MODEL THEORY OF GROUPS (MATH 223M, UCLA, SPRING 2018)

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Lecture notes in progress. All comments and corrections are very welcome.
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Contents

1. Introduction

1.1. **Motivation.**

- On the one hand, one is forced to study definable groups even if only interested in abstract model-theory of first-order structures (e.g., the beautiful result that totally categorical theories are not finitely axiomatizable, but are axiomatizable by a single sentence togher with the axiom schema of infinity — Zilber, Cherlin-Harrington-Lachlan, Hrushovski, Ahlbrandt-Ziegler — essentially shows that looks like vector spaces over a finite field, see [11] and references there).
- On the other hand, some of the most important applications of model theory are based on a detailed understanding of definable groups in certain structures (e.g. Hrushovski's proof of the Mordell-Lang conjecture for function fields via undertanding groups definable in differentially closed fields [5]).
- Instead of studying groups in a particular theory, one often restrict to groups definable in a certain "tame" class of theories (e.g. stable, NIP, etc). Amuzingly, it is possible to deduce quite a lot of structure from such general combinatorial conditions.
- Recovering groups from generic data - assume we have some operation which satisfies group axioms with some "positive probability". Can we find a group from which it comes? In a definable manner? Hrushovski's group chunk and group configuration theorems say that the answer is often yes (see [9] or Terry Tao's blog post).

1.2. **Preliminaries.** We recall briefly some basic facts and definitions in order to set up the notation for our discussion.

- $\mathcal{M} = (M, R_1, \ldots, f_1, \ldots, c_1, \ldots)$ denotes a first-order structure in a language $\mathcal{L}$.
- T denotes a first-order theory, $\mathcal{M} \models T$ if it satisfies all sentences in T.
- $\mathrm{Th}(\mathcal{M})$ denotes the complete theory of $\mathcal{M}$.
- Given $A \subseteq M$, $\phi(\bar{x})$ is an $\mathcal{L}(A)$-formula (formula over a set of parameters A) if it is of the form $\phi(\bar{x}) = \psi(\bar{x}, \bar{b})$ where $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{b}$ is tuple from A.
- We will abuse notation and write x, y, z, etc. to denote tuples of variables when there is no confusion.
- If $\Psi(x) \subseteq \mathcal{L}(A)$, a set of $\mathcal{L}(A)$ formulas and $a \in M$, we write $a \models \Psi(x) \iff a$ satisfies all formulas in Ψ.
- Given $B \subseteq M^{|x|}$, $\Psi(B) := \{b \in B : \mathcal{M} \models \Psi(a)\}$.
- $X \subseteq M^n$ is an *A-definable set* if $A = \psi(M)$ for some $\psi(x) \in \mathcal{L}(A)$.
- $\mathrm{Def}_n(A)$ is the boolean algebra of all A-definable subsets of M^n.
- Given $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$, write $\mathcal{M} \equiv \mathcal{N}$ to denote elementary equivalence ($\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$).
- Given a (partial) map $f : \mathcal{M} \to \mathcal{N}$, f is *elementary* if for all $a \in \mathrm{Dom}(f)$ and $\phi \in \mathcal{L}$, $\mathcal{M} \models \phi(a) \iff \mathcal{N} \models \phi(f(a))$.
- $\mathcal{M} \preceq \mathcal{N}$ is an *elementary substructure* if the embedding map is elementary.

Fact 1.1. *(Compactness theorem) Let $\mathcal{L}$ be any language and Ψ a set of $\mathcal{L}$-sentences (of any cardinality). If every finite $\Psi_0 \subseteq \Psi$ is consistent (i.e. there is some $\mathcal{L}$-structure $\mathcal{M} \models \Psi_0$), then Ψ is consistent.*

Fact 1.2. *(Löwenheim–Skolem theorem) Let $\mathcal{M} \models T$ be given, with $|\mathcal{M}| \geq \aleph_0$. Then for any cardinal $\kappa \geq |\mathcal{L}|$ there is some $\mathcal{N}$ with $|\mathcal{N}| = \kappa$ and such that:*

- *$\mathcal{M} \preceq \mathcal{N}$ if $\kappa > |\mathcal{M}|$,*

- $\mathcal{N} \preceq \mathcal{M}$ if $\kappa < |\mathcal{M}|$.
- For $A \subseteq M$, a *partial type* $\Phi(x)$ *over* A is *consistent* collection of $\mathcal{L}(A)$-formulas.
- $\Phi(x)$ is a *(complete) type* if $\phi(x) \in \Phi$ or $\neg\phi(x) \in \Phi$ for all $\phi \in \Phi$.
- For a tuple b in $\mathcal{M}$, $\operatorname{tp}(b/A) := \{\phi(x) : b \models \phi(x), \phi(x) \in \mathcal{L}(A)\}$.

Definition 1.3. Let κ be an infinite cardinal.

(1) $\mathcal{M}$ is *κ-saturated* if $\forall A \subseteq M$ with $|A| < \kappa$, every 1-type over A can be realized in $\mathcal{M}$.
(2) $\mathcal{M}$ is *κ-homogenous* if any partial elementary map from $\mathcal{M}$ to itself with a domain of size $< \kappa$ can be extended to an automorphism of $\mathcal{M}$.

Fact 1.4. *For any T and κ, there is a κ-saturated and κ-homogeneous model $\mathcal{M}$ of T.*

Definition 1.5. For $A \subseteq M$, we let $S_n(A)$ denote the space of all complete n-types over A.

This is the *Stone dual* of $\operatorname{Def}_n(A)$, hence compact (=compactness theorem), Hausdorff space with a basis of clopens given by the sets $\langle\phi(x)\rangle = \{p \in S_n(A) : \phi(x) \in p\}$ for $\phi(x) \in \mathcal{L}(A)$.

Exercise 1.6. Show:

- If $\mathcal{M}$ is κ-saturated and $A \subseteq M$, $|A| < \kappa$ then every n-type over A is realized in M.
- $(\mathbb{R}, +, \times, 0, 1)$ is not $\aleph_0$-saturated.

2. Examples of stable groups

2.1. Combinatorial definition of stability.

Definition 2.1. Let $M \models T$.

(1) A formula $\phi(x, y)$, with its variables partitioned into two groups x, y, has the *k-order property*, $k \in \omega$, if there are some $a_i \in M_x, b_i \in M_y$ for $i < k$ such that $M \models \phi(a_i, b_j) \iff i < j$.
(2) $\phi(x, y)$ has the *order property* if it has the k-order property for all $k \in \omega$.
(3) We say that a formula $\phi(x, y) \in L$ is *stable* if there is some $k \in \omega$ such that it does not have the k-order property.
(4) A theory is *stable* if it implies that all formulas are stable (note that this is indeed a property of a theory, if $\mathcal{M} \equiv \mathcal{N}$ then $\phi(x, y)$ has the k-order property in $\mathcal{M}$ if and only if it has the k-order property in $\mathcal{N}$).

Example 2.2. (1) Let T be the theory of dense linear orders without end-points. Then the formula $x < y$ is unstable.
(2) Let T be the theory of the Rado's random graph. Then the formula xEy is untable.

Exercise 2.3. If $\phi_1(x, y), \phi_2(x, y)$ are stable, then:

(1) $\neg\phi_1(x, y)$, $\phi(x, y) := \phi_1(x, y) \wedge \phi_2(x, y)$ are stable (hence all Boolean combinations preserve stability).
(2) $\phi_1^*(y, x) := \phi_1(x, y)$ is stable.

(Hint: use Ramsey's theorem.)

Definition 2.4. A group G is *stable* if it is definable in a stable theory, i.e. for some stable T and some $\mathcal{M} \models T$ we have: the underlying set G is a definable subset of M^n for some $n \in \omega$ and the group operation $\cdot : G(M^n) \times G(M^n) \to G(M^n)$ is a definable function. For simplicity of notation we assume that $n = 1, G(M) = M$ and that $G, \cdot$ are $\emptyset$-definable.

2.2. **Quantifier elimination and stability in modules.**

Definition 2.5. Let R be a (possibly non-commutative) ring with 1. An *R-module* $\mathcal{M}$ is a structure in a the language $\mathcal{L}_{R-\mathrm{Mod}} = \left\{0, +, -, (r)_{r \in R}\right\}$ with 0 a constant, $+, -$ binary function symbols, and r unary function symbols satisfying:

(1) $(M, +, -, 0)$ is an abelian group,
(2) $\forall x, y\, r(x+y) = r(x) + r(y)$,
(3) $\forall x\ (r+s)(x) = r(x) + s(x)$,
(4) $\forall x\ (rs)(x) = r(s(x))$,
(5) $\forall x\, 1(x) = x$

for all $r, s \in R$.

Definition 2.6. (1) An *equation* is an $\mathcal{L}_{R\text{-mod}}$-formula $\phi(\bar{x})$ of the form $r_1 x_1 + \ldots + r_m x_m = 0$.

(2) A *positive primitive* formula (a *pp*-formula) is of the form

$$\psi(\bar{x}) = \exists \bar{y}\left(\phi_1(\bar{x}, \bar{y}) \wedge \ldots \wedge \phi_n(\bar{x}, \bar{y})\right),$$

where $\phi_i(\bar{x}, \bar{y})$ are equations.

Theorem 2.7. *(Baur-Monk) For every ring R and any R-module $\mathcal{M}$, every $\mathcal{L}_{R\text{-mod}}$-formula is equivalent (modulo the theory of M) to a Boolean combination of pp-formulas.*

Remark 2.8. The *pp*-formulas that we obtain this way depend on the theory of $\mathcal{M}$, hence this quantifier elimination is not uniform over the class of R-modules.

(1) The class of pp-formula is closed under $\wedge$.
(2) A pp-formula $\phi(x_1, \ldots, x_n)$ defines a subgroup $\phi(M^n)$ of the product group M^n as
   (a) $\mathcal{M} \models \phi(0, \ldots, 0)$,
   (b) $\mathcal{M} \models \phi(x) \wedge \phi(y) \to \phi(x - y)$.
(3) If $\phi(x, y)$ is a pp-formula and $a \in M^{|y|}$, then $\phi(M, a)$ is either empty or a coset of $\phi(M, 0)$
   As $\mathcal{M} \models \phi(x, a) \to (\phi(y, 0) \leftrightarrow \phi(x + y, a))$.

Corollary 2.9. *Every theory of an R-module is stable (hence, by compactness, the common theory of all R-modules is stable). In paricular, every abelian group $\mathcal{G} = (G, +, -, 0)$ is stable (as a $\mathbb{Z}$-module).*

Proof. It is enough to show that pp-formulas are stable by Exercise 2.3. It follows from Remark 2.8(3) that if $\phi(x, y)$ is a pp-formula, then for any $a, b \in M^{|y|}$ we have that $\phi(M, a)$ and $\phi(M, b)$ are either equal or disjoint. This condition immediately implies that $\phi(x, y)$ doesn't have the 2-order property. $\square$

Fact 2.10. *(B.H. Neumann lemma) Let H_i denote subgroups of some abelian group. If $H_0 + a_0 \subseteq \bigcup_{i=1}^{n} (H_i + a_i)$ and $H_0 / (H_0 \cap H_i)$ is infinite for $i > k$, then $H_0 + a_0 \subseteq \bigcup_{i=1}^{k} (H_i + a_i)$.*

(Prove: a group covered by a finite number of cosets of subgroups is covered by the ones of finite index, and deduce it from this.)

The following combinatorial lemma is not difficult to check directly.

