Definition. A vector space V over a field F is a set with two operations:

(vector addition) For any x and y in V, there is a uniquely defined element $x+y$ in V.

(scalar multiplication) For any a in F and x in V, there is a uniquely defined element $a\cdot x$ in V.

Satisfying the following eight conditions (VS1) – (VS 8):

(VS1) $x+y=y+x$ for any x,y in V. (commutativity of addition)

(VS2) $(x+y)+z=x+(y+z)$ for any x,y,z in V (associativity of addition)

(VS3) There is an element in V denoted by 0 such that $x+0=x$ for all x in V.

(VS4) For each x in V there is some y in V such that $x+y=0$.

(VS5) $1\cdot x=x$ for all x in V (where 1 is the multiplicative identity of F).

(VS6) $a\cdot(b\cdot x)=(a\cdot b)\cdot x$ for all x in V and a,b in F. (associativity of scalar multiplication)

(VS7) $a\cdot(x+y)=a\cdot x+a\cdot y$ for all a in F and x,y in V

(VS8) $(a+b)\cdot x=a\cdot x+b\cdot x$ for all a,b in F and x in V.

} distributive laws

- Elements of V are called vectors, and elements of F are called scalars.
- Usually F will be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Example 1. Given a field F, consider the set

$F^n=\{(x_1,x_2,\dots,x_n): x_i\in F\}$ (the set of all n-tuples of elements from F).

We define the operations of vector addition and scalar multiplication in the following way:

$(x_1,\dots,x_n)+(y_1,\dots,y_n)=(x_1+y_1,\dots,x_n+y_n)$

$a\cdot(x_1,\dots,x_n)=(a\cdot x_1,\dots,a\cdot x_n)$,

where $(x_1,\dots,x_n),(y_1,\dots,y_n)$ are arbitrary elements of F^n and a is an arbitrary element of F.

(and x_i+y_i and $a\cdot x_i$ are calculated in F).

With these operations F^n is a vector space over the field F. (One has to check that (VS1) – (VS8) hold. Do it!)

In particular, if $n=2$ and $F=\mathbb{R}$, we obtain the familiar space $\mathbb{R}^2$ of vectors on the plane.

Example 2. Let F be a field, and let $P(F)$ denote the set of all polynomials with coefficients in F. That is, $P(F)$ consists of all expressions of the form

$p(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$

for some $n\geq 0$, with a_i in F for all $i=0,1,\dots,n$. If $a_i=0$ for all $i=0,1,\dots,n$ then $p(x)$ is called the zero polynomial.

The degree of a non-zero polynomial is the largest i such that $a_i\neq 0$. (the degree of the zero polynomial is defined to be -1)

Two polynomials $p(x)=a_nx^n+\dots+a_1x+a_0$ and $q(x)=b_mx^m+\dots+b_1x+b_0$ are equal if $m=n$ and $a_i=b_i$ for all $i=0,1,\dots,n$.

We define addition and scalar multiplication on $P(F)$ in the usual way:

Addition: Given $p(x),q(x)$ as above, suppose that $m\leq n$. Then we can also write $q(x)$ as $b_nx^n+\dots+b_1x+b_0$, where $b_{m+1}=b_{m+2}=\dots=b_n=0$

Then we define $p(x)+q(x)=(a_n+b_n)x^n+(a_{n-1}+b_{n-1})x^{n-1}+\dots+(a_1+b_1)x+(a_0+b_0)$.

Scalar multiplication: For any c in F, define $c\cdot p(x)=ca_nx^n+ca_{n-1}x^{n-1}+\dots+ca_1x+ca_0$.

(Equivalently, $p+q$ can be defined as the polynomial satisfying $(p+q)(x)=p(x)+q(x)$ for all x in F, and cp as the polynomial satisfying $(cp)(x)=c\cdot p(x)$ for all x in F).

With these operations, $P(F)$ is a vector space over F. (Again, need to check that (VS1) – (VS8) hold.)

Example 3. Let $M_2(\mathbb{R})$ denote the set of all 2×2 matrices with entries from $\mathbb{R}$.

We define addition and scalar multiplication in the familiar way:

$\begin{pmatrix} a & b \\ c & d\end{pmatrix}+\begin{pmatrix} e & f \\ g & h\end{pmatrix}=\begin{pmatrix} a+e & b+f \\ c+g & d+h\end{pmatrix}$ and $\alpha\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d\end{pmatrix}$ for any a,b,c,d,e,f,g,h,α in $\mathbb{R}$.

