LECTURE NOTES ON FORKING

ARTEM CHERNIKOV

Part 1. Preliminaries, model-theoretic notions of smallness, preindependence relations

1. Lecture 1

1.1. Types.

Most of the time we fix a complete countable first-order theory T in a language L, and we let $\mathbb{M}$ be a very saturated model of it, a "monster" model. All our sets and models are subsets and elementary submodels of $\mathbb{M}$. A set is called *small* if it is cardinality is smaller than the saturation of $\mathbb{M}$. For a set A, we let $\mathrm{Def}(A)$ be the boolean algebra of A-definable subsets of $\mathbb{M}$, i.e. sets of the form $\{x \in \mathbb{M} : \mathbb{M} \models \phi(x,a)\}$ for $\phi(x,y) \in L$ and a a tuple of elements from A. The space of types in the variable x over A, $S_x(A)$, is the Stone dual of $\mathrm{Def}(A)$. It is the space of ultrafilters on $\mathrm{Def}(A)$, a compact Hausdorff totally-disconnected space with the clopen basis given by $\langle \phi(x,a) \rangle = \{p \in S(A) : \phi(x,a) \in p\}$. Elements of $S(\mathbb{M})$ are called *global* types. We will write $a \equiv_C b$ if $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$. Then $a \equiv_C b$ if and only if there is an automorphism σ of $\mathbb{M}$ fixing C and such that $\sigma(a) = b$.

1.2. Indiscernible sequences.

Definition 1. Let O be a linear order. A sequence $(a_i)_{i \in O}$ of tuples is called *indiscernible* over a set B if for any $i_0 < \ldots < i_n \in O$ and $j_0 < \ldots < j_n \in O$ we have $a_{i_0} \ldots a_{i_n} \equiv_B a_{j_0} \ldots a_{j_n}$.

Example 2.
(1) A constant sequence is indiscernible over any set.
(2) In the theory of equality, any sequence is indiscernible.
(3) Any increasing (or decreasing) sequence of singletons in a linear order is indiscernible.
(4) Any basis in a vector space is an indiscernible sequence (and in fact is an indiscernible *set*, i.e. for any $i_0 \neq \ldots \neq i_n \in O$ and $j_0 \neq \ldots \neq j_n \in O$ we have $a_{i_0} \ldots a_{i_n} \equiv_B a_{j_0} \ldots a_{j_n}$).

The following is a standard method of finding indiscernible sequences in an arbitrary theory.

Fact 3. *Let $(a_i)_{i \in \lambda}$ be a sequence of tuples with $|a_i| < \kappa$ and a set B be given. If $\lambda \geq \beth_{\left(2^{\kappa + |B| + |T|}\right)^+}$ there is a B-indiscernible sequence $(a'_i)_{i \in \omega}$ such that for every $n \in \omega$ there are $i_0 < \ldots < i_n \in \kappa$ such that $a'_0 \ldots a'_n \equiv_B a_{i_0} \ldots a_{i_n}$.*

Proof. Using the Erdős-Rado theorem, see e.g. [BY03, Lemma 1.2]. □

Remark 4. In general, it is not always possible to find an *infinite* indiscernible subsequence, no matter how long the sequence we start with (unless it is a compact cardinal). This phenomena can occur even in NIP theories, (but not in strongly dependent theories, see [KS10]). In stable theories, if a sequence is sufficiently long, then one can actually find an infinite indiscernible subsequence (of the same length as the original sequence).

2. Notions of smallness for definable sets

2.1. Keisler measures.

Definition 5. (1) A *Keisler measure* is a finitely-additive probability measure on $\mathrm{Def}(\mathbb{M})$.
(2) A Keisler measure μ is invariant over A if $a \equiv_A b$ implies $\mu(\phi(x,a)) = \mu(\phi(x,b))$.

From now onwards by a measure we always mean a Keisler measure.

Example 6. Random graph, a natural measure given by saying that for any two points, the edge between them occurs with probability $\frac{1}{2}$.

Remark 7. Every Keisler measure can be extended in a unique way to a Borel measure on the σ-algebra generated by $\mathrm{Def}(\mathbb{M})$ by Caratheodory's theorem (given $B = \bigcup_{i\in\omega} B_i$ with $B_i, B \in \mathrm{Def}(\mathbb{M})$, by compactness we have that $B = \bigcup B_{i<n}$ for some $n \in \omega$).

2.2. Invariant types.

In the special case of a $0-1$ invariant Keisler measure we get the familiar notion of an invariant type.

Definition 8. A global type $p(x) \in S(\mathbb{M})$ is called *invariant* over C if it is invariant under all automorphisms of $\mathbb{M}$ fixing C. That is, for every $a \equiv_C b$ and $\phi(x,y) \in L$, $\phi(x,a) \in p \Leftrightarrow \phi(x,b) \in p$.

Fact 9. *Let $p(x)$ be a global type invariant over C. For $i \in \omega$, let $a_i \models p|_{Ca_{<i}}$. Then $\bar{a} = (a_i)_{i\in\omega}$ is an C-indiscernible sequence. Besides, for any other sequence $\bar{b} = (b_i)_{i\in\omega}$ such that $b_i \models p|_{Cb_{<i}}$, we have $\bar{a} \equiv_C \bar{b}$.*

How does one find invariant types in an arbitrary theory?

Fact 10. (1) *Let p be a partial type (over $\mathbb{M}$) finitely satisfiable in A (i.e. for every $\phi(x,a) \in p$ there is some $b \in A$ such that $\models \phi(b,a)$). Then there is $p' \in S(\mathbb{M})$ finitely satisfiable in A and such that $p \subseteq p'$.*
(2) *If M is a model, then every type $p \in S(M)$ is finitely satisfiable in M.*
(3) *Every global type finitely satisfiable in A is invariant over A.*
(4) *Combining, every type over a model has a global M-invariant extension.*

Remark 11. Note that the space of A-invariant global types is a closed subset of $S(\mathbb{M})$ (as it equals $\bigcap_{\phi\in L, a\equiv_A b\in\mathbb{M}} \langle \phi(x,a) \leftrightarrow \phi(x,b)\rangle$), thus compact. Similarly, the space of type types finitely satisfiable in A is a closed subset of A — it equals $\bigcap_{\phi(x,a)\in L(\mathbb{M}), \phi(A,a)=\emptyset} \langle \neg\phi(x,a)\rangle$. It can also be described as the closure of the set of types realized in A, i.e. of $\{\mathrm{tp}(a/\mathbb{M}) : a \in A\}$.

3. Lecture 2

3.1. Generically prime ideals.

Definition 12. Recall that a set $I \subseteq \mathrm{Def}(\mathbb{M})$ is an *ideal* if:
(1) $\emptyset \in I$,

(2) $\phi(x,a) \vdash \psi(x,b)$ and $\psi(x,b) \in I$ implies $\phi(x,a) \in I$,
(3) $\phi(x,a) \in I$ and $\psi(x,b) \in I$ implies $\phi(x,a) \vee \psi(x,b) \in I$.

An ideal I is invariant over A if $\phi(x,a) \in I$ and $a \equiv_A b$ implies $\phi(x,b) \in I$.

As usual, an ideal I in $\mathrm{Def}(\mathbb{M})$ is *prime* if whenever $\phi(x,a) \wedge \psi(x,b) \in I$, then either $\phi(x,a) \in I$ or $\psi(x,b) \in I$. However in a boolean algebra prime ideals correspond to complete types in $S_x(\mathbb{M})$ (as for any $\phi(x,b)$, $\phi(x,b) \wedge \neg\phi(x,b) \in I$, so either $\phi(x,b)$ or $\neg\phi(x,b)$ has to belong to I). We introduce a weaker notion.

Definition 13. An ideal I in $\mathrm{Def}(\mathbb{M})$ is *generically prime* if there is a cardinal κ such that for any $\{\phi_i(x,a_i)\}_{i\in\kappa}$ with $\phi_i(x,a_i) \notin I$, there are some $i < j \in \kappa$ with $\phi_i(x,a_i) \wedge \phi_j(x,a_j) \notin I$.

So a prime ideal is generically prime with $\kappa = 2$. Generically prime ideals are called "$S1$-ideals" in [Hru12].