Lemma 2.11. *Let $A_i, i \leq k$ be any sets. If A_0 is finite, then $A_0 \subseteq \bigcup_{i=1}^{k} A_i$ if and only if*

$$\sum_{\Delta \subseteq \{1,\ldots,k\}} (-1)^{|\Delta|} \left| A_0 \cap \bigcap_{i \in \Delta} A_i \right| = 0.$$

Proof of Theorem 2.7.

Fix an R-module M.

It is enough to show that if $\psi(x,y)$ is equivalent to a Boolean combination of pp-formulas in M, then so is $\forall x \psi$.

As pp-formulas are closed under conjunction, by propositional calculus ψ is equivalent to a conjunction of formulas of the form $\phi_0(x,y) \to \bigvee_{i=1}^{n} \phi(x,y)$, where $\phi_i(x,y)$ are pp-formulas.

We may hence assume that ψ itself is of this form.

Let $H_i := \phi_i(M,0)$, then $\phi_i(M,y)$ is either empty or a coset of H_i for each y.

Possibly re-enumerating ϕ_i, we may assume that $H/(H_0 \cap H_i)$ is finite for $i = 1, \ldots, k$ and infinite for $i = k+1, \ldots, n$, for some $0 \leq k \leq n$.

By Neumann's lemma, $M \models \forall x \psi \leftrightarrow \forall x (\phi_0(x,y) \to \phi_1(x,y) \vee \ldots \vee \phi_k(x,y))$.

Let $A_i := \phi_i(M,y) / (H_0 \cap \ldots \cap H_k)$.

Applying Lemma 2.11, $\phi_0(M,y) \cap \bigcap_{i \in \Delta} \phi_i(M,y)$ is empty or consists of N_Δ cosets of $H_0 \cap \ldots \cap H_k$, where

$$N_\Delta = \left| H_0 \cap \bigcap_{i \in \Delta} H_i / (H_0 \cap \ldots \cap H_k) \right|.$$

Hence

$$M \models \forall x \psi \leftrightarrow \sum_{\Delta \in \mathcal{N}} (-1)^{|\Delta|} N_\Delta = 0,$$

where $\mathcal{N} = \left\{ \Delta \subseteq \{1, \ldots, k\} : \exists x \left(\phi_0(x,y) \wedge \bigwedge_{i \in \Delta} \phi_i(x,y) \right) \right\}$.

Corollary 2.12. *(Exercise) Deduce from this full quantifier elimination for vector spaces (i.e. when R is a field).*

Corollary 2.13. *(Exercise, Szmielew) Let $\mathcal{A} = (A, +, -, 0)$ be an abelian group. Then every subset of A^n definable without parameters is a boolean combination of relations of the forms $\phi(A^n)$, where $\phi(\bar{x})$ is either $t(\bar{x}) = 0$ or $p^m \mid t(\bar{x})$ for some term t, prime p and positive integer m. (Hint: what can a pp-formula express in an abelian group?)*

2.3. **A type-counting criterion for stability.** Recall that, via Stone duality, the topological complexity of type spaces reflects the complexity of definable sets. We consider the most basic property of type spaces — their size.

Definition 2.14. For a complete first order theory T, let $f_T : \mathrm{Card} \to \mathrm{Card}$ be defined by $f_T(\kappa) = \sup\{|S_1(M)| : M \models T, |M| = \kappa\}$, for κ an infinite cardinal.

Exercise 2.15. Show that taking $f_T(\kappa) = \sup\{|S_n(M)| : M \models T, |M| = \kappa, n \in \omega\}$ gives an equivalent definition.

It is easy to see that $\kappa \leq f_T(\kappa) \leq 2^{\kappa+|T|}$ (every $p \in S_x(M)$ is determined by the collection of all sets of the form $\{a \in M_y : \phi(x,a) \in p\}$, where $\phi(x,y)$ varies over all formulas in L, which gives the upper bound; the lower bound is given by the set of realized types of the form $\{\mathrm{tp}(a/M) : a \in M\}$).

Let $\operatorname{ded} \kappa = \sup\{\lambda :$ there is a linear order of size κ with λ cuts$\}$ (Exercise: equivalently, we only have to consider dense linear orders without endpoints).

Lemma 2.16. $\kappa < \operatorname{ded} \kappa \leq 2^{\kappa}$.

Proof. To see $\kappa < \operatorname{ded} \kappa$: let μ be the minimal cardinal such that $2^{\mu} > \kappa$ (in particular $\mu \leq \kappa$), and consider the tree $2^{<\mu}$. Take the lexicographic ordering I on it, then $|I| \leq \kappa$ by the minimality of μ, but there are at least $2^{\mu} > \kappa$ cuts.

To see $\operatorname{ded} \kappa \leq 2^{\kappa}$, note that every cut is *uniquely* determined by the subset of elements in its lower half. □

Proposition 2.17. *Assume that T is unstable, then $f_T(\kappa) \geq \operatorname{ded} \kappa$ for all cardinals $\kappa \geq |T|$.*

Proof. Fix a cardinal κ. Let $\phi(x,y) \in L$ be a formula that has the k-order property for all $k \in \omega$. Then by compactness we have:

Claim. Let I be an arbitrary linear order. Then we can find some $\mathcal{M} \models T$ and $(a_i, b_i : i \in I)$ from M such that $\mathcal{M} \models \phi(a_i, b_j) \iff i < j$, for all $i, j \in I$.

Let I be an arbitrary dense linear order without endpoints of size κ, and let $(a_i, b_i : i \in I)$ in $\mathcal{M}$ be as given by the claim. By the downwards Löwenheim–Skolem theorem we can assume that $|\mathcal{M}| = \kappa$.

Given a cut $C = (A, B)$ in I, consider the set of $L(M)$-formulas

$$\Phi_C = \{\phi(x, b_j) : j \in B\} \cup \{\neg\phi(x, b_j) : j \in A\}.$$

Note that by compactness it is a partial type (consistency of finite subtypes is witnessed by the appropriate a_i's), let $p_C \in S_x(M)$ be a complete type over M extending $\Phi_C(x)$. Given two cuts C_1, C_2, we have $p_{C_1} \neq p_{C_2}$ (say $B_1 \subsetneq B_2$, then take $j \in B_2 \setminus B_1$, it follows that $\phi(x, b_j) \in p_{C_2}, \phi(x, b_j) \notin p_{C_1}$). As I was arbitrary, this shows that $\sup\{|S_x(M)| : M \models T, |M| = \kappa\} \geq \operatorname{ded} \kappa$.

Note that we may have $|x| > 1$, however using Exercise 2.15 we get $f_T(\kappa) \geq \operatorname{ded} \kappa$ as well. □

Theorem 2.18. *(Erdős-Makkai) Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(b_i : i < \omega)$ of elements of B and $(S_i : i < \omega)$ of elements of $\mathcal{F}$ such that one of the following holds:*

1. $b_i \in S_j \iff j < i$ *for all* $i, j \in \omega$,
2. $b_i \in S_j \iff i < j$ *for all* $i, j \in \omega$.

Proof. Choose $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = |B|$, such that any two finite subsets B_0, B_1 of B, if $\exists S \in \mathcal{F}$ with $B_0 \subseteq S, B_1 \subseteq B \setminus S$, then there is some $S' \in \mathcal{F}'$ with $B_0 \subseteq S', B_1 \subseteq B \setminus S'$ (possible as there are at most $|B|$-many pairs of finite subsets of B).

By assumption there is some $S^* \in \mathcal{F}$ which is not a Boolean combination of elements of $\mathcal{F}'$ (again there are at most $|B|$-many different Boolean combinations of sets from $\mathcal{F}'$).

We choose by induction sequences $(b'_i : i < \omega)$ in S^*, $(b''_i : i < \omega)$ in $B \setminus S^*$ and $(S_i : i < \omega)$ in $\mathcal{F}'$ such that:

- $\{b'_0, \ldots, b'_n\} \subseteq S_n$ and $\{b''_0, \ldots, b''_n\} \subseteq B \setminus S_n$,
- $b'_n \in S_i \iff b''_n \in S_i$ for all $i < n$.

Assume $(b'_i : i < n)$, $(b''_i : i < n)$ and $(S_i : i < n)$ have already been constructed. Since S^* is not a Boolean combination of $S_0, \ldots, S_{n-1}$, there are $b'_n \in S^*, b''_n \in B \setminus S^*$ such that for all $i < n$,

$$b'_n \in S_i \iff b''_n \in S_i.$$

By the choice of $\mathcal{F}'$, there is some $S_n \in \mathcal{F}'$ with $\{b'_0, \ldots, b'_n\} \subseteq S_n$ and $\{b''_0, \ldots, b''_n\} \subseteq B \setminus S_n$.

Now by Ramsey theorem we may assume that either: $b'_n \in S_i$ for all $i < n < \omega$, or $b'_n \notin S_i$ for all $i < n < \omega$ (color the set of pairs (i, n) for $i < n$ with two colors according to whether $b'_n \in S_i$ or $b'_n \notin S_i$). In the first case we set $b_i = b''_i$ and get (1), in the second case we set $b_i = b'_{i+1}$ and get (2). □

Definition 2.19. Let $\phi(x, y)$ be a formula, by a *complete ϕ-type* over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x, b), \neg\phi(x, b)$ where b ranges over A. Let $S_\phi(A)$ be the space of all complete ϕ-types over A.

Proposition 2.20. *Assume that $|S_\phi(B)| > |B|$ for some infinite set of parameters B. Then $\phi(x, y)$ is unstable.*

Proof. For any $a \in \mathbb{M}_x$, $\mathrm{tp}_\phi(a/B)$ is given by $S_a = \{b \in B :\models \phi(a, b)\} \subseteq B$.

Applying the Erdős-Makkai theorem to B and $\mathcal{F} = \{S_a : a \in M_x\}$ we obtain a sequence $(b_i : i < \omega)$ of elements of B and a sequence $(a_i : i < \omega)$ of elements of $\mathbb{M}_x$ such that either $b_i \in S_{a_j} \iff j < i$ or $b_i \in S_{a_j} \iff i < j$ for all $i, j \in \omega$. In the first case $\phi(x, y)$ is unstable by definition, in the second case by preservation of stability under boolean combinations. □

Definition 2.21. (1) We will say that T is κ-stable if $f_T(\kappa) = \kappa$.

(2) [Shelah] If T is countable and stable, then one of the following holds:
   (a) T is κ-stable for all infinite κ(i.e. T is *ω-stable*, or *$\aleph_0$-stable*),
   (b) T is κ-stable for all $\kappa \geq 2^{\aleph_0}$ (T is *superstable*),
   (c) Otherwise T is *strictly stable*.

Exercise 2.22. Find examples of abelian groups properly in each of these classes

2.4. **Algebraic groups.** We consider $\mathrm{Th}(\mathbb{C}, +, \times, 0, 1)$.

- Recall: a field K is algebraically closed if it contains a root for every non-constant polynomial in $K[x]$ (equivalently, K has no proper algebraic extensions).
- By the fundamental theorem of algebra, $\mathbb{C}$ is algebraically closed (and this condition is expressible as an infinite collection of first-order sentences).
- For $p = 0$ or prime, we denote by ACF_p the theory of algebraically closed fields of characteristic p.