With these operations, $M_2(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Analogously, the space $M_{m\times n}(\mathbb{R})$ of all $m\times n$ matrices is a vector space.

Example 4 The most boring vector space, aka the zero vector space.

- It consists of a single vector 0 (so $V=\{0\}$).

Operations of addition and scalar multiplication are defined by:

$0+0=0$

$\alpha \cdot 0 = 0$ for all α in F.
With these operations, V is a vector space over F. (check it!)

Basic properties of vector spaces

Now, we will see some of the basic properties of vector spaces (we will deduce them as logical consequences of the axioms (VS1)–(VS8).)

Theorem 1.1 (Cancellation law)
Let V be a vector space, and x, y, z be arbitrary elements of V.
If $x + z = y + z$, then $x = y$.

Proof. As V is a vector space, it satisfies all of the properties (VS1)–(VS8).
By (VS4) there exists an element $\tilde{z}$ in V such that $z + \tilde{z} = 0$. We have:

$$x \underset{\text{by (VS3)}}{=} x + 0 \underset{\text{by the choice of } \tilde{z}}{=} x + (z + \tilde{z}) \underset{\text{by (VS2)}}{=} (x+z) + \tilde{z} \underset{\text{by assumption}}{=} (y+z) + \tilde{z} \underset{\text{by (VS2) again}}{=} y + (z + \tilde{z}) \underset{\text{by the choice of } \tilde{z}}{=} y + 0 \underset{\text{by (VS3) again}}{=} y.$$

Thus $x = y$.

By commutativity of addition (VS1) we also have: if $z + x = z + y$, then $x = y$.

Corollary 1. The vector 0 described in (VS3) is unique. (and is called the zero vector of V).

Proof. Suppose that 0 and $0'$ are two elements in a vector space V that both satisfy (VS3).
Then for any x in V we have:

$$x + 0 \underset{\text{by (VS3) for } 0}{=} x \underset{\text{by (VS3) for } 0'}{=} x + 0'$$

By the cancellation law it follows that $0 = 0'$.

For example, in F^n the zero vector is $\overbrace{(0, 0, \dots, 0)}^{n \text{ times}}$, and in $M_2(\mathbb{R})$ the zero vector is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Corollary 2. For any x in V, the vector y described in (VS4) is unique.
(it is called the additive inverse of x and is denoted by $-x$).

Proof. Let x in V be arbitrary, and suppose that y_1 and y_2 are two elements in V both satisfying (VS4).
That is, $x + y_1 = 0 = x + y_2$.
By cancellation law this implies $y_1 = y_2$.

Finally, we state some further useful properties of vector spaces.

Theorem 1.2. Let V be a vector space over a field F.
For all x in V and a in F we have:

1) $\underset{\text{scalar in } F}{0} \cdot x = 0$ ← the zero vector of V.
2) $(-a) \cdot x = \underbrace{-(ax)}_{\text{the additive inverse of the vector } ax} = a \cdot (-x)$.
3) $a \cdot 0 = 0$ ← the zero vector in V

Proof. Problem Set 1.

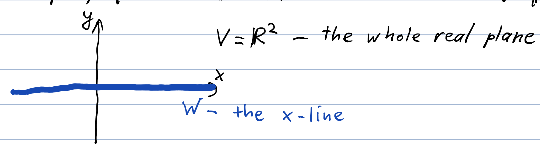
Subspaces

Definition. Let V be a vector space over a field F.
A subset W of V ($W \subseteq V$) is a **subspace** of V if W itself is a vector space over F, with respect to the addition and scalar multiplication defined on V.

That is, W satisfies all of the properties (VS1)–(VS8).

Example 1 Let $n \geq 1$ be an integer and F a field.
Recall that the vector space F^n is the set $\{(x_1, \dots, x_n) : x_i \in F\}$ with addition and scalar multiplication

given by $(x_1,\dots,x_n)+(y_1,\dots,y_n)=(x_1+y_1,\dots,x_n+y_n)$ and $a\cdot(x_1,\dots,x_n)=(ax_1,\dots,ax_n)$.
Consider the subset
$W=\{(x_1,\dots,x_{n-1},0) : x_i\in F\}$.
Then W is a subspace of F^n (we will prove it later).
For example, for $F=\mathbb{R}$ and $n=2$ we have $V=\mathbb{R}^2=\{(x_1,x_2) : x_1,x_2\in\mathbb{R}\}$, $W=\{(x_1,0): x_1\in\mathbb{R}\}$:



Example 2. Let F be a field, and recall that $P(F)$ is the vector space of all polynomials with coefficients from F.
For an integer $n\geq 0$, consider the subset $P_n(F)\subseteq P(F)$ consisting of all polynomials of degree $\leq n$.
Then $P_n(F)$ is a subspace of $P(F)$.