Proposition 14. *An A-invariant ideal I is generically prime if and only if it satisfies the following property: given an A-indiscernible sequence $(a_i)_{i\in\omega}$, if $\phi(x,a_0) \wedge \phi(x,a_1) \in I$ then $\phi(x,a_0) \in I$.*

Proof. Assume that we have an A-indiscernible sequence $(a_i)_{i\in\omega}$ such that $\phi(x,a_0) \wedge \phi(x,a_1) \in I$ but $\phi(x,a_0) \notin I$. By compactness, indiscernibility and invariance of I, for any κ we can find a sequence $(a_i)_{i\in\kappa}$ such that $\phi(x,a_i) \notin I$ and $\phi(x,a_i) \wedge \phi(x,a_j) \in I$ for all $i \neq j \in \kappa$, thus I is not generically prime.

Conversely, assume that I is not generically prime. Then for any κ we can find $(\phi_i(x,a_i))_{i\in\kappa}$ with $\phi_i(x,a_i) \notin I$ and $\phi_i(x,a_i) \wedge \phi_j(x,a_j) \in I$. Taking κ large enough and applying Fact 3 we find an A-indiscernible sequence starting with a_i, a_j for some $i \neq j$ and such that $\phi_i = \phi_j$. □

In fact a refinement of this proof shows that if an ideal I is generically prime and invariant over A, then one can always take $\kappa = \left(2^{|A|+|T|}\right)^+$ in the definition.

Proposition 15. *For any A-invariant measure μ, its 0-ideal is generically prime.*

Proof. Let (a_i) be A-indiscernible and assume that $\mu(\varphi(x,a)) > \frac{1}{k}$ for some $k \in \omega$ but $\mu(\varphi(x,a_0) \wedge \varphi(x,a_1)) = 0$. By A-invariance and the inclusion-exclusion formula we see that $\mu\left(\bigvee_{i<k+1} \varphi(x,a_i)\right) > 1$, a contradiction. □

3.2. Dividing and forking.

Definition 16. (1) A formula $\phi(x,a)$ *divides* over B if there is a sequence $(a_i)_{i\in\omega}$ and $k \in \omega$ such that $a_i \equiv_B a$ and $\{\phi(x,a_i)\}_{i\in\omega}$ is k-inconsistent. Equivalently, if there is an B-indiscernible sequence $(a_i)_{i\in\omega}$ starting with a and such that $\{\phi(x,a_i)\}_{i\in\omega}$ is inconsistent (by compactness and Fact 3).

(2) A formula $\phi(x,a)$ *forks* over B if it belongs to the ideal generated by the formulas dividing over B, i.e. if there are $\psi_i(x,c_i)$ dividing over B for $i < n$ and such that

$$\phi(x,a) \vdash \bigvee_{i<n} \psi_i(x,c_i).$$

(3) We denote by $\mathbf{F}(B)$ the ideal of formulas forking over B. It is invariant over B.

Example 17. In general there are formulas which fork, but do not divide. Consider the unit circle around the origin on the plane, and a ternary relation $R(x,y,z)$ on it which holds if and only if y is between x and z, ordered clock-wise. Check:

(1) This theory eliminates quantifiers.
(2) There is a unique 2-type $p(x,y)$ over $\emptyset$ consistent with "$x \neq y$".
(3) $R(a,y,c)$ divides over $\emptyset$ for any a,c.
(4) The formula "$x = x$" forks over $\emptyset$ (but it does not divide, of course — no formula can divide over its own parameters).

Definition 18. A (partial) type does not divide (fork) over B if it does not imply any formula which divides (resp. forks) over B.

Remark 19. Let $p(x)$ is a partial type. It does not fork over B if and only if it is contained in a global type which does not fork over B. This is because every ideal in a boolean algebra extends to a prime (equivalently maximal) ideal.

Proposition 20. $\mathbf{F}(B)$ *is contained in every generically prime ideal invariant over* B.

Proof. It is enough to show that if $\varphi(x,a)$ divides over B and I is a generically prime ideal, then $\varphi(x,a) \in I$. We use the equivalence from Proposition 14. Let $(a_i)_{i\in\omega}$ be indiscernible over B with $a_0 = a$ and $\{\varphi(x,a_i)\}_{i\in\omega}$ inconsistent. If $\varphi(x,a_0) \notin I$, then by induction using that I is generically prime (and that if $(a_i)_{i\in\omega}$ is indiscernible over B, then $(a_{2i}a_{2i+1})_{i\in\omega}$ is indiscernible over B), we see that $\bigwedge_{i<k}\varphi(x,a_i) \notin I$ for all $k \in \omega$. But as $\emptyset \in I$ this would imply that $\{\varphi(x,a_i)\}$ is consistent, a contradiction. □

Corollary 21. *In particular if* $p(x) \in S(\mathbb{M})$ *is invariant over* B*, then it does not fork over* B*.*

3.3. **Three fundamental ideals.** Notice that any intersection of B-invariant gp ideals is still B-invariant and gp. And the same for 0-ideals of B-invariant Keisler measures. Thus the following objects exist.

Definition 22. (1) Let $\mathbf{GP}(A)$ be the smallest generically prime ideal invariant over A.
(2) Let $\mathbf{0}(A)$ be the ideal of formulas which have measure 0 with respect to every A-invariant Keisler measure.

Summing up the previous observations, we have the following picture:

Proposition 23. *In any theory and for any set* A, $\mathbf{F}(A) \subseteq \mathbf{GP}(A) \subseteq \mathbf{0}(A)$.

Example 24. There are theories with $\mathbf{F}(A) \subsetneq \mathbf{GP}(A)$, equivalently theories in which the forking ideal is not generically prime. Look at the triangle-free random graph (i.e. the model completion of the theory of graphs saying that there are no triangles — it exists and eliminates quantifiers, an important property for us is that it embeds any finite graph without triangles). Then we have:

(1) $R(x,a)$ does not divide for any a (as any indiscernible sequence of singletons has to be an anti-clique).
(2) $R(x,a) \wedge R(x,b)$ divides for any $a \neq b$ (witnessed by a sequence (a_ib_i) such that $R(a_i,b_j) \Leftrightarrow i \neq j$).
(3) Thus for any infinite indiscernible sequence of singletons (a_i), $R(x,a_0)$ does not divide while $R(x,a_0) \wedge R(x,a_1)$ does.

Problem 25. Hrushovski had suggested an example of a (simple) theory in which $\mathbf{F}(\emptyset) = \mathbf{GP}(\emptyset) \subsetneq \mathbf{0}(\emptyset)$. I don't think there are known examples of $\mathbf{F}(A) \subsetneq \mathbf{GP}(A) \subsetneq \mathbf{0}(A)$ and of $\mathbf{F}(A) \subsetneq \mathbf{GP}(A) = \mathbf{0}(A)$. In NIP theories and in many natural simple theories, for example in pseudo-finite fields, we have that $\mathbf{F}(A) = \mathbf{0}(A)$.

In the following lectures our aim will be to prove that $\mathbf{F}(A) = \mathbf{GP}(A)$ in the class of NTP_2 theories (over an extension base).

4. Lecture 3

Given an ideal of "small" sets, we can define a "preindependence" relation (see e.g. [Adl05]).

4.1. **Preindependence relations.** We write $a \mathop{\smile\!\!\!\!\!\!\,|}^d_C b$, $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$ to denote that $\mathrm{tp}(a/bC)$ does not divide, respectively does not fork, over C. Of course if $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$ then $a \mathop{\smile\!\!\!\!\!\!\,|}^d_C b$.

Lemma 26. (1) *If $p \in S(M)$, $A \subset M$ and M is $|A|^+$-saturated, then p divides over A if and only if p forks over A.*
(2) *$a \mathop{\smile\!\!\!\!\!\!\,|}_C b$ if and only if $a \mathop{\smile\!\!\!\!\!\!\,|}^d_C M$ for some $|Cb|^+$-saturated model $M \supseteq Cb$.*

Proof. (1) By definition, if p forks over A then $p(x) \vdash \bigvee_{i \le n} \phi_i(x, a_i)$ for some $\phi_i(x, a_i)$ dividing over A, $a_i \in \mathbb{M}$. By compactness there is some finite $p_0 \subset p$ with parameters from a finite set B, such that still $p_0(x) \vdash \bigvee_{i \le n} \phi_i(x, a_i)$. By saturation of M there are $a'_{\le n}$ in M such that $a'_{\le n} \equiv_{AB} a_{\le n}$. Then we still have that $p_0(x) \vdash \bigvee_{i \le n} \phi_i(x, a'_i)$, so $\phi_k(x, a'_k) \in p$ for some $k \le n$. As $\phi_k(x, a'_k)$ still divides over A, by A-invariance of $\mathbf{F}(A)$, we conclude that $p(x)$ divides over A.