Fact 2.23. *[Tarski's quantifier elimination]* ACF_p *is a complete theory eliminating quantifiers in the language of rings $\mathcal{L}_{ring} = (0, 1, \times, +)$.*

Corollary 2.24. *Every definable (with parameters) subset of $\mathcal{M} \models T$ is either finite or cofinite.*

Proof. It follows from the QE that definable sets are precisely the constructible sets, i.e. Boolean combinations of algebraic sets. As every polynomial in a single variable of degree d has at most d roots, we get the result. □

Definition 2.25. Theories satisfying this condition are called *strongly minimal.*

Proposition 2.26. *If T is strongly minimal, then it is ω-stable.*

Proof. Each complete 1-type p over $\mathcal{M} \models T$ is either isolated by $x = a$ for some $a \in M$ (in case p contains some formula with finitely many solutions), or it is the unique non-algebraic type (axiomatized by $\{x \neq a : a \in M\}$. Hence $|S_1(M)| \leq |M|$. □

Example 2.27. Let K be an algebraically closed field. Then the following groups are definable:

(1) Algebraic matrix groups, e.g. $\mathrm{GL}(n, K)$ — the group of invertible $n \times n$ matrices, $\mathrm{SL}(n, K)$ — matrices with determinant 1, etc.
(2) Affine algebraic groups
(3) Abelian varieties (with the induced structure)

- [Macintyre, 71] All infinite ω-stable fields are algebraically closed.
- The Cherlin–Zilber conjecture (also called the algebraicity conjecture), due to Gregory Cherlin (1979) and Boris Zilber (1977), suggests that infinite (ω-stable) simple groups are simple algebraic groups over algebraically closed fields.

2.5. **Further examples of stable groups.**

(1) Free groups are stable (viewed as structures in the pure group language), in the pure group language (more generally, torsion-free hyperbolic groups are stable). This is a deep theorem of Sela (`https://en.wikipedia.org/wiki/Zlil_Sela`). Furthermore, if F_n is the free group on n generators, then we have $F_2 \prec F_3 \prec \ldots$, in particular they all have the same first-order theory. Non-abelian free groups are not superstable.
(2) [Mekler '81] Let $p > 2$ be prime. Let T be any theory in a finite relational language. There is a uniform construction of a group $G(\mathcal{M})$ for every $\mathcal{M} \models T$, a theory T^* of all groups $\{G(\mathcal{M}) : \mathcal{M} \models T\}$ and an interpretation Γ of T in T^* s.t.:
    - T^* is a theory of nilpotent groups of class 2 (i.e. $[[x_1, x_2], x_3] = e$) and of exponent p,
    - if $G \models T^*$, then $\exists \mathcal{M} \models T$ s.t. $G(\mathcal{M}) \equiv G$,
    - For $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \equiv \mathcal{N} \iff G(\mathcal{M}) \equiv G(\mathcal{N})$,
    - $\Gamma(G(\mathcal{M})) \cong \mathcal{M}$.

    Idea: Bi-interpret $\mathcal{M}$ with a graph C, then define a group $G(C)$ generated freely by the vertices of C, imposing that two generators commute $\iff$ they are connected by an edge in C.

    Fact. For any cardinal κ, $\mathrm{Th}(\mathcal{M})$ is κ-stable $\iff$ $\mathrm{Th}(G(\mathcal{M}))$ is κ-stable.
(3) For a field K, we let K^{alg} denote its algebraic closure (i.e. an algebraic extension of K which is algebraically closed, unique up to an isomorphism fixing K pointwise).

    Definition 2.28. A field K is *separably closed* if every polynomial $P(X) \in K[X]$ whose roots in K^{alg} are distinct, has at least one root in K.

(Equivalently, every irreducible polynomial over K is of the form $X^{p^k} - a$, where p is the characteristic)

- Any separably closed field of char 0 is algebraically closed.
- If $ch(K) = p$, then K^p is a subfield. If the degree of $[K : K^p]$ is finite, it is of the form p^e, and e is called the *degree of imperfection* of K. For any $e \in \mathbb{N}$, let $\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of char p with the degree of imperfection e, and let $\mathrm{SCF}_{p,\infty}$ be the theory of separably closed fields of char p with infinite degree of imperfection.
- These are all complete theories of separably closed fields, and they eliminate quantifiers after naming a basis and adding some function symbols to the language.
- [Wood, 79] All these theories are stable (and in the non-algebraically closed case, strictly stable, i.e. not superstable).

(a) **Open problem**. Is every field K with $\mathrm{Th}(K)$ stable is separably closed? (a positive answer is known for superstable fields [1] and some other special cases).

(b) Differentially closed fields are stable. A differential field is a field K equipped with a function symbol $d : K \to K$ for a derivation d, i.e. d is an additive map such that $d(r_1 r_2) = d(r_1) r_2 + r_1 d(r_2)$ (Leibniz rule). The theory of differentially closed fields DCF_0 is the theory of differential fields of characteristic 0 satisfying the following property: for $f \in K[x_0, \ldots, x_n] \setminus K[x_0, \ldots, x_{n-1}]$ and $g \in K[x_0, \ldots, x_{n-1}]$, $g \neq 0$, there is some $a \in K$ such that $f(a, da, \ldots d^n a) = 0$ and $g(a, da, \ldots, d^{n-1} a) \neq 0$. Any differential field can be extended to a model of DCF_0, and DCF_0 has QE. The theory DCF_0 is stable (using QE one can establish a bijection between n-types over F and the so-called *prime δ-ideals* in $F\{x_1, \ldots, x_n\}$, the ring of differential polynomials, and such ideals are always generated by finitely many differential polynomials — a form of Noetherianity. Thus there are few types.) There is a positive characteristic analogue. See e.g. [8].

2.6. **References.** The proofs of Theorem 2.7 and Theorem 2.18 are from [10]. See also my stability notes [2] for more details on the topic.

3. Chain conditions in stable groups

3.1. Two generalizations of stability: NIP and NSOP.

Definition 3.1. (1) A (partitioned) formula $\phi(x, y) \in L$ has the *strict order property* (relatively to T), or *SOP*, if there is an infinite sequence $(b_i)_{i \in \omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j \in \omega$.
(2) A theory T has SOP if some formula does.
(3) T is *NSOP* if it doesn't have the strict order property.

Definition 3.2. (1) A (partitioned) formula $\phi(x, y)$ has the *independence property*, or *IP*, if (in $\mathbb{M}$) there are infinite sequences $(b_i)_{i \in \omega}$ and $(a_s : s \subseteq \omega)$ such that $\models \phi(a_s, b_i) \iff i \in s$.
(2) A theory T has IP if some formula does, otherwise T is *NIP*.

Theorem 3.3. *[Shelah] T is unstable if and only if it has the independence property or the strict order property. (See e.g.* [2]*.)*

3.2. **Chain conditions.** By a *uniformly definable family* of subgroups of G we mean a family of subgroups $(H_i : i \in I)$ of G such that for some $\phi(x,y) \in L$ we have $H_i = \phi(\mathbb{M}, a_i)$ for some parameter a_i, for all $i \in I$.

Lemma 3.4. *Let G be an* NSOP *group. For every formula $\phi(x,y)$ there is some $n = n(\phi) \in \omega$ such that every chain $H_1 \subseteq H_2 \subseteq \ldots$ of subgroups of G uniformly defined by ϕ has length at most n.*

Proof. Immediate from the definition of NSOP. □

Lemma 3.5. *Let G be an* NIP *group. For every formula $\phi(x,y)$ there is some number $m = m(\phi) \in \omega$ such that if I is finite and $(H_i : i \in I)$ is a uniformly definable family of subgroups of G of the form $H_i = \phi(\mathbb{M}, a_i)$ for some parameters a_i, then $\bigcap_{i\in I} H_i = \bigcap_{i\in I_0} H_i$ for some $I_0 \subseteq I$ with $|I_0| \leq m$.*

Proof. Otherwise for each $m \in \omega$ there are some subgroups $(H_i : i \leq m)$ such that $H_i = \phi(\mathbb{M}, a_i)$ and $\bigcap_{i\leq m} H_i \subsetneq \bigcap_{i\leq m, i\neq j} H_i$ for every $j \leq m$. Let b_j be an element from the set on the right hand side and not in the set on the left hand side. Now, if $I \subseteq \{0, 1, \ldots, m\}$ is arbitrary, define $b_I := \prod_{j\in I} b_j$. It follows that $\models \phi(b_I, a_i) \iff i \notin I$. This implies that $\phi(x,y)$ is not NIP. □

Combining, we get:

Theorem 3.6. *(Baldwin-Saxl) Let G be a definable group. Then for any stable formula $\phi(x,y)$ there is some $k = k(\phi) \in \omega$ such that any intersection of ϕ-definable subgroups is equal to a subintersection of size at most k.*

Proof. By Lemma 3.5, every element of such a chain is an intersection of at most m_ϕ ϕ-definable subgroups, and so we may assume that the elements of the chain are themselves uniformly definable (and the formula is stable, as stability is closed under Boolean combinations). But then by Lemma 3.4 such a chain can only have length at most n_ψ where $\psi(x,\bar{y}) = \bigwedge_{i<m_\phi} \phi(x, y_i)$. □

Corollary 3.7. *It follows that if G is stable and $A \subset G$ then there is some finite $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$, where $C_G(A) = \{g \in G : g \cdot a = a \cdot g$ for all $a \in A\}$ is the centralizer of A in G.*

Proof. Apply Theorem 3.6 to the formula $\phi(x,y)$ given by $x \cdot y = y \cdot x$. □

Corollary 3.8. *Let G be a stable group, and let $A \subseteq G$ be an abelian subgroup (not necessarily definable). Then there is a* ***definable*** *abelian subgroup $A' \supseteq A$ of G.*

Proof. Let A' be the center of the centralizer of A. It is an abelian subgroup of G, and by Corollary 3.7 it is definable. □

Exercise 3.9. The same statement is true for nilpotent and solvable subgroups of fixed class.

3.3. **Connected component.**

Definition 3.10. Let $G = G(\mathbb{M})$ be a stable group.

(1) For $\phi(x,y) \in L$, let $G^0_\phi := \bigcap \{H \leq G : H = \phi(\mathbb{M}, a)$ for some a and $[G : H] < \infty\}$. By Theorem 3.6 G^0_ϕ is in fact a definable subgroup. Besides, it is clear from the definition that G^0_ϕ is $\mathrm{Aut}(\mathbb{M}/\emptyset)$-invariant, which implies that G^0_ϕ is $\emptyset$-definable, of finite index in G.

(2) Let $G^0 := \bigcap_{\phi \in L} G^0_\phi$ — the *connected component* of G. This is a normal subgroup of G (as the set of all definable subgroups of finite index is closed under conjugation) of *bounded* index (i.e. the index is small, compared to the saturation of the monster) type definable over $\emptyset$. In fact, $\left[G : G^0\right] < 2^{|T|}$.

Remark 3.11. The term "connected component" comes from algebraic geometry. Namely, if G is an algebraic group, then G^0 is precisely the connected component in the sense of Zariski geometry.

Fact: If G is ω-stable then G^0 is $\emptyset$-definable.

Exercise 3.12. Consider $G = (\mathbb{Z}, +)$. Calculate $G^0(\mathbb{M})$ — note that as a type-definable set, we want to calculate it in a monster model of T, as it simply has no points in the standard model.

4. Forking and definability for stable formulas

4.1. $\mathbb{M}^{\mathrm{eq}}$ and canonical parameters.
See [2, Section 1.3].

4.2. Ideals of definable sets, finite satisfiability and forking.

Definition 4.1. Recall that a set $I \subseteq \mathrm{Def}_x(\mathbb{M})$ is an *ideal* if:

(1) $\emptyset \in I$,
(2) $\phi(x, a) \vdash \psi(x, b)$ and $\psi(x, b) \in I$ implies $\phi(x, a) \in I$,
(3) $\phi(x, a) \in I$ and $\psi(x, b) \in I$ implies $\phi(x, a) \vee \psi(x, b) \in I$.

