MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 6

DUE THURSDAY, MAY 18

Problem 1. Do Exercise 1, Section 2.4. Justify each answer.

Problem 2.

(1) Which of the following pairs of vector spaces are isomorphic? Justify your answers.
  (a) F^3 and $P_3(F)$.
  (b) F^4 and $P_3(F)$.
  (c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$.
  (d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr}(A) = 0\}$ and $\mathbb{R}^4$.

(2) Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$. Construct an isomorphism from V to F^3.

Problem 3. Do Exercise 1, Section 2.5. Justify each answer.

Problem 4. For each of the following pairs of ordered bases β and β' for V, find the change of coordinates matrix that changes β'-coordinates into β-coordinates.

(1) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$, in $V = \mathbb{R}^2$.
(2) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$, in $V = \mathbb{R}^2$.
(3) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$, in $V = \mathbb{R}^2$.
(4) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$, in $V = P_2(\mathbb{R})$.

Problem 5. Given two matrices $A, B \in M_{n\times n}(F)$, we say that B is *similar* to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Similarity is an equivalence relation.

Recall that the trace of a matrix $A \in M_{n\times n}(F)$ is the sum of its diagonal entries, that is $\operatorname{tr}(A) = A_{1,1} + A_{2,2} + \ldots + A_{n,n}$. Prove the following statements:

(1) For any $A, B \in M_{n\times n}(F)$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
(2) If A and B are similar, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.
(3) Let V be a vector space with $\dim(V) = n$, and let β, β' be two ordered bases for V, and let $T \in \mathcal{L}(V)$ be arbitrary. Then $\operatorname{tr}\left([T]_\beta\right) = \operatorname{tr}\left([T]_{\beta'}\right)$.

Problem 6. Prove the following generalization of Theorem 2.23.

Let $T : V \to W$ be a linear transformation, $\dim(V), \dim(W) < \infty$. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q,$$

where Q is the matrix that changes β' coordinates into β-coordinates, and P is the matrix that changes γ'-coordinates into γ-coordinates.

Problem 7. Compute the determinants of the following matrices (and provide the details of your calculations).

(1) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$,

(2) $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$,

(3) $\begin{pmatrix} 14 & 80 & -14 & -76 & -4 \\ 0 & 2 & 1 & 3 & 0 \\ 1 & 0 & -2 & 2 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 \end{pmatrix}$.

Problem 8. Prove that $\det(cA) = c^n \det(A)$ for any $A \in M_{n\times n}(F)$ and $c \in F$.