(2) Right to left is obvious by (1) and definition of $\mathop{\smile\!\!\!\!\!\!\,|}$. For left to right, let M be an arbitrary $|Cb|^+$-saturated model containing Cb. If $\mathrm{tp}(a/bC)$ does not fork over C, then by Fact it is extends to a type $p(x) \in S(M)$ non-forking over C. Let $a' \models p$; as $a \equiv_{bC} a'$, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/bC)$ with $\sigma(a') = a$. But then we have $a \mathop{\smile\!\!\!\!\!\!\,|}_C \sigma(M)$ and $\sigma(M)$ is still a sufficiently saturated model containing bC. □

Lemma 27. *The following are equivalent:*

(1) $a \mathop{\smile\!\!\!\!\!\!\,|}^d_C b$
(2) *For any C-indiscernible sequence $\bar{b} = (b_i)_{i \in \omega}$ starting with b, we can find a sequence $\bar{b}' \equiv_{Cb} \bar{b}$ which is indiscernible over aC.*

Proof. Standard, see e.g. [Cas07, Lemma 3.4]. □

Proposition 28. *Properties of $\mathop{\smile\!\!\!\!\!\!\,|}$ in arbitrary theories ("non-commutative forking calculus"):*

(1) *Invariance under automorphisms: $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$ if and only if $\sigma(a) \mathop{\smile\!\!\!\!\!\!\,|}_{\sigma(C)} \sigma(b)$, for any $\sigma \in \mathrm{Aut}(\mathbb{M})$.*
(2) *Finite character: $a \not\mathop{\smile\!\!\!\!\!\!\,|}_C b$ implies that $a' \not\mathop{\smile\!\!\!\!\!\!\,|}_C b'$ for some finite $a' \subseteq a, b' \subseteq b$.*
(3) *Monotonicity: $aa' \mathop{\smile\!\!\!\!\!\!\,|}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$.*
(4) *Base monotonicity: $a \mathop{\smile\!\!\!\!\!\!\,|}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!\,|}_{Cb'} b$.*
(5) *Left transitivity: $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_{aC} b$ implies $aa' \mathop{\smile\!\!\!\!\!\!\,|}_C b$.*
(6) *Right extension: if $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$, then for any d there is $d' \equiv_{bC} d$ such that $a \mathop{\smile\!\!\!\!\!\!\,|}_C bd'$.*

Proof. (1), (2), (3) are clear from the definition of forking.

(6) By Lemma 26 there is some $|Cb|^{+}$-saturated model $M \supseteq Cb$ such that $a \mathop{\smile\!\!\!\!\!|}_C M$. But then for any d we can realize $\mathrm{tp}(d/bC)$ by some $d' \in M$ by saturation, and $a \mathop{\smile\!\!\!\!\!|}_C bd'$ by (3).

(5) First we check it for $\mathop{\smile\!\!\!\!\!|}^{d}$: by Lemma 27 we want to show that for every C-indiscernible sequence $\bar{b}$ starting with b, we can find some $\bar{b}' \equiv_{Cb} \bar{b}$ which is indiscernible over $aa'C$. So first by Lemma 27 and the assumption find some $\bar{b}_1 \equiv_{Cb} \bar{b}$ and indiscernible over aC, and then find $\bar{b}_2 \equiv_{Cab} \bar{b}_1$ and indiscernible over $aa'C$. But as $\bar{b}_2 \equiv_{Cb} \bar{b}_1 \equiv_{Cb} \bar{b}$ we are done. Now for forking. Let $M_1 \supseteq Cb$ be saturated enough with $a \mathop{\smile\!\!\!\!\!|}_C M_1$ (by Lemma 26), and let $M_2 \equiv_{abC} M_1$ such that $a' \mathop{\smile\!\!\!\!\!|}_{aC} M_2$ — exists by (6). It then follows that $a \mathop{\smile\!\!\!\!\!|}_C M_2$ by invariance, and together with $a' \mathop{\smile\!\!\!\!\!|}_{aC} M_2$ it implies $aa' \mathop{\smile\!\!\!\!\!|}_C^{d} M_2$. As M_2 is saturated enough it implies that $aa' \mathop{\smile\!\!\!\!\!|}_C M_2$ hence $aa' \mathop{\smile\!\!\!\!\!|}_C b$ by (3). □

We will call any relation on small subsets of $\mathbb{M}$ satisfying these properties a *preindependence* relation. Ideally one would like to work with such a relation axiomatically. But before that we have to establish its properties by doing some "dirty" combinatorial work.

Problem 29. Is this a complete list of axioms for forking? I.e., is it true that if some property of forking holds in all theories, then it should be possible to deduce it from these axioms. Of course one should formalize the question correctly first.

Exercise 30. Let $a \mathop{\smile\!\!\!\!\!|}_C^{u} b$ denote that $\mathrm{tp}(a/bC)$ if finitely satisfiable in C. Let $a \mathop{\smile\!\!\!\!\!|}_C^{i} b$ denote that $\mathrm{tp}(a/bC)$ has a global extension which is Lascar-invariant over bC. . Check that both are preindependence relations.

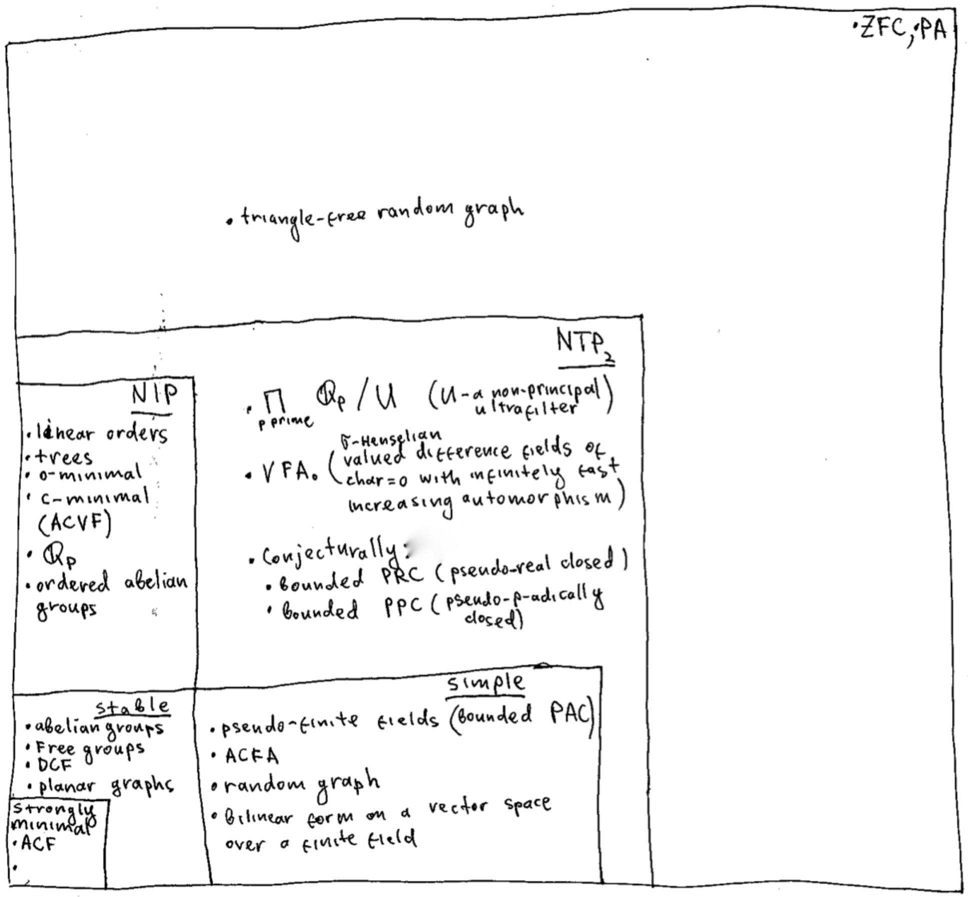
A preindependence relation is an *independence relation* if it satisfies symmetry ($a \mathop{\smile\!\!\!\!\!|}_C b \Leftrightarrow b \mathop{\smile\!\!\!\!\!|}_C a$). It does not hold in general, of course (consider dense linear order and $a < b < c$, it is easy to check that $ac \mathop{\smile\!\!\!\!\!|} b$ but $b \not\mathop{\smile\!\!\!\!\!|} ac$).