Lemma 4.2. *(Extension of a type avoiding an ideal) If a partial type $\pi(x)$ over a set A doesn't imply a formula from an ideal $\mathcal{I}$, then for any set $B \supseteq A$ there is a complete type $p(x)$ over B extending $\pi(x)$ not containing any formulas from $\mathcal{I}$.*

Proof. We claim that the set of formulas

$$\tau(x) := \pi(x) \cup \{\neg\phi(x, b) : b \in B \text{ and } \phi(x, b) \in \mathcal{I}\}$$

is consistent. If not, then by compactness there are finitely many formulas $\phi_i(x, b_i) \in \mathcal{I}$ such that $\pi(x) \vdash \bigvee \phi_i(x, b_i)$. As $\mathcal{I}$ is an ideal, this is a contradiction to the assumption on π.

Note: if $\phi(x, b) \in \mathcal{I}$ then $\neg\phi(x, b) \notin \mathcal{I}$ (as $x = x \notin \mathcal{I}$ by assumption since $\pi(x) \vdash x = x$).

Hence any complete type $p(x)$ over B extending $\tau(x)$ satisfies the requirement. □

Definition 4.3. (1) A formula $\phi(x, a)$ *divides* over B if there is a sequence $(a_i)_{i \in \omega}$ and $k \in \omega$ such that $a_i \equiv_B a$ and $\{\phi(x, a_i)\}_{i \in \omega}$ is k-inconsistent. Equivalently, if there is a B-indiscernible sequence $(a_i)_{i \in \omega}$ starting with a and such that $\{\phi(x, a_i)\}_{i \in \omega}$ is inconsistent (exercise).
(2) A formula $\phi(x, a)$ *forks* over B if it belongs to the ideal generated by the formulas dividing over B, i.e. if there are $\psi_i(x, c_i)$ dividing over B for $i < n$ and such that

$$\phi(x, a) \vdash \bigvee_{i<n} \psi_i(x, c_i).$$

Exercise 4.4. In general there are formulas which fork, but do not divide.

Definition 4.5. A (partial) type does not divide (fork) over B if it does not imply any formula which divides (resp. forks) over B.

Note: if $a \notin \mathrm{acl}(A)$ then $\mathrm{tp}(a/Aa)$ divides over A. Also, if $\pi(x)$ is consistent and defined over $\mathrm{acl}(A)$, then it doesn't divide over A.

Exercise 4.6. Let $p \in S_x(\mathbb{M})$ be a global type, and assume that it doesn't divide over a small set A. Then it doesn't fork over A.

Definition 4.7. A global type $p(x) \in S(\mathbb{M})$ is called *invariant* over C if it is invariant under all automorphisms of $\mathbb{M}$ fixing C. That is, for every $a \equiv_C b$ and $\phi(x,y) \in L$, $\phi(x,a) \in p \Leftrightarrow \phi(x,b) \in p$.

Let $p \in S_x(\mathbb{M}), q \in S_y(\mathbb{M})$ be two global, A-invariant types. Then we define their tensor product $p \otimes q \in S_{xy}(\mathbb{M})$ as follows:

given a formula $\phi(x,y) \in L(B)$, $A \subseteq B \subseteq \mathbb{M}$, we set $\phi(x,y) \in p \otimes q \iff \phi(x,b) \in p$ for some (equivalently, any, by invariance of p) $b \in \mathbb{M}_y$ such that $b \models q|_B$.

Remark 4.8.
1. Note that $p \otimes q$ is indeed a complete type, as
$$p \otimes q = \bigcup_{A \subseteq B \subset_{\text{small}} \mathbb{M}} \{\mathrm{tp}(ab/B) : a \models p|_{bB}, b \models q|_B\}.$$
2. If both p, q are A-invariant, then $p \otimes q$ is also A-invariant (Exercise).
3. The operation $\otimes$ is associative, i.e. $p \otimes (q \otimes r) = (p \otimes q) \otimes r$. The reason is that for any small set B, both products restricted to B are equal to $\mathrm{tp}(abc/B)$ for $c \models r, b \models q|_{Bc}, a \models p|_{Bbc}$.
4. However, $\otimes$ need not be commutative. Let T be DLO, and let $p = q$ be the type at $+\infty$, it is $\emptyset$-invariant. Then $p(x) \otimes q(y) \vdash x > y$, while $q(y) \otimes p(x) \vdash x < y$. (Check however that any two distinct types in DLO commute).
5. In fact, in the definition of the tensor product, we have only used that p is invariant.

Definition 4.9. Let $p \in S_x(\mathbb{M})$ be a global A-invariant type. Then for any $n \in \omega$ we define by induction $p^{(1)}(x_0) := p(x_0)$ and $p^{(n+1)}(x_0, \ldots, x_n) := p(x_n) \otimes p^{(n)}(x_0, \ldots, x_{n-1})$. We also let $p^{(\omega)}(x_0, x_1, \ldots) := \bigcup_{n \in \omega} p^{(n)}(x_0, \ldots, x_{n-1})$. For any set $B \supseteq A$, a sequence $(a_i : i \in \omega) \models p^{(\omega)}|_B$ is called a *Morley sequence* of p over B (indexed by ω).

Remark 4.10.
1. We can define $p^{(I)}$ for an arbitrary order type I in a natural way.
2. Note that for any $(a_i : i < \omega), (b_i : i < \omega) \models p^{(\omega)}|B$ we have that $(a_i : i < \omega) \equiv_B (b_i : i < \omega)$. In particular, any Morley sequence of p over B is B-indiscernible, by the associativity of $\otimes$.

Exercise 4.11. We say that a partialy type $\pi(x)$ is a finitely satisfiable in A if for any $\phi(x,b) \in \pi$ if every finite conjunction of formulas from π is realized by some $a \in A$.
1. If $\pi(x)$ is finitely satisfiable in A, then there exists a complete global type extending $\pi(x)$ and finitely satisfiable in A.
2. Every global type finitely satisfiable in A is invariant over A.
3. Every global type invariant over a set A doesn't fork over A.

4.3. **Definability of types and stable formulas.**

Definition 4.12. (1) Let $\phi(x,y) \in L$ be given. A type $p(x) \in S_\phi(A)$ is *definable over* B if there is some $L(B)$-formula $\psi(y)$ such that for all $a \in A$,

$$\phi(x,a) \in p \iff \models \psi(a).$$

(2) A type $p \in S_x(A)$ is definable over B if $p|_\phi$ is definable over B for all $\phi(x,y) \in L$.
(3) A type is *definable* if it is definable over its domain.
(4) We say that types in T are *uniformly definable* if for every $\phi(x,y)$ there is some $\psi(y,z)$ such that every type can be defined by an instance of $\psi(y,z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that $\phi(x,a) \in p \iff \models \psi(a,b)$, for all $a \in A$.

Example 4.13. Consider $(\mathbb{Q},<) \models \mathrm{DLO}$, and let $p = \mathrm{tp}(\pi/\mathbb{Q})$ ($\pi \in \mathbb{R} \succ \mathbb{Q}$). It is easy to check by QE that p is not definable.

Fact 4.14. *Let $\phi(x,y)$ be a stable formula. Then all ϕ-types are uniformly definable (see* [2, Section 2.3]*).*

We prove a stronger statement, but for a restricted family of parameter sets.

Proposition 4.15. (1) *Let $\phi(x,y)$ be stable and $q(x) \in S_x(\mathbb{M})$ be a global type. Then for any small set A there is a finite sequence $(c_i : i < n)$ with $c_i \models q|_{Ac_{<i}}$ such that $q|_\phi$ is defined by a positive Boolean combination of the formulas $\phi^*(y,c_i) = \phi(c_i,y)$.*
(2) *If q is finitely satisfiable in a set B, then $q|_\phi$ is definable by a positive Boolean combination of the formulas $\phi^*(y,b)$ with $b \in B$.*

Proof. The proof is just a slight rephrasement of the proof of Erdos-Makkai above.

1) Suppose towards contradiction that there is no such finite sequence (c_i) for q, A and ϕ.

Then for all $n \in \omega$ we can construct inductively $(b_n,b'_n)_{n<\omega}$ and $(c_n)_{n<\omega}$ in $\mathbb{M}$ with $c_n \models q|_{Ac_{<n}}$ such that:

(1) $\phi(x,b_i)$ and $\neg\phi(x,b'_i)$ belong to p for every $i \in \omega$,
(2) $\phi(c_i,b_j) \to \phi\left(c_i,b'_j\right)$ holds for every $i<j$,
(3) $\phi(c_i,b_j)$ and $\neg\phi\left(x_i,c'_j\right)$ hold for every $i \geq j$.

Assume we have consructed $(b_i,b'_i,c_i : i<n)$. As $q|_\phi$ is not definable by a positive Boolean combination of the formulas $\phi^*(c_i,y)$ for $i<n$, there are some tuples b_n,b'_n in $\mathbb{M}$ such that $\phi(x,b_n) \in p$, $\phi(x,b'_n) \notin p$ and $\phi(c_i,b_n) \to \phi(c_i,b'_n)$ for all $i<n$. Taking any $c_n \models q|_{Ab_{\leq n}b'_{\leq n}c_{<n}}$ we obtain the desired sequence.

Now by Ramsey, passing to an infinite subsequence we may assume that either $\models \phi(c_i,b_j)$ for all $i<j$, or $\models \neg\phi(c_i,b_j)$ for all $i<j$. In the first case, the sequence $(c_i,b'_i)_{i\in\omega}$ witnesses that $\phi(x,y)$ is not stable, in the second case the sequence $(c_i,b_{i+1})_{i\in\omega}$ witnesses this.

2) Using finite satisfiability of $q$in B, in the construction above we find c_n in B such that $c_n \models (q|_\phi)_{b_{\leq n}b'_{\leq n}c_{<n}}$ (which is equivalent to a finite conjunction of formulas from q, hence realized in B). □

Corollary 4.16. *If $\phi(x,y)$ is stable, $\mathcal{M} \preceq \mathbb{M}$ and $p \in S_\phi(M)$, then p is definable by a positive Boolean combination of the formulas $\phi^*(y,b)$ with $b \in M$.*

Proof. As remarked before, every partial type p over a model is finitely satisfiable in it, and extends to a global type q finitely satisfiable in it. Then 2) in the previous proposition applies. □

Proposition 4.17. *(Existence of definable extensions) Let $\phi(x,y)$ be stable and $p(x) \in S(A)$. Then there is $q \in S_\phi(\mathbb{M})$ such that $p(x) \cup q(x)$ is consistent and q is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.*

Proof. Let $X := \{q(x) \in S_\phi(\mathbb{M}) : p(x) \cup q(x) \text{ is consistent}\} \subseteq S_\phi(\mathbb{M})$. Note that X is the image under the restriction map $S_x(\mathbb{M}) \to S_\phi(\mathbb{M})$ of the closed set of all types in $S_x(\mathbb{M})$ extending $p(x)$. Since the restriction map is closed, X is a closed subset of $S_\phi(\mathbb{M})$ (which is a Stone space with a basis of clopens given by the sets $[\psi] = \{q \in S_\phi(\mathbb{M}) : q \vdash \psi\}$, ψ a Bolean combination of ϕ-formulas.

