Midterm I Review Sheet

Vector spaces

Definition. A vector space V over a field F is a set with two operations, addition and scalar multiplication, (so for any x, y in V and $a \in F$, $x+y$ and ax are in V) such that the following conditions hold.

(VS1) $x+y = y+x$ for all x, y in V (commutativity)

(VS2) $(x+y)+z = x+(y+z)$ for all x,y,z in V (associativity)

(VS3) There exists an element 0 in V such that $x+0 = x$ for all $x \in V$. (identity)

(VS4) For each x in V there is an element y in V such that $x+y=0$ (y is an inverse of x)

(VS5) $1 \cdot x = x$ for all x in V (where 1 is the multiplicative identity of F).

(VS6) $a(bx) = (ab)x$ for all x in V and a, b in F

(VS7) $a(x+y) = ax+ay$ for all a in F and x,y in V.

(VS8) $(a+b)x = ax+bx$ for all a,b in F and x in V.

(VS7)–(VS8): distributive laws

Elements of V are called vectors.

Elements of F are called scalars.

Theorem 1.1 (cancellation law)

Let V be a v.s. and let $x, y, z \in V$.

If $x+z = y+z$, then $x=y$.

Corollary.

1) In any vector space V, there is a unique element 0 satisfying (VS3) — the zero vector of V.

2) For any v.s. V and any x in V, there is a unique element y in V satisfying (VS4). It is called the inverse of x, and denoted by $-x$.

Theorem 1.2. Let V be a v.s. over F.

For all x in V and a in F we have:

1) $0 \cdot x = 0$ (Note: the 1st 0 is a scalar in F, the 2nd one is the zero vector in V).

2) $(-a)\cdot x = -(ax) = a(-x)$

3) $a \cdot 0 = 0$ (Note: this is the zero vector of V on both sides).

Subspaces

Definition. Let V be a v.s. A subset $W \subseteq V$ is a subspace of V if W itself is a v.s. with respect to the addition and scalar multiplication defined on V.

Theorem 1.3. Let V be a v.s., and let $W \subseteq V$ be a subset of V.

Then W is a subspace of V if and only if all of the following conditions hold:

(a) $0 \in W$

(b) $x+y \in W$ for all $x, y \in W$ (W is closed under addition)

(c) $c \cdot x \in W$ for all $c \in F$ and $x \in W$ (W is closed under scalar multiplication).

Theorem 1.4. Let V be a v.s. over F.

If $W_1, \dots, W_n$ are subspaces of V, then the set $W = W_1 \cap W_2 \cap \dots \cap W_n$ is also a subspace of V.

Linear combinations

Definition. Let V be a v.s., and let $S \subseteq V$ be a non-empty subset of V.

1) A vector v in V is a linear combination of S if one can write

$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$

for some vectors $u_1, \dots, u_n$ in S and some scalars $a_1, \dots, a_n$ in F.

2) The span of S, denoted $\text{Span}(S)$, is the subset of V consisting precisely of all linear combinations of S. That is,

$\text{Span}(S) = \{a_1u_1 + \dots + a_nu_n : n \in \mathbb{N},\ a_i \in F,\ u_i \in S\}$.

For convenience, we define $\text{Span}(\emptyset) = \{0\}$.

Theorem 1.5. Let S be any subset of a v.s. V. Then:

1) $\text{Span}(S)$ is a subspace of V.
2) Any subspace of V that contains S must also contain $\text{Span}(S)$.

Definition. Let V be a v.s. and S a subset of V.
We say that S generates (or spans) V if $\text{Span}(S) = V$.

Definition. A subset S of a v.s. V is linearly dependent if there exist a finite number of distinct vectors $u_1, \dots, u_n$ in S and scalars $a_1, \dots, a_n \in F$, with at least one $a_i \neq 0$, such that

$a_1u_1 + \dots + a_nu_n = 0$.

We call S linearly independent if it is not linearly dependent.

Theorem 1.6. Let V be a v.s. and $S_1 \subseteq S_2 \subseteq V$ be two subsets of V.

1) If S_1 is lin. dependent, then S_2 is also linearly dependent.
2) If S_2 is lin. indep., then S_1 is also lin. indep.

Theorem 1.7. Let S be a lin. indep. subset of a vector space V.
Let v be any vector in V not contained in S.
Then $S \cup \{v\}$ is lin. dep. if and only if $v \in \text{Span}(S)$.

Bases and dimension.

Definition. A basis for a v.s. V is a subset of V which is lin. indep. and generates V.

Theorem 1.8. A subset $\{u_1, \dots, u_n\}$ of a v.s. V is a basis if and only if every vector $v \in V$ can be written uniquely in the form

$v = a_1u_1 + \dots + a_nu_n$,

where $a_i \in F$.

(so "uniquely" here means that there is only one possible choice of the scalars $a_1, \dots, a_n \in F$ satisfying the equality)

Theorem 1.9. If a v.s. V is generated by a finite subset S, then some subset of S is a basis for V.
It follows that every finitely generated v.s. has a basis.

Theorem 1.10. (Replacement theorem)
Let V be a v.s. generated by a set $G \subseteq V$ with $|G| = n$, and let L be a lin. indep. subset of V, $|L| = m$.
Then $m \leq n$, and there exists $H \subseteq G$ with $|H| = n - m$ such that $L \cup H$ generates V.

Corollary 1. Let V be a finitely generated v.s. Then every basis for V has the same number of elements.

Definition. A v.s. V is finite-dimensional if it has a finite basis.
The (unique) number of vectors in a basis for V is called the dimension of V, denoted $\dim(V)$.
If there is no finite basis, then V is infinite-dimensional.

Corollary 2. Let V be a v.s. of dimension n. Then:

a) Any generating set for V must contain at least n vectors.
b) Any lin. indep. subset of V with n elements is a basis.
c) Every lin. indep. subset of V can be extended to a basis for V.

Theorem 1.11. Let W be a subspace of a v.s. V with $\dim(V) < \infty$.
Then $\dim(W) \leq \dim(V)$.
Moreover, if $\dim(W) = \dim(V)$, then $V = W$.