Proposition 31. *The following are equivalent:*

1. $\mathbf{F}(C)$ *is generically prime.*
2. $\mathop{\smile\!\!\!\!\!|}$ *satisfies the chain condition over* C*: for any* C*-indiscernible sequence* $\bar{b} = (b_i)_{i\in\omega}$ *starting with* b *and* $a \mathop{\smile\!\!\!\!\!|}_C b$*, we can find a sequence* $\bar{b}' \equiv_{Cb} \bar{b}$ *which is indiscernible over* aC ***and in addition*** $a \mathop{\smile\!\!\!\!\!|}_C \bar{b}$.

Proof. See [BYC12, Lemma 2.2]. □

4.2. **Shelah's classification theory.** Here is a (incomplete) map of the space of complete countable first-order theories (the word "space" is not metaphorical, because in fact first-order theories naturally form a Polish space, and then one could study the descriptive complexity of classes of theories described here).



- Most theories are *wild* of course, meaning that combinatorics of definable sets leaves no hope for an analysis (perhaps one could even prove that a "random" first-order theory is wild in a precise sense). For example theories such as ZFC or Peano arithmetic are highly undecidable and encode all mathematics, while (generalized) stability theory only deals with theories devoid of the Gödelian phenomena.
- At the opposite end: Morley proved that if T has a unique model of some uncountable cardinality κ (up to an isomorphism), then it has a unique model of each uncountable cardinality. He conjectured that the function $f_T(\kappa) = |\{M \models T : |M| = \kappa\}|$ is non-decreasing on uncountable cardinals.
- Historically stability theory started with Shelah's approach to Morley's conjecture: he decided to describe all possible functions $f_T(\kappa)$ for all T's.
- The approach was to isolate "dividing lines": usually a combinatorial property of a first-order theory, formulated in terms of ability to encode certain combinatorial configuration (linear order, tree, random graph, etc) such that for theories satisfying it one can prove a non-structure theorem (e.g. that $f_T(\kappa) = 2^\kappa$, that is maximal) while for theories not satisfying it one can develop some structure theory (better understanding of types, definable sets and models, e.g. restricting possible values of $f_T(\kappa)$).

- This programme culminated successfully: after isolating sufficiently many refining dividing lines Shelah proved Morley's conjecture [She90], and later all possible f_T's were described [HHL00]. For this work it was enough to consider stable theories, and a lot of tools were developed: forking calculus and ranks, multiplicity theory, weight, etc).
- However, most theories are not stable, so people started generalizing techniques from stability theory to larger contexts (simple theories [Wag00], NIP [Adl08], and now also NTP_2 [Che12]).
- Power of these dividing lines comes from the fact that they are somehow "canonical", for example admit equivalent characterizations by internal combinatorial properties, like not being able to encode certain graphs, as well as by "geometric" properties (properties of $\mathop{\smile\!\!\!\!\!\!\mid}$, behavior of indiscernible sequences or interaction between models).
- For example, specifically for forking, later in the course we will see the following characterizations:
  - T is NTP_2 if and only if every strictly invariant sequence witnesses dividing.
  - T is simple if and only if $\mathop{\smile\!\!\!\!\!\!\mid}$ is symmetric, if and only if $\mathop{\smile\!\!\!\!\!\!\mid}$ has *local character* (i.e. for any a and B there is some $B_0 \subseteq B$, $|B_0| \leq |T|$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$), if and only if it is NTP_2 and $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the *independence theorem* (3-amalgamation).
  - T is NIP if and only if $\mathop{\smile\!\!\!\!\!\!\mid}$ is *exponentially bounded* (for a set A, there are at most $2^{|A|}$ global types non-forking over A) if and only if it is NTP_2 and $\mathop{\smile\!\!\!\!\!\!\mid}$ is *bounded* (i.e. every type has only boundedly many global non-forking extensions).

Part 2. Forking in NTP_2 theories

5. Lecture 4

5.1. NTP_2 theories.

Definition 32. (1) A formula $\phi(x,y)$ has TP_2 (the tree property of the second kind) if there are $(a_{i,j})_{i,j\in\omega}$ and $k \in \omega$ such that:
   (a) $\{\phi(x,a_{i,j})\}_{j\in\omega}$ is k-inconsistent for every $i \in \omega$,
   (b) $\{\phi(x,a_{i,f(i)})\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$.

(2) A formula is NTP_2 if it is not TP_2. A theory is NTP_2 if it implies that every formula is NTP_2.

Definition 33. Assume that we are given sequences $(\bar{a}_i)_{i\in\kappa}$ with $\bar{a}_i = (a_{i,j})_{j\in\omega}$. We say that $(\bar{a}_i)_{i\in\kappa}$ are *mutually-indiscernible* over A if for every $i \in \kappa$, $\bar{a}_i$ is indiscernible over $A\bar{a}_{\neq i}$.

There is a natural generalization of Fact 3.

Fact 34. *Assume that we are given sequences $(\bar{a}_i)_{i\leq n}$ for $n \in \omega$ with $\bar{a}_i = (a_{i,j})_{j\in\lambda}$ and a set A. if λ is sufficiently large with respect to $|A| + |a_{i,j}|$ then we can find $(\bar{a}'_i)_{i\leq n}$ which are mutually indiscernible over A and such that:*

- *for any m there are $(j_{i,k})_{i\leq n,k\leq m}$ in λ such that* $a_{0,0} \dots a_{0,m} \dots a_{n,0} \dots a_{n,m} \equiv_A a_{0,j_{0,0}} \dots a_{0,j_{0,m}} \dots a_{n,0} \dots a_{n,j_{n,m}}$.

Proof. By a repeated use of Fact 3, taking λ so that we can iterate it n times. □

Remark 35. In fact if T has TP_2, then there is a formula and an array as in the definition with x singleton, $k = 2$ and $(\bar{a}_i)_{i\in\omega}$ mutually indiscernible, see [Che12, Section 1]. This is very useful when we want to show that some particular structure is NTP_2. Also if $\phi_1(x, y_1)$ and $\phi_2(x, y_2)$ are NTP_2 then $\phi(x, y_1 y_2) = \phi_1(x, y_1) \vee \phi_2(x, y_2)$ is NTP_2. This is the only boolean operation preserving NTP_2 (see Example 37).

Example 36. The following theories are NTP_2 (and don't fit into any smaller class in the picture):

1. Let U be an ultrafilter on the set of prime numbers. Let $K = \prod_{p \text{ prime}} \mathbb{Q}_p/U$, then $\mathrm{Th}(K)$ is NTP_2 (in your favorite language for valued fields) , see[Che12, Section 6].
2. Certain σ-Henselian valued difference fields of characteristic 0 with contractive automorphisms, see [CH12].

Example 37. On the other hand, the triangle free random graph has TP_2. We can find $(a_{i,j} b_{i,j})_{i,j<\omega}$ such that $R(a_{i,j}, b_{i,k})$ for every i and $j \neq k$, and this are the only edges around. But then $\{xRa_{i,j} \wedge xRb_{i,j}\}_{j<\omega}$ is 2-inconsistent for every i as otherwise it would have created a triangle, while $\{xRa_{i,f(i)} \wedge xRb_{i,f(i)}\}_{i<\omega}$ is consistent for any $f : \omega \to \omega$. Note that the formula xRy is NTP_2, thus demonstrating that a conjunction of two NTP_2 formulas need not be NTP_2 (and so for the negation).

5.2. **Towards forking=dividing over models in** NTP_2**.** All the material in this section is from [CK12].

Definition of dividing says "exists an indiscernible sequence such that...", but it is not true that all indiscernible sequences give the same answer.

Exercise 38. Find an example of $\phi(x, a)$ dividing over B and a B-indiscernible sequence $(a_i)_{i\in\omega}$ starting with a but such that $\{\phi(x, a_i)\}_{i\in\omega}$ is *consistent*.

We are going to show that certain kind of indiscernible sequences in a "sufficiently free" position always give the right answer (i.e. witness dividing in case the formula divides).

Definition 39. A global type $p(x) \in S(\mathbb{M})$ is *strictly invariant* over a small set A if:

- it is invariant over A,
- for $B \supseteq A$, if $a \models p|_B$ then $B \mathop{\smile\!\!\!\!\!\!|}_A a$.