Now we run a Cantor-Bendixon analysis on X. Set $X^{(0)} := X$ and let $X^{(i+1)}$ be the set of accumulation points in $X^{(i)}$. I.e., $q(x) \in X^{(i+1)}$ if $q \in X^{(i)}$ and for any $\psi(x)$ a Boolean combination of ϕ-formulas, if $q \vdash \psi$ then there is some $q' \in X^{(i)}$ such that $q' \vdash \psi$ and $q' \neq q$. Observe inductively that each $X^{(i)}$ is closed in $S_\phi(\mathbb{M})$.

Claim. $X^{(n+1)}$ is empy for some n.

Otherwise, for any $n \in \omega$, there is some ϕ-type $p \in X^{(n)}$ such that for any $\psi(x)$ a Boolean combination of ϕ-formulas with $p \vdash \psi$, there is some $p' \in X^{(n)}, p' \neq p$ such that $p' \vdash \psi$. As $\phi(x,a) \in p, \neg\phi(x,a) \in p'$ for some a, we have that $\psi(x) \wedge \phi(x,a)$ and $\psi(x) \wedge \neg\phi(x,a)$ are each consistent, but contradictory. Arguing inductively, we construct a binary tree $\left\{\phi\left(x, a_{\eta|i}\right)^{\eta(i)} : \eta \in 2^n, i < n\right\}$ such that each branch is consistent, but the conjunction of any two branches is inconsistent (as always, $\phi^0 = \phi$ and $\phi^1 = \neg\phi$). As $n \in \omega$ was arbitrary, by compactness for any cardinal κ we can construct a binary tree $\left\{\phi\left(x, a_{\eta|i}\right)^{\eta(i)} : \eta \in 2^\kappa, i < \kappa\right\}$ with the same property. Let A contain all of the parameters $a_{\eta|i}$, then $|A| \leq 2^{<\kappa}$ and $|S_\phi(A)| = 2^\kappa$. Taking any κ such that $2^{<\kappa} < 2^\kappa$, we get a contradiction to stability of ϕ by Proposition 2.20.

Now as $X^{(n+1)} = \emptyset$, all types in $X^{(n)}$ are isolated. Then compactness of $X^{(n)}$ implies that it is finite.

Given $q \in X^{(n)}$, it is definable (over $\mathbb{M}$) by Corollary 4.16. As $X^{(n)}$ is A-invariant, the orbit of q under $\mathrm{Aut}(\mathbb{M}/A)$ is finite, hence the canonical parameter of $d_\phi q$ is in $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Lemma 4.18. *Let $\phi(x,y)$ be stable, and let $p(x)$ and $q(y)$ be global types. Then $d_p\phi(y) \in q(y) \iff d_q\phi^*(x) \in p(x)$ for any definitions such that $d_p\phi(b) \iff \phi(x,b) \in p(x)$ and $d_q\phi^*(a) \iff \phi^*(a,y) \in q(y)$ for all a,b in $\mathbb{M}$.*

Proof. Let A be a small set such that $p, q, \phi(x,y)$ are all definable over A. We define a sequence $(a_i, b_i : i \in \omega)$ recursively. Given $(a_i, b_i : i < n)$, let $b_n \models q|_{Aa_0 \ldots a_{n-1}}$ and let $a_n \models p|_{Ab_0 \ldots b_n}$. Then for $i < j$ we have

$$\models \phi(a_i, b_j) \iff \phi(a_i, y) \in q \iff \models d_q^*\phi(a_i) \iff d_q^*\phi(x) \in p,$$

and for $j \leq i$ we have

$$\models \phi(a_i, b_j) \iff \phi(x, b_j) \in p \iff \models d_p\phi(b_j) \iff d_p\phi(y) \in q.$$

Since $\phi(x,y)$ does not have the order property, the claim follows. □

Definition 4.19. A *generalized ϕ-type* over a set A is a maximal consistent collection of formulas with parameters in A which are equivalent to a Boolean combination of ϕ-formulas (with parameters anywhere in $\mathbb{M}$).

Exercise 4.20. If M is a model, then a generalized ϕ-type over M is equivalent to a ϕ-type over M.

Corollary 4.21. *(Uniqueness of definable extensions over algebraically closed sets) Let $\phi(x,y)$ be stable. A complete generalized ϕ-type over a set $A = \operatorname{acl}^{\mathrm{eq}}(A)$ has a unique A-definable global ϕ-type extension.*

Proof. Let $A = \operatorname{acl}^{\mathrm{eq}}(A)$, let p be a complete generalized ϕ-type over A.

Existence follows by Lemma 4.17 applied to any completion of p over A.

Towards uniqueness, let $p_1, p_2 \in S_\phi(\mathbb{M})$ be two extensions of p definable over A.

Let $\phi(x,b)$ be any instance with $b \in \mathbb{M}$, let $q(y) := \operatorname{tp}(b/A)$.

Again by Lemma 4.17 (applied to the stable formula ϕ^*), there is $\bar{q} \in S_{\phi^*}(\mathbb{M})$ definable over A and such that $q(y) \cup \bar{q}(y)$ is consistent. Let $q' \in S_y(\mathbb{M})$ be any completion of $q(y) \cup \bar{q}(y)$, note that $q'|_{\phi^*} = \bar{q}$.

As q', p_1, p_2 are all definable over A, by Lemma 4.18 for each i we have:

$$\phi(x,b) \in p_i \iff d_{p_i}\phi^*(x) \in q' \iff d_{q'}\phi(y) \in p_i',$$

where p_i' is an arbitrary completion of p_i.

On the other hand, by Lemma 4.15 applied to q', the formula $d_{q'}\phi^*(x)$ is equivalent to a positve Boolean combination of ϕ-formulas with parameters in $\mathbb{M}$ (any two definitions of the same type are equivalent). As both p_i' extend the generalized ϕ-type p over A, we have

$$d_{q'}\phi^*(x) \in p_i' \iff d_{q'}\phi^*(x) \in p.$$

Combining, $\phi(x,b) \in p_1 \iff \phi(x,b) \in p_2$, as wanted. □

Lemma 4.22. *Let $\phi(x,y)$ be stable. If $p \in S_\phi(\mathbb{M})$ is definable over M and consistent with a partial type $\pi(x)$ over M, then $\pi(x) \cup p(x)$ is finitely satisfiable in M.*

Proof. First note that $\pi(x) \cup p|_M(x)$ is finitely satisfiable in M. By Lemma 4.2, let $q \in S_x(\mathbb{M})$ be finitely satisfiable in M and extendion $\pi(x) \cup p|_M(x)$. Then $q|_\phi$ is definable over M by Proposition 4.15.

Finally, using 4.21, we have $q|_\phi = p$. □

Now we demonstrate that all our notions of "generic" extensions of a type coincide for stable formulas.

Theorem 4.23. *Let $\phi(x,y)$ be stable, and $a \in \mathbb{M}_y$. TFAE:*

1. *$\phi(x,a)$ is satifiable in every model containing A;*
2. *$\phi(x,a)$ doesn't fork over any model containing A;*
3. *$\phi(x,a)$ doesn't divide over A;*
4. *there is a positive Boolean combination of A-conjugates of $\phi(x,a)$ which is equivalent to a consistent formula with parameters over A;*
5. *there is a global complete ϕ-type p containing $\phi(x,a)$ which is definable over* $\operatorname{acl}^{\mathrm{eq}}(A)$.

Proof. Already know (1) $\implies$ (2) $\implies$ (3).

(3) $\implies$ (4). We use the previous lemmas without mentioning. Suppose $\phi(x,a)$ doesn't divide over A, let $q(y) := \mathrm{tp}(a/A)$, and $q'(y) \in S_{\phi^*}(\mathbb{M})$ such that $q(y) \cup q'(y)$ is consistent and q' is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. In particular, q' is definable over some/any model $M \supseteq A$, hence $q(y) \cup q'(y)$ is finitely satisfiable in M. Hence there is $\bar{q} \in S_y(\mathbb{M})$ extending $q(y) \cup q'(y)$ and finitely satisfiable in M.

By Lemma 4.15, there is a sequence $(c_i : i < n)$ with $c_i \models \bar{q}|_{Mc_{<i}}$ such that $\bar{q}|_{\phi^*} = q'$ is definable by a positive Boolean combination $\psi(x)$ of the formulas $\phi(x,c_i)$. By saturation of $\mathbb{M}$, we can extend this to a sequence $(c_i : i < \omega)$ such that $c_i \models \bar{q}|_{Mc_{<i}}$ for all $i < \omega$. Then $(c_i : i < \omega)$ is M-indiscernible by Remark 4.10, hence A-indiscernible. As $\bar{q} \supseteq q = \mathrm{tp}(a/A)$, we have $c_i \equiv_A a$ and so $\psi(x)$ is consistent since $\phi(x,a)$ doesn't divide over A.

On the other hand, as $\bar{q}|_{\phi^*} = q'$ we have that $\psi(x)$ is equivalent to a formula $\chi(x)$ with parameters over $\mathrm{acl}^{\mathrm{eq}}(A)$ since any two definitions of the same type are equivalent. Let $\chi'(x)$ be the disjunction of all the finitely many (since its parameters are in $\mathrm{acl}^{\mathrm{eq}}(A)$, so have finite orbits) A-conjugates of $\chi(x)$. But then $\chi'(x)$ is as wanted.

(4) $\implies$ (5). Let $\chi(x)$ be as given by (4). There is $\bar{q} \in S_\phi(\mathbb{M})$ consistent with $\chi(x)$ and definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Hence some A-conjugate of $\phi(x,a)$ belongs to $\bar{q}$, and so applying an A-automorphism the formula $\phi(x,a)$ belongs to some A-conjugate of $\bar{q}$. As every A-automorphism permutes $\mathrm{acl}^{\mathrm{eq}}(A)$ and preserves definability of a type, we get that $\phi(x,a)$ belongs to some global ϕ-type definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

(5) $\implies$ (1) A type p given by (5) is definable over any model M containing A (since it necessarily contains $\mathrm{acl}^{\mathrm{eq}}(A)$), hence it is finitely satisfiable in A by Lemma 4.22. ☐

Definition 4.24. Let $\Delta = \{\phi_i(x,y_i) : i \leq n\}$ by a finite set of formulas.

1. By a Δ-*formula* over A we mean a formula of the form $\pm\phi_i(x,a)$ with $a \in A$.
2. A complete Δ-type over A is a maximal consistent set of Δ-formulas over A.

We observe that Δ-types can be coded as ϕ-types for an appropriate choice of ϕ, so the previously developed theory applies to Δ-types as well.

Exercise 4.25. Given a finite set of formulas Δ as above, there is formula $\psi_\Delta(x, y_0, \ldots, y_n, z, z_0, \ldots, z_{2n})$ such that:

1. If A has at least two elements, then each Δ-formula over A is equivalent to a positive ψ_Δ-formula over A.
2. Any consistent positve ψ_Δ-formulas over A is equivalent to a Δ-formula over A.
3. If all formulas in Δ are stable, then ψ_Δ is stable.