Example 3. For any vector space V, V itself and $\{0\}$ are both subspaces of V.

Example 4. Let S be a non-empty set and F a field.
Let $\mathcal{F}(S,F)$ denote the set of all maps from S to F.
(A map $f: S\to F$ takes x in S as an input and returns $f(x)$ in F as an output.
Synonym: function).
Two maps f and g in $\mathcal{F}(S,F)$ are equal if $f(x)=g(x)$ for all x in S.
For any f and g in $\mathcal{F}(S,F)$ and c in F we define $f+g$ and $c\cdot f$ in $\mathcal{F}(S,F)$ by taking
$(f+g)(x)=f(x)+g(x)$ for each x in S (where "+" and "$\cdot$" on the right side are calculated in F).
$(c\cdot f)(x)=c\cdot f(x)$
With these operations $\mathcal{F}(S,F)$ is a vector space (check!)
In particular, $\mathcal{F}(\mathbb{R},\mathbb{R})$ consists of all real-valued functions defined on real numbers.
Let $C(\mathbb{R})$ denote the set of all continuous real functions.
Then $C(\mathbb{R})$ is a subset of $\mathcal{F}(\mathbb{R},\mathbb{R})$, and we will see that it is a subspace as well.

Let now V be a vector space, and let $W\subseteq V$ be a subset of V. According to the definition, in order to show that W is a subspace, we have to check that all of the properties (VS1)–(VS8) hold <u>in W</u>, with respect to addition and scalar multiplication defined on <u>V</u>.
Turns out that we actually need to check less in this situation.

Theorem 1.3. Let V be a vector space over F, and let $W\subseteq V$ be a subset of V. (need not be a subspace!)
Then W is a subspace of V if and only if the following holds for W:
(a) $0\in W$ (that is, the zero vector of V is in W).
(b) If $x,y\in W$ then $x+y\in W$ (that is, W is closed under addition)
(c) If $x\in W$ and $c\in F$ then $c\cdot x\in W$ (that is, W is closed under scalar multiplication)

Proof.
1) First we assume that W is a subspace of V and show that (a), (b), (c) hold.
"W is a subspace of V" means that it is a vector space under the operations of addition and scalar multiplication defined on V.
In particular:
- for any x,y in W and c in F, $x+y$ and $c\cdot x$ must also be in W. So (b) and (c) hold.
- (VS3) holds <u>in W</u>, that is there is some $0'\in W$ such that $x+0'=x$ for all <u>$x\in W$</u>.

In particular, $0'+0'=0'$.
But as $0'\in V$, also $0'+0=0'$ (since $0\in V$ satisfies (VS3) <u>in V</u>.)
By cancellation law in the vector space V, this implies that $0'=0$. In particular, $0\in W$ and (a) holds.

2) Conversely, suppose (a), (b), (c) holds, and we will show that then W is not just a subset of V, but also a subspace. For this we have to check that W satisfies (VS1)–(VS8).
- As V is a vector space, (VS1), (VS2), (VS5), (VS6), (VS7), (VS8) hold for all elements of V. As W is a subset of V,

they hold in particular for all elements of W. (so we get all these properties in W for free).

- Remains to check that (VS3) and (VS4) hold in W.
  - (VS3): holds by (a).
  - (VS4): Let $x \in W$ be arbitrary. We need to find some $y \in W$ such that $x+y=0$. We know that in V there is such an element, namely the additive inverse $-x$. We show that actually $-x \in W$ as well.
    As -1 is a scalar in F, by (c) we have $(-1)\cdot x \in W$.
    As V is a vector space, it satisfies $-x = (-1)\cdot x$, by Theorem 1.2.
    Thus $-x \in W$, as wanted.

Now we return to the examples and verify our claims using Theorem 1.3.

Example 1. $V = F^n$, $W = \{(x_1, \dots, x_{n-1}, 0) : x_i \in F\} \leq V$.