Lemma 40. *If $p(x) \in S(\mathbb{M})$ is invariant over A, $\bar{a} = (a_i)$ is an A-indiscernible sequence and $b \models p|_{A\bar{a}}$ then $\bar{a}$ is indiscernible over bA.*

Proof. For any $\phi \in L(A)$ and $i_0, \dots, i_n$ we have $\models \phi(b, a_0, \dots, a_n) \Leftrightarrow \phi(x, a_0, \dots, a_n) \in p \Leftrightarrow \phi(x, a_{i_0}, \dots, a_{i_n})$ (by invariance of p and indiscernibility of $\bar{a}$ over A) $\Leftrightarrow \phi(b, a_{i_0}, \dots, a_{i_n})$. □

Lemma 41. *Let $p(x) \in S(\mathbb{M})$ be strictly invariant over A, $(a_i)_{i\in\omega}$ is a Morley sequence in p over A. Let $\bar{a}_i$ be an A-indiscernible sequence starting with a_i. Then there are $(\bar{a}'_i)_{i\in\omega}$ mutually indiscernible over A and such that $\bar{a}'_i \equiv_{a_i A} \bar{a}_i$ (so in particular they have the same first elements).*

Proof. Enough to show for finite $(\bar{a}_i)_{i<n}$ for all $n \in \omega$, by compactness (as we can write down a partial type $P(\bar{y}_0, \bar{y}_1, \ldots)$ expressing that $(\bar{y}_i)_{i\in\omega}$ are mutually indiscernible and $\bar{y}_i \equiv_{a_i A} \bar{a}_i$, and then it's enough to show that every finite part of it is consistent). So assume we have already found $\bar{a}'_0, ..., \bar{a}'_{n-1}$, and lets choose $\bar{a}'_n$. As $a_n \models p|_{a_{<n}A}$, there are $\bar{a}''_0 ... \bar{a}''_{n-1} \equiv_{Aa_0 \ldots a_{n-1}} \bar{a}'_0 ... \bar{a}'_{n-1}$ and such that $a_n \models p|_{A\bar{a}''_{<n}}$ (take some $a'_n \models p|_{A\bar{a}'_{<n}}$ and let $\bar{a}''_i$ be the image of $\bar{a}'_i$ under an $Aa_{<i}$-automorphism sending a'_n to a_n). In particular $\bar{a}''_{<n}$ are still mutually indiscernible over A. Then for any $i < n$, as $\bar{a}''_i$ is indiscernible over $\bar{a}''_{\neq i}A$ and p is still invariant over $\bar{a}''_{\neq i}A$, it follows by Lemma 40 that $\bar{a}''_i$ is indiscernible over $\bar{a}''_{\neq i}a_n A$. On the other hand, as p is strictly invariant over A, we have $\bar{a}''_{<n} \downarrow_A a_n$, so in particular $\bar{a}''_{<n} \downarrow^d_A a_n$. By Lemma 27 there is $\bar{a}''_n \equiv_{Aa_n} \bar{a}_n$ and indiscernible over $\bar{a}''_{<n}A$. So we have:

- for all $i \leq n$: $\bar{a}''_i$ is indiscernible over $\bar{a}''_{<i}a_{>i}A$.

Then (e.g. by [CH12, Lemma 3.5(2)]) we find $(\bar{a}'''_i)_{i\leq n}$ mutually indiscernible over A and such that $\bar{a}'''_i \equiv_{Aa_i} \bar{a}''_i \equiv_{Aa_i} \bar{a}_i$. □

Theorem 42. *Let T be NTP_2, $p(y)$ be a global type strictly invariant over $M \models T$, and let $\bar{a} = (a_i)_{i\in\omega}$ be a Morley sequence in p over M. Then for any $\phi(x,y) \in L(M)$ and $a \models p|_M$, if $\phi(x,a)$ divides over M then $\{\phi(x,a_i)\}_{i\in\omega}$ is inconsistent.*

Proof. Assume not, that is $\{\phi(x,a_i)\}_{i\in\omega}$ is consistent. As $\phi(x,a_0)$ divides over M, let $\bar{a}_0 = (a_{0,j})_{j\in\omega}$ be an M-indiscernible sequence witnessing this, i.e. $\{\phi(x,a_{0,j})\}_{j\in\omega}$ is k-inconsistent for some $k \in \omega$. For each $i \in \omega$ choose some $\bar{a}_i$ such that $a_i\bar{a}_i \equiv_M a_0\bar{a}_0$. By Lemma 41 we can find $\bar{a}'_i \equiv_{a_i M} \bar{a}_i$ such that $(\bar{a}'_i)_{i\in\omega}$ are mutually indiscernible over M. Then we have:

- $\left\{\phi\left(x, a'_{i,j}\right)\right\}_{j\in\omega}$ is k-inconsistent for all $i \in \omega$ (as $\bar{a}'_i \equiv \bar{a}_i$),
- $\left\{\phi\left(x, a'_{i,f(i)}\right)\right\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$ ($\left\{\phi\left(x, a'_{i,0}\right)\right\}_{i\in\omega}$ is consistent by the assumption as $a'_{i,0} = a_i$, then use mutual indiscernibility and induction to show that $a_0a_1a_2 \ldots \equiv_M a_{f(0)}a_{f(1)}a_{f(2)} \ldots$, which is enough).

But this shows that $\phi(x,y)$ has TP_2 — a contradiction. □

But do strictly invariant types always exist?

Example 43.
1. Let p be a global type invariant over A and assume that $\downarrow$ satisfies symmetry. Then p is strictly invariant.
2. Let $A \subseteq M$ and M is $|A|^+$-saturated. Assume that $p \in S(\mathbb{M})$ is invariant over A. Then it is strictly invariant over M.
3. In fact, for any A and $p(x)$ invariant over A, there is some $M \supseteq A$ with $|M| = |A| + |T|$ such that p is strictly invariant over M.

Proof. (2) Of course p is still invariant over M. So let $B \supseteq M$ and let $a \models p|_B$. We show that $\mathrm{tp}(B/aM)$ is finitely satisfiable in M (which implies $B \downarrow_M a$ as we saw that every finitely satisfiable type is non-forking). So let $\phi(x,y,z) \in L$, $b \in B$ and $c \in M$ finite be given. By saturation of M we can find $b' \in M$ with $b' \equiv_{Ac} b$. Now if $\models \phi(a,b,c)$, then by invariance of p over Ac (and the fact that $a \models p|_B$) it follows that $\models \phi(a,b',c)$ holds — so $\mathrm{tp}(B/aM)$ is indeed finitely satisfiable in M. □

However, we had to increase our model in order to find a strictly invariant type.

Question: Given $p(x) \in S(M)$, can we always find a global type extending p and strictly invariant over M?

The answer is *no* in general, but *yes* in NTP_2!

Towards it, we begin by observing that dividing of a formula over a model is always witnessed not just by an indiscernible sequence, but actually by a Morley sequence of some finitely satisfiable type.

Proposition 44. *Let T be NTP_2, $M \models T$ and assume that $\phi(x,a)$ divides over M. Then there is a global type $p(y) \in S(\mathbb{M})$ extending $\mathrm{tp}(a/M)$, finitely satisfiable in M and such that for any Morley sequence $\bar{a} = (a_i)_{i\in\omega}$ in p over M (i.e. $a_i \models p|_{Ma_{<i}}$) the set $\{\phi(x,a_i)\}_{i\in\omega}$ is inconsistent.*

Proof. Let κ be a cardinal large enough compared to $2^{|M|}$. Let $\bar{a} = (a_j)_{j\in\kappa}$ be an M indiscernible sequence starting with $a = a_0$ and witnessing that $\phi(x,a)$ divides over M, i.e. $\{\phi(x,a_j)\}_{j\in\kappa}$ is k-inconsistent. Let $N \succ M$ be $|M|^+$-saturated of size $\leq 2^{|M|}$ and such that $\mathrm{tp}(\bar{a}/N)$ is finitely satisfiable in M (exists by Fact 10). As κ was large enough compared to $|N|$, by Fact 3 we can extract from $\bar{a}$ a sequence $\bar{a}' = \left(a'_j\right)_{j\in\omega}$ which is indiscernible over N; we still have $a \equiv_M a'_0$ and $\{\phi(x,a'_i)\}_{i\in\omega}$ is k-inconsistent. Replace $\bar{a}$ by $\bar{a}'$. Let $P(\bar{x}) = \mathrm{tp}(\bar{a}/N)$, it is finitely satisfiable in M and $P|_{x_i} = P|_{x_j} = p$ for all i,j; $p(x) \in S(N)$ is finitely satisfiable in M, of course.