Proposition 4.26. *Let $\phi(x,y)$ and $\psi(x,z)$ be stable formulas, and suppose that both $\phi(x,a)$ and $\psi(x,b)$ divide over A. Then $\phi(x,a) \vee \psi(x,b)$ also divides over A.*

Proof. WLOG $A = \mathrm{acl}^{\mathrm{eq}}(A)$. Let $\delta(x;y,z) := \phi(x,y) \vee \psi(x,z)$, then δ is stable. Suppose that $\delta(x;a,b)$ doesn't divide over A. By Theorem 4.23, there is some $p \in S_\delta(\mathbb{M})$ such that p is definable over A. Let $p_0 := p|_A$, so $p_0 \in S_\delta(A)$. Let $\Delta(x;y,z) := \{\phi(x,y), \psi(x,z), \delta(x,y,z)\}$, and let $p_1 \in S_\Delta(A)$ contain p_0. Then

p_1 has a unique A-definable global extension $q \in S_\Delta(\mathbb{M})$ (since Lemma 4.21 holds for Δ-types, by applying it to ψ_Δ given by Exercise 4.25). Thus $q|_\delta = p$. So $\delta(x; a, b) \in q$, hence either $\phi(x, a) \in q$ or $\psi(x, b) \in q$. Hence one of these formulas doesn't divide over A by Theorem 4.23. □

Corollary 4.27. *If $\phi(x, y)$ is stable, then for any a and A, either $\phi(x, a)$ or $\neg\phi(x, a)$ doesn't divide over A.*

Definition 4.28. Let $\mathrm{FER}_\phi(A)$ denote the collection of formulas $E(x_1, x_2)$ over A which define an equivalence relation with finitely many classes, such that for each a, $E(x, a)$ is equivalent to a Boolean combination of ϕ-formulas over A.

Proposition 4.29. *(Finite equivalence relations theorem) Let $\phi(x, y)$ be stable. Let $p(x)$ be a generalized ϕ-type over A. Let $X := \{q(x) \in S_\phi(\mathbb{M}) : q$ is a non-forking extension of $p\}$. Then*

1. *X is finite,*
2. *$\mathrm{Aut}(\mathbb{M}/A)$ acts transitively on X,*
3. *there is some $E(x_1, x_2) \in \mathrm{FER}_\phi(A)$ such that for all $q_1, q_2 \in X$, $q_1 = q_2$ if and only if $q_1(x_1) \cup q_2(x_2) \vdash E(x_1, x_2)$.*

Proof. Claim. Let $Y := \left\{q|_{\mathrm{acl}^{\mathrm{eq}}(A)} : q \in X\right\}$. Then $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(A)/A)$ acts transitively on Y.

Let $p_1(x) \in S_x(A)$ extending p and $q(x) \in Y$ be arbitrary. Then $p_1(x) \cup q(x)$ is consistent: otherwise $p_1(x) \vdash \neg\psi(x)$ for some $\psi(x) \in q$. Let $\chi(x)$ be the finite disjunction of all conjugates of $\psi(x)$ under $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(A)/A)$. Then $p_1 \vdash \neg\chi(x)$. But χ is a ϕ-formula over A which is in q, hence also in p — this is a contradiction. Hence $p_1(x) \cup q(x)$ is consistent, and this proves the claim.

Now (2) follows by uniqueness of definable extensions of types over algebraically closed sets, and (2) $\Rightarrow$ (1).

(3) A X is finite, using uniqueness of definable extensions of types over algebraically closed sets again, there is a finite collection $\Phi(x)$ of ϕ-formulas over $\mathrm{acl}^{\mathrm{eq}}(A)$ such that for $q_1, q_2 \in X$, $q_1 = q_2 \iff$ for each $\phi(x) \in \Phi$, $\phi(x) \in q_1 \iff \phi(x) \in q_2$. Possibly adding finitely many formulas, we may assume that Φ is closed under A-automorphisms. Let $E(x_1, x_2)$ be the formula $\bigwedge\{\phi(x_1) \leftrightarrow \phi(x_2) : \phi \in \Phi\}$. Then E is as wanted. □

4.4. **References.** In this section we have followed [6, Section 5], [7] and [2].

5. Fundamental theorem of stable group theory

5.1. **Setting: definable homogeneous spaces and equivariant formulas.** Fix a theory T. We will work with a definable homogeneous space — i.e. a definable group $G(z)$ with a definable transitive (every element can be sent to any other one) action on a definable set $S(x)$ (i.e. the graphs of multiplication on G and of the action of G on S are definable relations). We assume that everything is $\emptyset$-definable.

Definition 5.1. Let $\delta(x, y)$ be a formula such that $\models \delta(x, y) \to S(x)$.

1. We say that $\delta(x, y)$ is a (G-) *equivariant* formula if for every $a \in \mathbb{M}$ and $c \in G$ ($=G(\mathbb{M})$) there is some $b \in \mathbb{M}$ such that $\models \delta(cx, a) \leftrightarrow \delta(x, b)$.
2. Note: every instance of any (stable) formula is equivalent to an instance of a (stable) equivariant formula.

(3) If $X \subseteq S$ is defined by $\delta(x, a)$ for some a, and $c \in G$, then cX is defined by $\delta\left(c^{-1}x, a\right)$.
(4) It follows that if $\phi(x)$ is a δ-formula over M, then for any $c \in G(M)$, $\phi(cx)$ is also a δ-formula. Hence if $p(x) \in S_\delta(M)$ and $c \in G(M)$, then $cp := \left\{\phi\left(c^{-1}x\right) : \phi(x) \in p\right\} \in S_\delta(M)$. Hence $G(M)$ acts on $S_\delta(M)$ by homeomorphisms.

Our aim is to understand this action.

5.2. **Generic sets.**

Definition 5.2. (1) We say that $X \subseteq S$ is *generic* if finitely many G-translates of X cover S.
(2) Hence if $\phi(x)$ is a δ-formula, it is generic if $S(x) \vdash \bigvee_{i=1}^{n} \phi\left(c_i^{-1}x\right)$ for some $c_i \in G$.
(3) A partial type $p \in S_\delta(\mathbb{M})$ is generic if every formula in p is generic.

Remark 5.3. The term "generic" here is inspired by algebraic geometry. In topological dynamics, such sets are called *syndetic.*

We will prove some results about syndetic sets. Our method is to convert a question about a definable group action into a question about the action of an automorphism group of some theory — to which the results about forking from the previous section can be applied.

Proposition 5.4. *Let $\phi(x, y)$ be a stable equivariant formula and let $X = \phi(x, b)$ for some b. Then either X or $S \setminus X$ is generic.*

Proof. We define an auxiliary two-sorted structure $\mathbb{M}_0 = (S, G, R)$ which is a reduct of $\mathbb{M}$. Namely, $R(x, y) \subseteq S \times G$ is a binary relation defined by $\mathbb{M}_0 \models R(x, y) \iff \mathbb{M} \models x \in S \wedge y \in G \wedge y^{-1}x \in X$. Let $T_0 := \mathrm{Th}(\mathbb{M}_0)$. We have:
(1) $R(x, y)$ is a stable formula in T_0 (follows from stability and equivariance of $\phi(x, y)$);
(2) Given $g \in G$, let σ_g be the map taking $s \in S \mapsto gs$ and $h \in G \mapsto gh$. It is easy to see that $\sigma_g \in \mathrm{Aut}(\mathbb{M}_0)$.
(3) Since G acts transitively on S, it follows from (2) that in T_0 there is a unique complete type over $\emptyset$ implying $S(x)$.

Let $1 \in G$ be the identity element, note that $R(\mathbb{M}_0, 1) = X, \neg R(\mathbb{M}_0, 1) = S \setminus X$. Obviously, any $\mathrm{Aut}(\mathbb{M}_0)$-conjugate of $R(x, 1)$ is of the form $R(x, g)$ for some $g \in G$.

By Corollary 4.27 either $R(x, 1)$ or $\neg R(x, 1)$ doesn't divide over $\emptyset$ in T_0. Assume the first case (the second case is analogous, showing that $S \setminus X$ is generic).

Then by Theorem 4.23, there is some *positive* Boolean combination of $\mathrm{Aut}(\mathbb{M}_0/\emptyset)$-conjugates of $R(x, 1)$ which is consistent and $\emptyset$-definable. Since $S(x)$ determines a unique complete type, it follows that $S(x)$ is equivalent to a *positive* Boolean combination of formulas of the form $R(x, g)$ with $g \in G(\mathbb{M}_0)$, hence there are some $g_0, \ldots, g_n \in G$ such that $S(x) \vdash \bigvee_{i \leq n} R(x, g_i)$, hence X is generic. □

Proposition 5.5. *Let $\phi(x, y)$ be a stable equivariant formula and let $X = \phi(x, b)$ for some b. TFAE:*
(1) *X is generic;*
(2) *$g\phi(x, b)$ doesn't fork over $\emptyset$ for every $g \in G$;*
(3) *for any indiscernible sequence $(g_i : i \in \omega)$ from G, the set $\{g_iX : i \in \omega\}$ is consistent.*

Proof. (2) $\Rightarrow$ (1) Assume first that X is not generic, but $g\phi(x,b)$ doesn't fork over $\emptyset$ for all $g \in G$. By the previous proof, in T_0 the formula $R(x,1)$ divides over $\emptyset$. That is, there is an infinite $\emptyset$-indiscernible (in the sense of $\mathbb{M}_0$) sequence $(g_i : i \in \omega)$ such that $\{R(x,g_i) : i \in \omega\}$ is k-inconsistent, for some $k \in \omega$. Then by compactness, for any small cardinal κ we can find a sequence $(g_i : i < \kappa)$ with the same property. Take some $\kappa > 2^{|T|}$.

By assumption, each of the formulas $g\phi(x,b)$ doesn't fork over $\emptyset$, hence it belongs to some global ϕ-type p_i non-forking over $\emptyset$ — and there are at least κ many pairwise distinct p_i's by the previous paragraph. By Corollary 4.21, each p_i is determined by its restriction to $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. However there are at most $2^{|T|}$ complete types over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, giving a contradiction.

(1) $\Rightarrow$ (2) We introduce another auxiliary structure $\mathbb{M}_1$ that will be a reduct of $\mathbb{M}^{\mathrm{eq}}$. Let $E(y_1,y_2)$ be the formula $\forall x(\phi(x,y_1) \leftrightarrow \phi(x,y_2))$. Let Γ denote the sort S_E in $\mathbb{M}^{\mathrm{eq}}$, and let $\epsilon(x,y)$ be the T^{eq}-formula $\exists y(\phi(x,y) \wedge w = y/E)$ (so Γ correspond to the collection of subsets of S given by $g\phi(\mathbb{M},a), a \in \mathbb{M}, g \in G$ and ϵ is the membership relation).

Consider $\mathbb{M}_1 = (S,\Gamma,\epsilon)$ as a two-sorted structure, and let $T_1 := \mathrm{Th}(\mathbb{M}_1)$. As in the proof of Proposition 5.4 we have:

1. $\epsilon(x,y)$ is stable in T_1 (using stability and equivariance of ϕ);
2. for any $g \in G$, the permutation τ_g given by $b \mapsto gb$ for $b \in S$ extends to a unique automorphism of $\mathbb{M}$ (given by $Y \in \Gamma \mapsto gY$);
3. hence in T_1, the formula $S(x)$ determines a complete type over $\emptyset$.
4. Note that every type $q \in S_\epsilon(\mathbb{M}_1)$ can be identified with the type $q' \in S_\phi(\mathbb{M})$ given by $\phi(x,a) \in q'$ whenever $\epsilon(x,a) \in q$.

By (3), there is a unique $p \in S_\varepsilon(\emptyset)$, in T_1.

By Proposition 4.17, let $p \in S_\varepsilon(\mathbb{M}_1)$ extending $S(x)$ be definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, in the sense of T_1.

