\!\!\,|}\, c'$.
- Finally, must check $\tilde{g}_s$ is gen. trans.
- Let $x\models p$, let $b_2'\models r|_x$, $b_1'\models r|_{xb_2'}$.
- Let $y:=\tilde{f}_{b_2'}(x)$, $z:=\tilde{f}_{b_1'}^{-1}(y)$.
- Then $y\mathop{\smile\!\!\!\!\!\!\,|}\, x$ by gen. trans., and $b_1'\mathop{\smile\!\!\!\!\!\!\,|}\, yx$, so $y\mathop{\smile\!\!\!\!\!\!\,|}\, xb_1'$, i.e. $y\models q|_{xb_1'}$, so $z\models p|_{xb_1'}$, in particular $x\mathop{\smile\!\!\!\!\!\!\,|}\, z$.
- Since $b_1'b_2'\equiv b_1b_2$, this proves gen. trans. of $\tilde{g}_s$.

$\square$

- In an ω-stable theory, we can obtain all these objects definably.

Fact 5.13. *Supp.* $\mathbb{M}$ *is* ω*-stable and* $G\subseteq\mathbb{M}$ *is an* $\bigwedge$*-definable group. Then* G *is definable.*

Exercise 5.14. Give an example of a type definable group in some theory T which is not definable.

5.3. **The group configuration.**

Definition 5.15. A tuple (a,b,c,x,y,z) is a *group configuration* if

- any non-collinear triple is independent,
- $\mathrm{acl}^{\mathrm{eq}}(ab)=\mathrm{acl}^{\mathrm{eq}}(bc)=\mathrm{acl}^{\mathrm{eq}}(ac)$,
- $\mathrm{acl}^{\mathrm{eq}}(ax)=\mathrm{acl}^{\mathrm{eq}}(ay)$ and $\mathrm{acl}^{\mathrm{eq}}(a)=\mathrm{Cb}(xy/a)$. Similarly for bzy and czx.

Example 5.16. Supp. (G, S) is a connected $\bigwedge$-definable (over $\emptyset$) homogeneous space. Let (a, b, x) be an independent triple with a, b generics of G and x a generic og S. Let $c = ba$, $y = cx$, $z = bx$ (then also $y = az$). Then (a, b, c, x, y, z) is a group configuration (of (G, S)).

- Remarkably, the converse is also true:

Theorem 5.17. *[Hrushovski] Supp. (a, b, c, x, y, z) is a group configuration. Then, after possibly expanding the language by parameters B with $B \mathop{\smile\!\!\!\!\!\!|}\; abcxyz$, there is a connected $\bigwedge$-definable homogeneous space (G, S) and a group configuration (a', b', c', x', y', z') of (G, S) such that a' is interalgebraic with a, b' is interalgebraic with b, etc.*

Proof. Idea:

(1) Reduce $\mathrm{acl}^{\mathrm{eq}}$ to $\mathrm{dcl}^{\mathrm{eq}}$: replacing the elements of the original group configuration with interalgebraic ones, we may assume that (b, z, y) define a generically transitive canonical family of invertible germs via Proposition 5.6.
(2) Prove that the family of germs obtained this way satisfies the independence assumption of Lemma 5.12 — thus obtaining a $\bigwedge$-definable homogeneous space.
(3) Connect the resulting homogeneous space to the original group configuration.

□

For the details, see Pillay, "Geometric stability theory", Chapter 5.4.

Remark 5.18. See also `https://terrytao.wordpress.com/2013/11/16/qualitative-probability-theory-t` for a different exposition.

Corollary 5.19. *Let D be a non-trivial, locally modular, strongly minimal set. Then there is an infinite group definable in $\mathbb{M}^{\mathrm{eq}}$.*

Proof.
- Expand by parameters to make D modular (possibly by definition).
- By non-triviality (after possibly expanding by more parametersm $\exists a, b, c \in D$ s.t.

$$\mathrm{RM}\,(ab) = \mathrm{RM}\,(bc) = \mathrm{RM}\,(ca) = \mathrm{RM}\,(abc) = 2.$$

- Let $b'c' \models \mathrm{stp}\,(bc/a)\,|_a$.
- Then $\mathrm{RM}\,(bcb'c') = 2$.
- Then by modularity $\mathrm{RM}\,(\mathrm{acl}\,(bc') \cap \mathrm{acl}\,(b'c)) = 1$.
- Then there is $d \in (\mathrm{acl}\,(bc') \cap \mathrm{acl}\,(b'c)) \setminus \mathrm{acl}\,(\emptyset)$.
- Then (a, b, c, b', c', d) is a group configuration.
- By Theorem 5.17 in $\mathbb{M}^{\mathrm{eq}}$ there is a $\bigwedge$-definable group, which is definable by Fact 5.13.

□

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