Now we choose $\bar{a}_i$ in N such that $\bar{a}_i \models P|_{M\bar{a}_{<i}}$ — possible by saturation of N. We still have that $\{\phi(x,a_{i,j})\}_{j\in\omega}$ is k-inconsistent for all $i \in \omega$. But as $\phi(x,y)$ is NTP_2, it follows that there is some $f : \omega \to \omega$ such that $\left\{\phi\left(x,a_{i,f(i)}\right)\right\}_{i\in\omega}$ is inconsistent. By the construction we have that $a_{i,f(i)} \models p|_{M\left\{a_{j,f(j)}\right\}_{j<i}}$ and $p(x) \in S(N)$ is finitely satisfiable in M, then an arbitrary global extension of p which is finitely satisfiable in M is as wanted (as all Morley sequences of an M-invariant type have the same type over M by Fact 9). □

6. Lecture 5

Originally existence of global strictly invariant types was established in [CK12] using the so-called Broom Lemma. The following is a simplified proof of a special case of this lemma in NTP_2 theories, due to Adler.

Lemma 45. *(Weak Broom Lemma) Let $p(x)$ be a partial type over $\mathbb{M}$, invariant over M (which as usual means that $\phi(x,a) \in p \Leftrightarrow \phi(x,a') \in p$ for any $a \equiv_M a'$ and ϕ). Suppose that $p(x) \vdash \psi(x,b) \vee \bigvee_{i<n} \phi_i(x,c)$, where $b \mathop{\smile\!\!\!\!\!\!\downarrow}^u_M c$ and each $\phi_i(x,c)$ divides over M. Then $p(x) \vdash \psi(x,b)$.*

Proof. We prove it by induction on n, the case $n=0$ is trivial. Suppose that it holds for n, and $p(x) \vdash \psi(x,b) \vee \bigvee_{i\leq n} \phi_i(x,c)$, where $b \mathop{\smile\!\!\!\!\!\!\downarrow}^u_M c$ and each $\phi_i(x,c)$ divides over M. Let $\bar{c} = (c_i)_{i\in\omega}$ be a Morley sequence over M of some global type finitely satisfiable in M extending $\mathrm{tp}(c/M)$, and such that $\{\phi_n(x,c_i)\}_{i\in\omega}$ is k-inconsistent (exists by Proposition 44). As $b \mathop{\smile\!\!\!\!\!\!\downarrow}^u_M c = c_0$, we may assume that $b \mathop{\smile\!\!\!\!\!\!\downarrow}^u_M \bar{c}$ (by Exercise 30). Then in particular $\bar{c}$ is indiscernible over bM, by Lemma

40. Then by invariance of p over M we have

$$p(x) \vdash \psi(x,b) \vee \bigwedge_{j<k} \bigvee_{i\leq n} \phi_i(x,c_j).$$

By the choice of k it follows that

$$p(x) \vdash \psi(x,b) \vee \bigvee_{i<n, j<k} \phi_i(x,c_j).$$

Claim. $bc_{>j} \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_j$ for every $j \in \omega$.

We are going to use Exercise 30 freely. By the choice of $\bar{c}$ we have $b \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M \bar{c}$, so $b \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_{\geq j}$ by monotonicity, so $b \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_{Mc_{>j}} c_j$ by base monotonicity. On the other hand we know that $(c_i)_{i\in\omega}$ is a Morley sequence, so $c_i \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_{<i}$. It follows from this (exercise) that $c_{>j} \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_j$. Combining we get $bc_{>j} \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_j$ by left transitivity.

Applying the inductive assumption (taking $b' = b(c_j)_{1\leq j<k}$ and $\psi'(x,b') = \psi(x,b) \vee \bigvee_{1\leq j<k} \bigvee_{i<n} \phi_i(x,c_j)$, as $p(x) \vdash \psi'(x,b') \vee \bigvee_{i<n} \phi_i(x,c_0)$ and $b' \mathop{\smile\!\!\!\!\!\mid}\nolimits^u_M c_0$ by the Claim) we get:

$$p(x) \vdash \psi(x,b) \vee \bigvee_{1\leq j<k} \bigvee_{i<n} \phi_i(x,c_j)$$

Repeating the argument we get:

$$p(x) \vdash \psi(x,b) \vee \bigvee_{2\leq j<k} \bigvee_{i<n} \phi_i(x,c_j)$$

$$\vdots$$

$$p(x) \vdash \psi(x,b) \vee \bigvee_{k-1\leq j<k} \bigvee_{i<n} \phi_i(x,c_j)$$

$$p(x) \vdash \psi(x,b).$$

□

Remark 46. Hrushovski observed a similarity between the Broom Lemma and Neumann's lemma from group theory, see [Hod93, Lemma 4.2.1]. It would be very curious to make this parallel more precise.

Corollary 47. *If $p(x)$ is a* ***consistent*** *global partial type invariant over M, then it does not fork over M.*

Proof. By Lemma 45 taking $x \neq x$ as $\psi(x,b)$. □

Remark 48. We knew it in the special case of complete global types by Corollary 21.

Now we are ready to show that every type over a model has a global strictly invariant extension.

Theorem 49. *Let T be* NTP$_2$ *and $M \models T$. Then every type over M has a global extension strictly invariant over M.*

Proof. Given $p(x) \in S(M)$ let $a \models p$. Consider the partial type

$$\begin{gathered} p(x) \\ \cup \\ \{\neg\phi(x,b) : \phi(a,y) \text{ forks over } M,\, \phi \in L(M),\, a \in \mathbb{M}\} \\ \cup \\ \{\psi(x,c) \leftrightarrow \psi(x,c') : c \equiv_M c',\, \psi \in L(M),\, c, c' \in \mathbb{M}\}. \end{gathered}$$

It is enough to show that this type is consistent, as then any of its completions will be a global extension of p strictly invariant over M. If not, then by compactness (and the fact that any disjunction of forking formulas is still a forking formula) we get:

$$p(x) \vdash \phi(x,b) \vee \bigvee_{i<n} \left(\psi_i(x,c_i) \not\leftrightarrow \psi_i(x,c_i')\right)$$

where $\phi(a,y)$ forks over M and $c_i \equiv_M c_i'$. Since $\phi(a,y)$ forks over M, the partial type $q(y) = \{\phi(a',y) : a' \equiv_M a\}$ also forks over M. As $q(y)$ is invariant over M, by Corollary 47 it is inconsistent. Then by compactness there are $a_0, \ldots, a_{m-1} \equiv_M a$ such that $\{\phi(a_i,y)\}_{i<m}$ is inconsistent. But as M is a model, by Fact 10 the type $\mathrm{tp}(a_0 \ldots a_{m-1}/M)$ has a global extension $p^*(x_0, \ldots, x_{m-1})$ invariant over M. Each $p^*|_{x_j}$ is invariant over M, and $p^*|_{x_j} \supset p(x_j) \vdash \phi(x_j,b) \vee \bigvee_{i<n}(\psi_i(x_j,c_i) \not\leftrightarrow \psi_i(x_j,c_i'))$. It follows that $p^*(x_0, \ldots, x_{m-1}) \vdash \phi(x_0,b) \wedge \ldots \wedge \phi(x_{m-1},b)$ — a contradiction. □

Finally we show that dividing and forking over a model are the same.

Theorem 50. *Let T be* NTP$_2$ *and $M \models T$. Then $\phi(x,a)$ divides over M if and only if it forks over M.*

Proof. Assume that $\phi(x,a)$ forks over M, i.e. $\phi(x,a) \vdash \bigvee_{i\leq n} \phi_i(x,b_i)$ and $\phi_i(x,b_i)$ divides over M. By Theorem 49, let $p(y, z_{\leq n})$ be a global type extending $\mathrm{tp}(ab_0 \ldots b_n/M)$ and strictly invariant over M. Let $(a_j b_{0,j} \ldots b_{n,j})_{j\in\omega}$ be a Morley sequence of p over M. By Theorem 42 there is some $k \in \omega$ such that $\{\phi_i(x,b_{i,j})\}_{j\in\omega}$ is k-inconsistent for each $i \leq n$. As $\phi(x,a_j) \vdash \bigvee_{i\leq n} \phi_i(x,b_{i,j})$ for all $j \in \omega$, it follows by the pigeonhole principle that $\{\phi(x,a_j)\}_{j\in\omega}$ is inconsistent. As $(a_j)_{j\in\omega}$ is an M-indiscernible sequence starting with a this shows that $\phi(x,a)$ divides over M. □

Fact 51. *The following are equivalent for an arbitrary theory T:*

1. *T is* NTP$_2$.
2. *For every model $M \models T$ and $\phi(x,a)$ dividing over M, if $(a_i)_{i\in\omega}$ is a Morley sequence of some global strictly M-invariant extension of* $\mathrm{tp}(a/M)$ *then $\{\phi(x,a_i)\}_{i\in\omega}$ is inconsistent.*

We had demonstrated that (1) implies (2), and for (2) implies (1) see [Che12].