- Let $p' \in S_\phi(\mathbb{M})$ be the associated type, then p' is $\emptyset$-definable in T.
  Indeed, note that any automorphism from $\mathrm{Aut}(\mathbb{M})$ induces an automorphism in $\mathrm{Aut}(\mathbb{M}_1)$. As p' is definable over $\mathbb{M}$ by stability of ϕ, the canonical parameter of its definition must have a finite orbit under the action of $\mathrm{Aut}(\mathbb{M})$ since this holds for the canonical parameter of p under the action of $\mathrm{Aut}(\mathbb{M}_1)$.

Now suppose that X is generic. Then there is some translate $h\phi(x,b) \in p'$, hence $h\epsilon(x,b) \in p$. Thus $\epsilon(x,b) \in \tau_{h^{-1}}(p)$, and $\tau_{h^{-1}}(p)$ is $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable as a conjugate of a type with the same property. Hence $h^{-1}p'$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, and so $\phi(x,b)$ doesn't divide over $\emptyset$. As for any $g \in G$, the set gX is generic, the argument applied to it gives (2).

(2) $\iff$ (3) follows from proof (Exercise). □

Corollary 5.6. *Let $\phi(x,y)$ be a stable equivariant formula. Let X be the set of generic types in $S_\phi(\mathbb{M})$.*

1. *X is non-empty and finite;*
2. *G acts transitively on X;*
3. *There is $E_\phi \in \mathrm{FER}_\phi(\emptyset)$ such that E_ϕ is G-invariant (i.e. $E(a,b) \implies E(ga,gb)$ for any $g \in G$) and for $p_1,p_2 \in X$, $p_1 = p_2 \iff p_1$ and p_2 contain the same E_ϕ-class;*
4. *A ϕ-formula $\psi(x)$ is generic if and only if $\psi(x) \in p$ for some $p \in X$.*

Proof. Essentially follows from Proposition 4.29 applied in $\mathbb{M}_1$.

By (3) in the proof of Proposition 5.5, there is a unique $p \in S_\varepsilon(\emptyset)$ in T_1. Applying Proposition 4.29 to obtain the finite non-empty set $X_1 = \{q \in S_\epsilon(\mathbb{M}_1) : q \text{ doesn't fork over } \emptyset\}$ and $E_1 \in \mathrm{FER}_\epsilon(\emptyset)$ such that the types in X_1 are distinguished by the E_1-classes they contain; and $\mathrm{Aut}(\mathbb{M}_1)$ acts transitively on X_1.

As E_1 is $\emptyset$-definable, by (2) in the proof of Proposition 5.5 E_1 is G-invariant. In particular, since the action of G is transitive, we have that

- G acts transitively on the E_1-classes, and thus on X_1.

It remains to show the following.

Claim. If $\psi(x)$ is a ϕ-formula over $\mathbb{M}$ (hence, it is equivalent to an ϵ-formula over $\mathbb{M}_1$), then $\psi(x)$ is generic $\iff$ $\psi(x) \in q$ for some $q \in X$.

$\Rightarrow$ Already saw it in the last paragraph of proof of Proposition 5.5.

$\Leftarrow$ Suppose $\psi(x) \in q$ for some $q \in X_1$. By the bullet, some finite union $\chi(x)$ of G-translates of $\psi(x)$ is contained in every type in X_1. Thus $\neg\chi(x)$ belongs to no type in X_1. By the first implication, $\neg\chi(x)$ is not generic. But then by Proposition 5.4 $\chi(x)$ is generic. Hence $\psi(x)$ is also generic. □

Remark 5.7. These results with minor modifications apply to *type-definable* homogeneous spaces.

Exercise 5.8. We worked with an action of G on S by multiplication on the left. Identical results hold with respect to an action on the right.

Assume that G is a stable group acting on itself by both left and right multiplication. Show that then a set X is left-generic if and only if it is right-generic.

5.3. **The fundamental theorem of stable groups (local version).** Now we restrict to the case $S = G$, i.e. the group is acting on itself. Recall from Section 3.3 that $G^0 = \bigcap_{\phi \in L} G^0_\phi$ and G^0_ϕ is $\emptyset$-definable.

Theorem 5.9. *(Local version) Let $\delta(x, y)$ be a stable equivariant formula.*

1. *Let $E_\delta(\mathbb{M}, 1)$ be the E_δ-class of the identity in G (where E_δ is from Corollary 5.6).*
   Then $E_\delta(\mathbb{M}, 1)$ is a subgroup of G of finite index, and the E_δ-classes in G are precisely the left cosets of $E_\delta(\mathbb{M}, 1)$.
2. *Each left coset of $E_\delta(\mathbb{M}, 1)$ is contained in a unique generic type $p \in S_\delta(\mathbb{M})$.*
3. *For any $\phi(x) \in \mathrm{Def}_\delta(G)$ and any left coset C of $E_\delta(\mathbb{M}, 1)$ in G, exactly one of $C \cap \phi(G)$ or $C \cap \neg\phi(G)$ is generic.*
4. *For any $\phi(x) \in \mathrm{Def}_\delta(G)$, if Y is the union of the cosets of $E_\delta(\mathbb{M}, 1)$ whose intersection with ϕ is generic, then $\phi(G)\,\Delta Y$ is not generic.*
5. $E_\delta(\mathbb{M}, 1) = G^0_\delta$.
6. *There is a unique*

Proof. 1) Since E_δ is G-invariant, we see that $x \in g$ □

Definition 5.10. Let $p \in S_G(\mathbb{M})$ and let $\phi(x, y)$ be a formula. We define stabilizers $\mathrm{Stab}_\phi(p) := \{g \in G : \forall y\, (\phi(x, y) \in p \iff \phi(g \cdot x, y) \in p)\} = \{g \in G : gp|_\phi = p|_\phi\}$ and $\mathrm{Stab}_\phi(p) := \bigcap_{\phi \in L} \mathrm{Stab}_\phi(p) = \{g \in G : gp = p\}$. Both are subgroups of G.

By the definability of the type p it follows that $\mathrm{Stab}_\phi(p)$ is a definable group (as $\{g \in G : \forall y\, (d_p\phi(x, y) \leftrightarrow d_p\phi(g \cdot x, y))\}$), and so $\mathrm{Stab}(p)$ is type-definable.

Consider a formula $\phi(x;y,u) = \phi'(u \cdot x, y)$. Then any translate of an instance of ϕ is again an instance of ϕ, and any definable set is defined by an instance of such formula.

Proposition 5.11. (1) *For any formula ϕ, the set $\{p|_\phi : p$ is generic$\}$ is finite.*
(2) $\mathrm{Stab}_\phi(p) \subseteq G^0_\phi$ *and* $\mathrm{Stab}(p) \subseteq G^0$ *for any* $p \in S_G(\mathbb{M})$.
(3) *If p is generic then* $\mathrm{Stab}_\phi(p)$ *has finite index in G, and* $\mathrm{Stab}(p) = G^0$.

Proof. (1) The type p contains the information about its coset modulo $G^0(\phi(ux,y))$. Let $\psi(x)$ define G^0_ϕ. There is some $b \in G$ with $\psi(b^{-1}x) \in p$ (as p has to be in some coset of G^0_ϕ). If $g \in \mathrm{Stab}_\phi(p)$ then $\psi(b^{-1}gx) \in p$. Hence $b^{-1}gb$ and $g \in G^0_\phi$.

(2) As generic types do not fork over $\emptyset$, their number is bounded by **??**. Hence there are only finitely many generic ϕ-types, as otherwise could produce unboundedly many by compactness.

(3) Follows from (2) as a translate of a generic is generic. □

Theorem 5.12. *(Global version)*
(1) *There is a unique generic in every coset of G^0, G/G^0 is a profinite group acting transitively on its generics, p is generic iff* $\mathrm{Stab}(p) = G^0$ *(and generics form the unique minimal flow).*
(2) *This action is uniquely ergodic, with the measure supported on the orbit of the generic types.*
(3) *The following are equivalent for a definable set $\phi(x)$:*
   (a) *$\phi(x)$ is generic, in the sense of the definition above.*
   (b) *no translate of $\phi(x)$ forks over $\emptyset$.*
   (c) *$\phi(x)$ has positive measure with respect to all G-invariant Keisler measures.*

Proof. Let M be a small model containing representatives of every coset of G^0, possible since there are boundedly many of them.

Note that $G^0(x)$ is a generic partial type (as each G^0_ϕ is of finite index, so finitely many translates cover G). Hence there is some generic type p in G^0 — we call it *principal generic*. By translation, in every coset of G^0 there is a generic. By translation, enough to show that there is only one generic in G^0.

Choose realizations a and b of two generic types concentrated on G^0 and independent over M. Since $\mathrm{Stab} = G^0$, b and ab realize the same type over Ma, and so over M. Similarly, a and ab have the same type over M. Hence a and b have the same type over M. Works for any model M, so determines the complete type.

If p is generic, we already know $\mathrm{Stab}(p) = G^0$. Assume $\mathrm{Stab}(p) = G^0$. Then this is true for every heir of p, by definability. Let a realize p and b realize the principal generic over Ma. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$ and $a, b \cdot a$ have the same type over Mb. Furthermore, $b \cdot a$ is generic over Ma. Therefore a is generic over M. □

5.4. **Stable type definable groups are intersections of definable groups.**

Exercise 5.13. Consider $\mathcal{M} = (\mathbb{R}, +, <)$, $G(M) = (\mathbb{R}, +)$, and let $G = G(\mathbb{M})$. Let $\Phi(x) = \left\{-\frac{1}{n} < x < \frac{1}{n}\right\}$ be a partial type. Then $\Phi(\mathbb{M})$ is a subgroup of G consisting of the infinitesimal elements. Show that in any expansion of $\mathcal{M}$, $\Phi(\mathbb{M})$ is **not** an intersection of definable subgroups of G.

Note that $\mathrm{Th}(\mathcal{M})$ is not stable in this example. We show that such a situation cannot occur in a stable theory.

5.5. **References.** In this section we have followed [7], [9] and [3].

6. NIP groups

6.1. Connected components, revised.

Definition 6.1. Let $\mathbb{M} \models T$ be arbitrary, and let A be a small set of parameters. We write $\kappa(\mathbb{M})$ to denote the saturation cardinal of $\mathbb{M}$, and assume that it is a strong limit. Define:

(1) $G_A^0 = \bigcap \{H \leq G : H \text{ is definable over } A, \text{ of finite index}\}$.
(2) $G_A^{00} = \bigcap \{H \leq G : H \text{ is type-definable over } A, \text{ of bounded index}\}$. Then G_A^{00} is the smallest subgroup of G of bounded index, type-definable over A.
(3) $G_A^{000} = G_A^{\infty} = \bigcap \{H \leq G : H \text{ is } A\text{-invariant, of bounded index}\}$. Then G_A^{∞} is the smallest subgroup of G of bounded index, invariant over A.

We have $G_A^{\infty} \subseteq G_A^{00} \subseteq G_A^0$, and all three are normal subgrous of G of bounded index.

Exercise 6.2. (1) Give an example of a group G definable in an NIP theory such that $G_A^0 \neq G_A^{00}$.
(2) Give an example of a definable group G in which G_A^{00} depends on the set of parameters A.

Exercise 6.3. Show that if a type-definable subroup has finite index, then it is definable.