We check that W satisfies (a), (b), (c), and so indeed W is a subspace of V by Theorem 1.3.

(a) The zero vector of V is $\overbrace{(0,0,\dots,0)}^{n}$, it is in W (taking $x_1 = \dots = x_{n-1} = 0$).

(b),(c) W is closed under addition and scalar multiplication.

Given any $(x_1, \dots, x_{n-1}, 0), (y_1, \dots, y_{n-1}, 0)$ in W and a in F we have:

$(x_1, \dots, x_{n-1}, 0) + (y_1, \dots, y_{n-1}, 0) = (x_1 + y_1, \dots, x_{n-1} + y_{n-1}, \underbrace{0+0}_{=0})$ — also in W.

$a \cdot (x_1, \dots, x_{n-1}, 0) = (a \cdot x_1, \dots, a \cdot x_{n-1}, \underbrace{a \cdot 0}_{=0})$

Similarly, $W = \{(0, x_1, \dots, x_{n-1}) : x_i \in F\}$ is also a subspace of F^n.

Example 2. $V = P(F)$, $W = P_n(F)$ — the set of all polynomials of degree at most n.

(a) The zero vector in $P(F)$ is the zero polynomial $p_0(x) = a_n x^n + \dots + a_1 x + a_0$ with $a_n = \dots = a_1 = a_0 = 0$. Its degree is by definition -1, so p_0 is in $P_n(F)$.

(b), (c) If both $p(x)$ and $q(x)$ are in $P_n(F)$, that is they have degree at most n, then $p(x) + q(x)$ also has degree at most n, and thus $p+q$ is in $P_n(F)$ as well.
If $p(x)$ has degree $\leq n$ and $a \in F$, then $a \cdot p(x)$ has degree $\leq n$ as well, and so $a \cdot p \in P_n(F)$.

Example 4. $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ — all real-valued functions, $W = C(\mathbb{R})$ — continuous real-valued functions.

The zero vector in $\mathcal{F}$ is the function given by $f(x) = 0$ for all x in $\mathbb{R}$.

By basic calculus we know that all constant functions are continuous, that sum of any two continuous functions is continuous, and that a product of a constant function and a continuous function is also a continuous.

Thus $C(\mathbb{R})$ satisfies the conditions (a), (b), (c) in Theorem 1.3.

We can form new subspaces from the old ones.

Theorem 1.4. Let V be a vector space over F.
If $W_1, \dots, W_n$ are subspaces of V, then the set $W = W_1 \cap W_2 \cap \dots \cap W_n$ is also a subspace of V.

Proof. We check that W satisfies (a), (b), (c) and apply Theorem 1.3.

By assumption each of W_i, $i = 1, \dots, n$, is a subspace, and so satisfies (a), (b), (c).

(a) As each of W_i satisfies (a), we have $0 \in W_i$ for all $i = 1, \dots, n$.
But this means that $0 \in W_1 \cap \dots \cap W_n = W$ as well.

(b) Let $x, y \in W$, which means precisely that $x, y \in W_i$ for all $i = 1, \dots, n$.
As each of W_i satisfies (b), $x + y \in W_i$ for each $i = 1, \dots, n$.
Hence $x + y \in W_1 \cap \dots \cap W_n$.

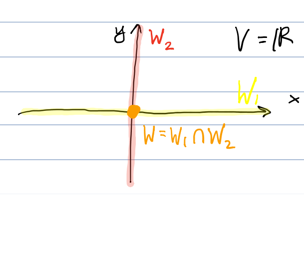
(c) If $x \in W$, then $x \in W_i$ for each $i = 1, \dots, n$.
Then for any $c \in F$, $c \cdot x \in W_i$ for each $i = 1, \dots, n$ (as W_i satisfies (c)).
Hence $c \cdot x \in W_1 \cap \dots \cap W_n = W$.

Example. Let $V = \mathbb{R}^2$.

Let $W_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ and let $W_2 = \{(0, x_2) : x_2 \in \mathbb{R}\}$.

We already know that both W_1 and W_2 are subspaces of V.

Then $W_1 \cap W_2 = \{(0,0)\}$ — the zero subspace of $\mathbb{R}^2$.



However, the union $W = W_1 \cup W_2$ of two subspaces W_1, W_2 of V need not be a subspace of V in general! (see Problem Set 1).