6.1. Extension bases.

Definition 52. A set A is called an *extension base* if every $p(x) \in S(A)$ does not fork over A. Equivalently, every type in $S(A)$ has a global extension non-forking over A. A theory is called extensible if every set is an extension base.

Remark 53. (1) A is an extension base if and only if $\mathrm{acl}(A)$ is an extension base.

(2) T is extensible if and only if for every set A, every 1-type $p \in S_1(A)$ has a global non-forking extension.

Proof. (1) Assume that A is not an extension base, say $p(x) \in S(A)$ forks over A, i.e $p(x) \vdash \bigvee_{i<n} \phi_i(x, a_i)$ where $\phi_i(x, a_i)$ divides over A. Let $p'(x) \in S(\mathrm{acl}(A))$ be an arbitrary extension of p. Then still $p'(x) \vdash \bigvee_{i<n} \phi_i(x, a_i)$. Notice that each of $\phi_i(x, a_i)$ divides over $\mathrm{acl}(A)$ (exercise: $\bar{a}$ is indiscernible over A if and only if it is indiscernible over $\mathrm{acl}(A)$), so p' forks over $\mathrm{acl}(A)$.

(2) Follows from left transitivity of $\mathop{\smile\!\!\!\!\!\!\vert}$: if $a \mathop{\smile\!\!\!\!\!\!\vert}_A A$ and $b \mathop{\smile\!\!\!\!\!\!\vert}_{aA} aA$ then $ab \mathop{\smile\!\!\!\!\!\!\vert}_A A$. □

Example 54. Some examples of extension bases:

(1) Any model in any theory is an extension base (because every type $p(x) \in S(M)$ has a global extension finitely satisfiable in M by Fact 10).
(2) Any simple theory is extensible ([Wag00]).
(3) Any *o*-minimal theory is extensible.
(4) Any *c*-minimal theory is extensible (follows from the existence of generic 1-types over algebraically closed sets and Remark 53(2)).
(5) Any ordered *dp*-minimal theory is extensible (e.g. linear order with a dense-codense predicate named).
(6) Any theory with definable Skolem functions is extensible (e.g. $\mathbb{Q}_p$ in the language of rings; follows by (1), Remark 53(1) and the fact that $\mathrm{acl}(A) \prec \mathbb{M}$ in a theory with Skolem functions).

As we saw, no type $p \in S(A)$ can divide over A. Then of course, if forking=dividing over A, then A is an extension base. We show that the converse is true in NTP_2 theories.

Lemma 55. *Assume that $A \subseteq B$ and that $B \mathop{\smile\!\!\!\!\!\!\vert}^d_A c$. Then $\phi(x, c)$ divides over A if and only if it divides over B.*

Proof. If $\phi(x, c)$ divides over B, then it divides over any subset of B by the definition. On the other hand, assume that it divides over A, then there is an A-indiscernible sequence $\bar{c} = (c_i)$ such that$c = c_0$ and $\{\phi(x, c_i)\}_{i \in \omega}$ is inconsistent. By Lemma 27 we can find some $\bar{c}' \equiv_{Ac} \bar{c}$ and such that $\bar{c}'$ is indiscernible over B. But then $\{\phi(x, c'_i)\}_{i \in \omega}$ is inconsistent (as $\bar{c} \equiv_M \bar{c}'$), and so $\bar{c}'$ witnesses that $\phi(x, c)$ divides over B. □

Theorem 56. *Let T be* NTP_2 *and A an extension base. Then $\phi(x, c)$ divides over A if and only if it forks over A.*

Proof. Assume that $\phi(x, a)$ forks over A, say $\phi(x, a) \vdash \bigvee_{i \leq n} \psi_i(x, b_i)$ where $\psi_i(x, b_i)$ divides over A. Let $M \supseteq A$ be an arbitrary model. As A is an extension base, there is some $M' \equiv_A M$ such that $M' \mathop{\smile\!\!\!\!\!\!\vert}_A ab_0 \ldots b_n$. By Lemma 55 $\psi_i(x, b_i)$ divides over M' for each i, so $\phi(x, a)$ forks over M'. But then $\phi(x, a)$ divides over M' by Theorem 50, thus it divides over A by Lemma 55. □

7. Lecture 6

In this lecture we will finally prove that in an NTP_2 theory, if A is an extension base then $\mathbf{F}(A)$ is a generically prime ideal. But we have to do some more work first.

Problem 57. It is still open if $\mathbf{F}(A)$ is generically prime for any set A in an NTP_2 theory.

7.1. **Array dividing.** The material here is from [BYC12]. For the clarity of exposition (and since this is all that we will need) we only deal in this section with 2-dimensional arrays. All our results generalize to n-dimensional arrays by an easy induction (or even to λ-dimensional arrays for an arbitrary ordinal λ, by compactness; see [**?**, Section 1]).

Definition 58. (1) We say that $(a_{ij})_{i,j\in\kappa}$ is an *indiscernible array* over A if both $\left((a_{ij})_{j\in\kappa}\right)_{i\in\kappa}$ and $\left((a_{ij})_{i\in\kappa}\right)_{j\in\kappa}$ are indiscernible sequences. Equivalently, all $n\times n$ sub-arrays have the same type over A, for all $n<\omega$. Equivalently, $\mathrm{tp}(a_{i_0j_0}a_{i_0j_1}...a_{i_nj_n}/A)$ depends just on the quantifier-free order types of $\{i_0,...,i_n\}$ and $\{j_0,...,j_n\}$. Notice that, in particular, $\left(a_{if(i)}\right)_{i\in\kappa}$ is an A-indiscernible sequence of the same type for any strictly increasing function $f:\kappa\to\kappa$.

(2) We say that an array $(a_{ij})_{i,j\in\kappa}$ is *very indiscernible* over A if it is an indiscernible array over A, and in addition its rows are mutually indiscernible over A, i.e. $(a_{ij})_{j\in\kappa}$ is indiscernible over $(a_{i'j})_{i'\in\kappa\setminus\{i\},j\in\kappa}$ for each $i\in\kappa$.

Definition 59. We say that $\varphi(x,a)$ *array-divides* over A if there is an A-indiscernible array $(a_{ij})_{i,j\in\omega}$ such that $a_{00}=a$ and $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is inconsistent.

Definition 60. Given an array $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ and $k\in\omega$, we define:

(1) $\mathbf{A}^k=\left(a'_{ij}\right)_{i,j\in\omega}$ with $a'_{ij}=a_{(ik)j}a_{(ik+1)j}\dots a_{(ik+k-1)j}$.
(2) $\mathbf{A}^{\mathrm{T}}=(a_{ji})_{i,j\in\omega}$, namely the transposed array.
(3) Given a formula $\varphi(x,y)$, we let $\varphi^k(x,y_0\dots y_{k-1})=\bigwedge_{i<k}\varphi(x,y_i)$.
(4) Notice that with this notation $\left(\mathbf{A}^k\right)^l=\mathbf{A}^{kl}$ and $\left(\varphi^k\right)^l=\varphi^{kl}$.