Remark 6.4. Let $G = \Phi(\mathbb{M})$ be a type-definable group. By compactness we have:

(1) there is some formula $\phi(x)$ such that $\phi(\mathbb{M}) \supseteq G$ and for any $a, b, c \models \phi(x)$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and $a \cdot 1 = 1 \cdot a = a$.
(2) for every formula $\phi_0(x)$ containing G there is some formula $\phi_1(x)$ containing G such that $\forall a, b \models \phi_1(x)$ we have $a \cdot b \models \phi_0(x)$ and $\forall a \models \phi_0(x)$ there is a unique $b \models \phi_1(x)$ such that $a \cdot b = b \cdot a = 1$.
(3) Iterating (2), we can choose formulas $\phi_2(x), \phi_3(x), \ldots$ such that $\bigcap_{i<\omega} \phi_i(x)$ is a type-definable group containing G.
(4) Applying this to every formula in Φ, we see that G is the intersection of type-definable groups, each defined by a countable intersection of formulas.

Theorem 6.5. *If T is* NIP, *then $G_A^{00} = G_{\emptyset}^{00}$ for every small set A (and we will write just G^{00}).*

Proof. The proof is essentially a "type-definable" version of the Baldwin-Saxl theorem from before, but requires some extra tricks to deal with type-definability.

Assume that for every small set A there is some small set $B \supseteq A$ such that $G_A^{00} \supsetneq G_B^{00}$ (otherwise, if it stabilizes for some small A, taking the intersection of all of the boundedly many conjugates of G_A^{00} we get that $G_{\emptyset}^{00} \subseteq G_A^{00}$ — as wanted.).

Then for any $\kappa < \kappa(\mathbb{M})$ we can find an arbitrarily large collection $(H_i : i < \kappa)$ of pairwise-distinct type definable subgroups of G of bounded index.

By Remark 6.4, we may assume that each H_i is defined as an intersection of countably many formulas.

By pigeonhole, Ramsey and compactness, there is a partial type $\Phi(x, \bar{y})$ over $\emptyset$ and an indiscernible sequence $\left(\bar{b}_i : i \in \mathbb{Q}\right)$, with each $\bar{b}_i$ a countable tuple, such that the groups $H_i = \Phi\left(\mathbb{M}, \bar{b}_i\right)$ are pairwise distinct, each of bounded index. We may also assume that for each $\bar{b}$, $\Phi\left(x, \bar{b}\right)$ is a subgroup of G.

Claim. $\bigcap_{j\in\mathbb{Q}\setminus\{i\}} H_j \not\subseteq H_i$ for any $i \in \mathbb{Q}$.

Proof. Assume that this fails for some $i \in \mathbb{Q}$. Then by compactness for any $\kappa < \kappa(\mathbb{M})$ we can find tuples $\left(\bar{b}_\alpha : \alpha < \kappa\right)$ such that the sequence $\left(\bar{b}_j : j < i\right) + \left(\bar{b}_\alpha : \alpha < \kappa\right) + \left(\bar{b}_j : j > i\right)$ is indiscernible (in particular the groups $\{\Phi(\mathbb{M}, b_\alpha) : \alpha < \kappa\}$ are pairwise distinct), and $\bigcap_{j\in\mathbb{Q}\setminus\{i\}} H_j \subseteq \Phi(\mathbb{M}, b_\alpha)$ for any $\alpha < \kappa$. But $\bigcap_{j\in\mathbb{Q}\setminus\{i\}} H_j$ has bounded index in G, hence there can be only boundedly many pairwise distinct subgroups of G containing it. Since $\kappa < \kappa(\mathbb{M})$ is arbirary, we get a contradiction.

Now for each $i < \omega$, there is some $a_i \in \left(\bigcap_{j\neq i} H_j\right) \setminus H_i$, and by Ramsey, automorphism and indiscernibility of $\left(\bar{b}_i : i \in \mathbb{Q}\right)$ we can choose it so that the sequence $\left(a_i, \bar{b}_i : i \in \mathbb{Q}\right)$ is indiscernible. Hence there is some $\phi(x, \bar{y}) \in \Phi(x, \bar{y})$ such that $\models \phi\left(a_i, \bar{b}_j\right) \iff i \neq j$. Let $\theta(x, \bar{y}) \in \Phi(x, \bar{y})$ be such that $\models \bigwedge_{i<3} \theta(x_i, \bar{y}) \to \phi(x_0 \cdot x_1 \cdot x_2, \bar{y})$. For $I = \{i_1, \ldots, i_n\} \subseteq \mathbb{Q}$, let $a_I := a_{i_1} \cdot \ldots \cdot a_{i_n}$. Then $\models \theta(a_I, b_i) \iff i \notin I$ (if $i \notin I$ — clear, otherwise any a_{i_k} can be written as $c_0 \cdot a_I \cdot c_1$ with $c_0, c_1 \in H_{i_k}$). This shows that $\theta(x, \bar{y})$ has IP. □

Now we consider the "invariant" connected component.

Exercise 6.6. Let H be an invariant subgroup of G.

Then it has unbounded index if and only if there is an indiscernible sequence $(a_i : i < \omega)$ of elements of G which lie in pairwise-distinct cosets of H, if and only if there is a small model M and some $a \equiv_M b$ from G with $ab^{-1} \notin H$ (Hint: use Erdos-Rado).

Deduce that if H has bounded index, then it is at most $2^{2^{|T|}}$.

Remark 6.7.

(1) Let M be a small model and consider the set $X_M := \left\{a \cdot b^{-1} : a \equiv_M b\right\}$. Then $\langle X_M\rangle$, the subgroup generated by X_M, is M-invariant and of bounded index.
On the other hand, if H is an M-invariant subgroup of bounded index, by the exercise if $a \equiv_M b$ then $ab^{-1} \in H$. Hence H contains the subgroup generated by X_M. Combining, G^∞_M is precisely the subgroup of G generated X_M.
(2) Given a set of formulas $\Phi(x, \bar{y})$, where $\bar{y}$ is a tuple of variables corresponding to an enumeration of M, we also define $X^\Phi_M = \left\{a \cdot b^{-1} :\models \phi(a, M) \leftrightarrow \phi(b, M) \text{ for all } \phi \in \Phi\right\}$.
(3) If A is any set of parameters and $c \in G$, then $c(X_A)c^{-1} \subseteq (X_A)^2$.
Indeed, given $a \equiv_A b$, there is some $d \in G$ such that $ac \equiv_A bd$. Then $c\left(ab^{-1}\right)c^{-1} = (ca)(db)^{-1} \cdot dc^{-1} \in (X_A)^2$.

Theorem 6.8. *If T is* NIP*, then $G^\infty_A = G^\infty_\emptyset$ for every small set A (and we will write just G^∞).*

Proof. Assume not, then for some small model M we have $G^\infty_M \neq G^\infty_\emptyset$. Let $\lambda := |M|$. The intersection $\bigcap\{G^\infty_{M'} : M' \text{ is a model of size } \lambda\}$ is an $\emptyset$-invariant subgroup of G. If it has bounded index, then $G^\infty_\emptyset \subseteq G^\infty_M$, hence $G^\infty_\emptyset = G^\infty_M$ contrary to the assumption.

It follows that for any $\kappa < \kappa(\mathbb{M})$ we can find a sequence $(M_i : i < \kappa)$, $|M_i| = \lambda$, such that $G^\infty_{M_i}$ does not contain $\bigcap_{j<i} G^\infty_{M_j}$.

By Erdős-Rado, we can find such a sequence $(M_i : i < \omega)$ which is moreover indiscernible (with a fixed enumeration of M_i as a tuple of length λ).

For each $i<\omega$, let $c_i \in \left(\bigcap_{j<i} G^{\infty}_{M_i}\right)\setminus G^{\infty}_{M_i}$, and we may assume that the sequence $((M_i,c_i) : i<\omega)$ is indiscernible.
***

In particular, for some $m<\omega$, for all $j<i$ we have $c_i \in \left(X_{M_j}\right)^m$. By compactness, there is some finite set $\Phi(x,\bar{y})$ of formulas such that $c_i \notin \left(X^{\Phi}_{M_i}\right)^{m+4}$.

By the Remark, there is some finite set $\Phi'(x,\bar{y})$ such that for all $c\in G$ we have $c\left(X^{\Phi'}_{M_i}\right)c^{-1} \subseteq \left(X^{\Phi}_{M_i}\right)^2$.

Now, give a finite sequence $I=(i_1,\ldots,i_n)$ of distinct elements of ω, define

$$c_{I,0} := c_{2i_1+1}\cdot \ldots \cdot c_{2i_n+1} \text{ and } c_{I,1} := c_{2i_1}\cdot \ldots \cdot c_{2i_n}.$$

Fix $j<\omega$.

Claim 1. If $j\notin I$, then $c_{I,0}\cdot c^{-1}_{I,1} \in X_{M_{2j}} \subseteq X^{\Phi'}_{M_{2j}}$.

Proof. Indeed, if $j\notin I$, then by indiscernibility of the sequence $c_{I,0} \equiv_{M_{2j}} c_{I,1}$, hence $c_{I,0}\cdot c^{-1}_{I,1} \in X_{M_{2j}}$.

Claim 2. If $j\in I$, then $c_{I,0}\cdot c^{-1}_{I,1} \notin X^{\Phi'}_{M_{2j}}$.

Proof. Assume that $j\in I$, but the conclusion doesn't hold. Write the index set as $I=I_1+j+I_2$. Then we have the following straightforward calculations:

$$c_{I,0}\cdot c^{-1}_{I,1} = c_{I,0}\cdot c_{2j+1}\cdot c_{I_2,0}\cdot c^{-1}_{I_2,1}\cdot c^{-1}_{2j}\cdot c^{-1}_{I_1,1},$$

$$c_{I,0}\cdot c^{-1}_{I,1}\cdot c_{I_1,1}\cdot c_{2j} = c_{I_1,0}\cdot c_{2j+1}\cdot c_{I_2,0}\cdot c^{-1}_{I_2,1},$$

$$c_{2j} = c^{-1}_{I_1,1}\cdot c_{I,1}\cdot c^{-1}_{I,0}\cdot c_{I_1,0}\cdot c_{2j+1}\cdot c_{I_2,0}\cdot c^{-1}_{I_2,1} = \left(c^{-1}_{I_1,1}\left(c_{I,1}c^{-1}_{I,0}\right)c_{I_1,1}\right)\cdot\left(c^{-1}_{I_1,1}c_{I_1,0}\right)\cdot c_{2j+1}\cdot\left(c_{I_2,0}c^{-1}_{I_2,1}\right).$$

By assumption $c_{2j+1}\in\left(X_{M_{2j}}\right)^m$, and $c^{-1}_{I_1,1}\left(c_{I,1}c^{-1}_{I,0}\right)c_{I_1,1} \in c^{-1}_{I_1,1}X^{\Phi'}_{M_{2j}}c_{I_1,1} \subseteq \left(X^{\Phi}_{M_{2j}}\right)^2$ and both $c^{-1}_{I_1,1}c_{I_1,0}$ and $c_{I_2,0}c^{-1}_{I_2,1}$ are in $X_{M_{2j}}$. Combining, $c_{2j}\in\left(X^{\Phi}_{M_{2j}}\right)^{m+4}$ — contradicting the choice of the c_i's.

By Claims 1 and 2, the formula $\psi(x,\bar{y}) := \exists x_1x_2\left(\bigcap_{\phi\in\Phi'}\left(\phi(x_1,\bar{y})\leftrightarrow\phi(x_2,\bar{y})\right)\wedge x=x_1x_2^{-1}\right)$ is not NIP. $\square$

Exercise 6.9. Show that in any theory, if G^{∞}_A doesn't depend on A, then G^{00}_A also doesn't depend on A.

Remark 6.10. There are examples of groups definable in NIP theories for which $G^{00}\neq G^{\infty}$, see [4]. However, we will see that they coincide for a large class of NIP groups that includes all stable groups.

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