Algebraic description of the operations in $\mathcal{L}(V,W)$.

Last time we saw:
- every lin. transformation can be represented by a matrix,
- linear transformations from V to W form a vector space $\mathcal{L}(V,W)$, under pointwise addition and scalar mult.

These operations on $\mathcal{L}(V,W)$ correspond to matrix addition and scalar mult. on the representations.

Theorem 2.8

Let V, W be fin. dim. v.s. with ordered bases β and γ, respectively.
Let $T, U: V \to W$ be linear. Then:

a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. operations on matrices!

b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars $a \in F$.

Proof.

a) Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$.
There exist unique scalars a_{ij} and b_{ij} $(1 \le i \le m,\ 1 \le j \le n)$ s.t.:
$T(v_j) = \sum_{i=1}^m a_{ij} w_i$ and $U(v_j) = \sum_{i=1}^m b_{ij} w_i$ for $1 \le j \le n$.
Hence
$$(T+U)(v_j) = \sum_{i=1}^m (a_{ij} + b_{ij}) w_i.$$
Thus, for the matrix $[T+U]_\beta^\gamma$ we have
$$\left([T+U]_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = \left([T]_\beta^\gamma + [U]_\beta^\gamma\right)_{ij}.$$

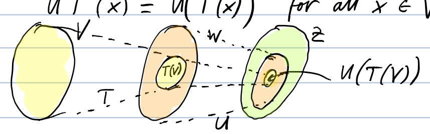
b) Similar (Exercise.)

Example. Let $T, U: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2),$$
$$U(a_1, a_2) = (a_1 - a_2,\ 2a_1,\ 3a_1 + 2a_2).$$
Let β, γ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, resp. Then
$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$ – calculated in the previous example
$$[U]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

Applying definition, we have
$$(T+U)(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2) + (a_1 - a_2, 2a_1, 3a_1 + 2a_2) = (2a_1 + 2a_2,\ 2a_1,\ 5a_1 - 2a_2).$$ So
$$[T+U]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix} = [T]_\beta^\gamma + [U]_\beta^\gamma$$ – as Theorem 2.8 predicts.

Composition of lin. transf.'s and matrix multiplication.

Definition. Let $T: V \to W$ and $U: W \to Z$ be two lin. transf.'s of v.s.'s.
Their composition, denoted by UT, is a function from V to Z defined by
$UT(x) = U(T(x))$ for all $x \in V$.



Theorem 2.9. Let V, W, Z be v.s. over F.
Let $T: V \to W$ and $U: W \to Z$ be linear.
Then $UT: V \to Z$ is linear.

Proof.
Let $x, y \in V$ and $a \in F$. Then
$$UT(ax+y) = U(T(ax+y)) \overset{(T \text{ is lin.})}{=} U(aT(x) + T(y)) \overset{(U \text{ is lin.})}{=} aU(T(x)) + U(T(y)) = a(UT)(x) + UT(y).$$

- See Problem Set 4 for more basic properties of the composition.

• Assume that V, W, Z are v.s. over F, and let
$\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$, $\gamma = \{z_1, \dots, z_p\}$ be ordered bases for V, W and Z, respectively.
Let $T: V \to W$ and $U: W \to Z$ be linear.
Let $A = [U]_\beta^\gamma$ and $B = [T]_\alpha^\beta$ be their matrix representations.
We have $UT: V \to Z$ — their composition.

• Let's calculate its matrix representation $[UT]_\alpha^\gamma$.
For $1 \le j \le n$, we have

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{k=1}^m B_{kj} w_k\right) = \sum_{k=1}^m B_{kj} U(w_k) = \sum_{k=1}^m B_{kj}\left(\sum_{i=1}^p A_{ik} z_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m A_{ik} B_{kj}\right) z_i = \sum_{i=1}^p C_{ij} z_i$$

where

$$C_{ij} = \sum_{k=1}^m A_{ik} B_{kj}.$$

Hence $[UT]_\alpha^\gamma = C = (C_{ij})_{\substack{1\le i\le p \\ 1\le j\le n}}$.

This computation motivates the definition of matrix multiplication.

Definition. Let A be an $m\times n$ matrix, and B an $n\times p$ matrix. We define the product of A and B, denoted AB, to be the $m\times p$ matrix such that

$$(AB_{ij}) = \sum_{k=1}^n A_{ik} B_{kj} \quad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

Example.

1) $$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 + 2\cdot 2 + 1\cdot 5 \\ 0\cdot 4 + 4\cdot 2 + (-1)\cdot 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}.$$

2) Matrix multiplication is not commutative.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$ so it is possible that $AB \ne BA$.

3) Recall the definition of the transpose of a matrix from Problem Set 2:
If $A \in M_{m\times n}(F)$, then its transpose $A^t \in M_{n\times m}(F)$ is given by $(A^t)_{ij} = A_{ji}$ for all $1 \le i \le m$, $1 \le j \le n$.
We show that

$$(AB)^t = B^t A^t$$

Indeed, we have

$$(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^n A_{jk} B_{ki} = \sum_{k=1}^n B_{ki} A_{jk} = \sum_{k=1}^n (B^t)_{ik} (A^t)_{kj} = (B^t A^t)_{ij}.$$

Returning to our previous calculation, we can now state it in a compact form using matrix multiplication.

Theorem 2.11.
Let V, W and Z be fin. dim. v.s. with ordered bases α, β and γ, respectively.
Let $T: V \to W$ and $U: W \to Z$ be lin. transformations. Then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Corollary. Let V be a fin. dim. v.s. with an ordered basis β.
Let $T, U \in \mathcal{L}(V)$. Then $[UT]_\beta = [U]_\beta [T]_\beta$.