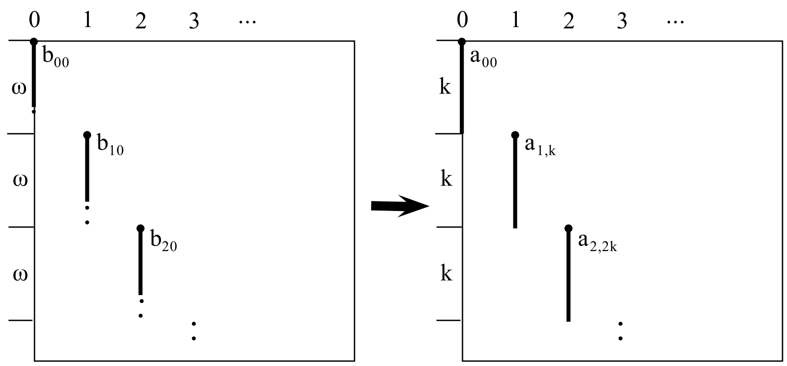
Lemma 61. (1) *If $\mathbf{A}$ is a B-indiscernible array, then $\mathbf{A}^k$ (for any $k\in\omega$) and $\mathbf{A}^{\mathrm{T}}$ are B-indiscernible arrays.*
(2) *If $\mathbf{A}$ is a very indiscernible array over B, then $\mathbf{A}^k$ is a very indiscernible array over B (for any $k\in\omega$).*

Lemma 62. *Assume that T is NTP_2 and let $(a_{ij})_{i,j\in\omega}$ be a very indiscernible array. Assume that the first column $\{\varphi(x,a_{i0})\}_{i\in\omega}$ is consistent . Then the whole array $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is consistent.*

Proof. Let $\varphi(x,y)$ and a very indiscernible array $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be given. By compactness, it is enough to prove that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent for every $k\in\omega$. So fix some k, and let $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ — it is still a very indiscernible array by Lemma 61. Besides $\left\{\varphi^k(x,b_{i0})\right\}_{i\in\omega}$ is consistent. But then $\left\{\varphi^k(x,b_{ij})\right\}_{j\in\omega}$ is consistent for some $i\in\omega$ (as otherwise φ^k would have TP_2 by the mutual indiscernibility of rows), thus for $i=0$ (as the sequence of rows is indiscernible). Unwinding, we conclude that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent. □

Lemma 63. *Let $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be an indiscernible array and assume that the diagonal $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent. Then for any $k\in\omega$, if $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ then the diagonal $\left\{\varphi^k(x,b_{ii})\right\}_{i\in\omega}$ is consistent.*

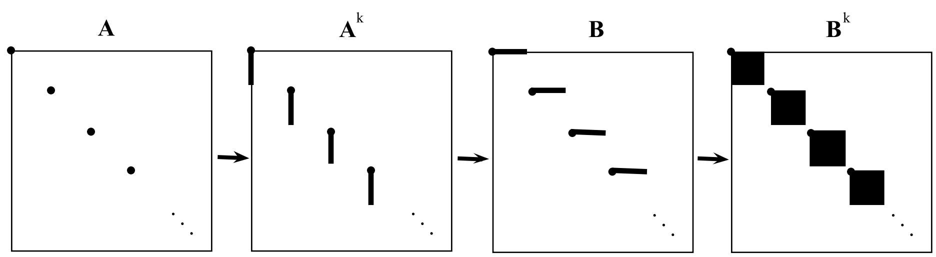
Proof. By compactness we can extend our array $\mathbf{A}$ to $(a_{ij})_{i\in\omega\times\omega,j\in\omega}$ and let $b_{ij}=a_{i\times\omega+j,i}$. It then follows that $(b_{ij})_{i,j\in\omega}$ is a very indiscernible array and that $\{\varphi(x,b_{i0})\}_{i\in\omega}$ is consistent. But then $\{\varphi(x,b_{ij})\}_{i,j\in\omega}$ is consistent by Lemma 62 , and we can conclude by indiscernibility of $\mathbf{A}$.



∎

Proposition 64. *Assume T is NTP$_2$. If $(a_{ij})_{i,j\in\omega}$ is an indiscernible array and the diagonal $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent, then the whole array $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is consistent. Moreover, this property characterizes NTP$_2$.*

Proof. Let $\kappa\in\omega$ be arbitrary. Let $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^k(x,b_{ii})\right\}_{i\in\omega}$ is consistent by Lemma 63. As $\mathbf{B}=\left(\mathbf{A}^k\right)^T$ has the same diagonal, using Lemma 63 again we conclude that if $\mathbf{B}^k=(c_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^{k^2}(x,c_{ii})\right\}_{i\in\omega}$ is consistent. In particular $\{\varphi(x,a_{ij})\}_{i,j<k}$ is consistent. Conclude by compactness.



"Moreover" follows from the fact that if T has TP$_2$, then there is a very indiscernible array witnessing this (see Remark 35). ∎

Corollary 65. *Let T be NTP$_2$. Then $\varphi(x,a)$ divides over A if and only if it array-divides over A.*

Proof. If $(a_{ij})_{i,j\in\omega}$ is an A-indiscernible array with $a_{00}=a$, then $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent since $(a_{ii})_{i\in\omega}$ is indiscernible over A and $\varphi(x,a)$ does not divide over A, apply Proposition 64. ∎

Exercise 66. Find an example of a formula which array-divides but does not divide.

7.2. **Generic primality of forking ideals.**

Proposition 67. *Let T be NTP$_2$ and $M \vDash T$. If $(\bar{a}_i)_{i\in\omega}$ is indiscernible over M and $\phi(x, a_0)$ does not divide over M, then $\phi(x, a_0) \wedge \phi(x, a_1)$ does not divide over M.*

Proof. Assume not, let κ be very large compared to $|M|$, and let $\bar{a}_0 = (a_{0j})_{j\in\kappa}$ be indiscernible over M, $\varphi(x, a_{00})$ does not divide over M, but $\varphi(x, a_{00}) \wedge \varphi(x, a_{01})$ does. By Theorem 49, we know that $\mathrm{tp}(\bar{a}_0/M)$ has a global extension strictly invariant over M, let's call it $Q(\bar{y})$ where $\bar{y} = (y_j)_{j\in\omega}$. Let $(\bar{a}_i)_{i\in\omega}$ be a Morley sequence of $Q(\bar{y})$ over M. Note that:

(1) $\phi(x, a_{0j}) \wedge \phi(x, a_{0j'})$ divides over M for any $j \neq j' \in \kappa$ (by indiscernibility of $\bar{a}_0$ over M and the assumption),
(2) $q_{j,j'}(y_j, y_{j'}) = Q(\bar{y}) \restriction_{y_j, y_{j'}}$ is still a global type strictly invariant over M and $(a_{i,j} a_{i,j'})_{i\in\omega}$ is a Morley sequence of $q_{j,j'}$ over M,
(3) $\{\phi(x, a_{i,j}) \wedge \phi(x, a_{i,j'})\}_{i\in\omega}$ is inconsistent for any $j \neq j' \in \kappa$ (combining (1), (2) and Theorem 42).

As κ was sufficiently large with respect to $|M| + \aleph_0$, by Fact 3 we can extract an M-indiscernible sequence $\left((a'_{ij})_{i\in\omega}\right)_{j\in\omega}$ from the sequence of columns $\left((a_{ij})_{i\in\omega}\right)_{j\in\kappa}$, such that type of every finite subsequence of it over M is already present in the original sequence. Then also $\left((a'_{ij})_{j\in\omega}\right)_{i\in\omega}$ is an indiscernible sequence over M (as $(\bar{a}_i)_{i\in\omega}$ was an indiscernible sequence over M). It follows that $\left(a'_{ij}\right)_{i,j\in\omega}$ is an M-indiscernible array and that $\left\{\varphi(x, a'_{ij})\right\}_{i,j\in\omega}$ is inconsistent (by (3)), thus $\varphi(x, a_{00})$ array-divides over M, thus divides over M, by Corollary 65 — a contradiction. □

Theorem 68. *If T is NTP$_2$ and A is an extension base, then $\mathbf{F}(A)$ is generically prime.*

Proof. Let C be an extension base and $\bar{a} = (a_i)_{i\in\omega}$ be an A-indiscernible sequence. As C is an extension base, we can find $M \supseteq C$ such that $M \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}$. It follows by Lemma 55 that for any $n \in \omega$, $\bigwedge_{i<n} \varphi(x, a_i)$ divides over C if and only if it divides over M. It follows from Proposition 67 that if $\varphi(x, a_0)$ does not divide over C, then $\{\varphi(x, a_i)\}_{i\in\omega}$ does not divide over C.

To conclude recall that $\psi(x, b)$ divides over A if and only if it forks over A (by Theorem 56) and apply Proposition 14 to $\mathbf{F}(A)$. □

8. Further topics

1. Forking in NIP
   - (a) Non-forking = invariance.
   - (b) Bound on the number of non-forking extensions.
   - (c) NIP = NTP$_2$ + boundedness of non-forking (see the proof in).
   - (d) Measures. Measurability of forking.
   - (e) Generically stable types.
2. Forking in simple theories
   - (a) Symmetry, transitivity, local character are all equivalent to simplicity.
   - (b) Independence theorem, higher amalgamation.
   - (c) Canonicity: forking is the only independence relation satisfying the independence theorem.

(d) Lascar strong types, hyperimaginaries, canonical bases.
(e) Ranks, supersimplicity.

(3) Forking in stable theories
   (a) Stationarity.
   (b) Definable types.
   (c) ...

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