Example. Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the lin. transf. defined by
$U(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(t)\,dt$.
Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$ be the standard ordered bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively.
We have:

$U(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$
$U(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$
$U(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$
$U(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2.$

Hence $$[U]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Similarly, for T we have:

$T(1) = x = 0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$

$T(x) = \frac{1}{2}x^2 = 0\cdot 1 + 0\cdot x + \frac{1}{2}x^2 + 0\cdot x^3$ Hence $[T]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

$T(x_2) = \frac{1}{3}x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \frac{1}{3}x^3$

Thus

$$[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]_\beta,$$ where $I: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is the identity transformation.

This confirms the fundamental theorem of calculus in a special case!

Definition The $n\times n$ identity matrix I_n is defined by $(I_n)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$.

Hence $I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc.

We summarize basic properties of matrix multiplication.

Theorem 2.12. Let $A \in M_{m\times n}(F)$, $B, C \in M_{n\times p}(F)$, and $D, E \in M_{q\times m}(F)$. Then

a) $A(B+C) = AB + AC$ and $(D+E)A = DA + EA$.

b) $a(AB) = (aA)B = A(aB)$ for any scalar $a \in F$.

c) $I_m A = A = A I_n$.

d) If $\dim(V) = n$ and $I: V \to V$ is the identity transformation, then $[I]_\beta = I_n$ for any ordered basis β for V.

Proof.

See textbook.

Compare to the basic properties of the composition of lin. transformations (Theorem 2.10).

Calculating value of a lin. transf. using its matrix representation

Theorem 2.14.

Let $T: V \to W$ be linear, V, W fin. dim. v.s.'s with ordered bases β and γ, respectively.

Then, for each $u \in V$ we have

$$[\underbrace{T(u)}_{\text{vector in } W}]_\gamma = \overbrace{[T]_\beta^\gamma}^{m\times n \text{ matrix}} \overbrace{[u]_\beta}^{n\times 1 \text{ matrix}}.$$

its coordinate vector, viewed as an $m\times 1$ matrix

matrix multiplication

Proof.

Suppose $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$ – ordered bases for V and W, respectively.

Let $x \in V$, say $x = a_1 v_1 + \dots + a_n v_n$.

That is, $[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

Let $B = [T]_\beta^\gamma$. Then

$$T(x) = a_1 T(v_1) + \dots + a_n T(v_n) = a_1\left(\sum_{i=1}^m B_{i1} w_i\right) + \dots + a_n\left(\sum_{i=1}^m B_{in} w_i\right) = \sum_{i=1}^m \left(\sum_{j=1}^n a_j B_{ij}\right) w_i.$$

Hence

$$[T(x)]_\gamma = \begin{pmatrix} \sum_{j=1}^n a_j B_{1j} \\ \vdots \\ \sum_{j=1}^n a_j B_{mj} \end{pmatrix} = B\cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$ – as wanted.

Example. Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f'(x)$.

Then
$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ – calculated in a previous example, β, γ – standard ordered bases.

Let $p(x) \in P_3(\mathbb{R})$ be arbitrary, for example $p(x) = 2 - 4x + x^2 + 3x^3$.

Then $T(p(x)) = p'(x) = -4 + 2x + 9x^2$.

Hence:

$$[T(p(x))]_\gamma = [p'(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}.$$

But also

$$[T]_\beta^\gamma [p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$ – illustrating Theorem 2.14.

Associating a linear transformation to a matrix

Definition Let $A \in M_{m\times n}(F)$. We denote by L_A the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$.
regarded as column vectors.
We call L_A a left-multiplication transformation.

Example.
Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \in M_{2\times 3}(\mathbb{R})$, hence $L_A : \mathbb{R}^3 \to \mathbb{R}^2$.

If $x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ then $L_A(x) = Ax = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Theorem 2.15 (Properties of L_A)
Let $A \in M_{m\times n}(F)$. Then $L_A : F^n \to F^m$ is linear.
If $B \in M_{m\times n}(F)$ and β, γ are the standard ordered bases for F^n and F^m, resp., then:

a) $[L_A]_\beta^\gamma = A$.
b) $L_A = L_B \iff A = B$.
c) $L_{A+B} = L_A + L_B$, $L_{aA} = a \cdot L_A$ for all $a \in F$.
d) If $T: F^n \to F^m$ is lin., then there is a unique $C \in M_{m\times n}(F)$ s.t. $T = L_C$. In fact, $C = [T]_\beta^\gamma$.
e) If $E \in M_{n\times p}(F)$, then $L_{AE} = L_A L_E$.
f) If $m = n$, then $L_{I_n} = I_{F^n}$.

Proof. Linearity of L_A is clear by Theorem 2.12.
a) The j^{th} column of $[L_A]_\beta^\gamma$ is $L_A(e_j) = Ae_j$, which is also the j^{th} column of A.
So $[L_A]_\beta^\gamma = A$.
b) "$\Leftarrow$": clear
"$\Rightarrow$": If $L_A = L_B$, then by (a), $A = [L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$.
d) Let $T: F^n \to F^m$ be lin, let $C = [T]_\beta^\gamma$.
By Theorem 2.14,
$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$, or $T(x) = Cx = L_C(x)$ for all $x \in F^n$.
So $T = L_C$. The uniqueness of C follows from (b).
e) $(AE)e_j$ = the j^{th} column of $AE = A(Ee_j)$ – both equalities are easy to see by writing out the products.
Thus $L_{AE}(e_j) = (AE)e_j = A(Ee_j) = L_A(Ee_j) = L_A(L_E(e_j))$.
Hence $L_{AE} = L_A L_E$ (by the corollary to Theorem 2.6, if to linear transf.'s agree on a basis then they are equal).
(c), (f) – Exercise.

Theorem 2.16. (Matrix multiplication is associative)
Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$, $C \in M_{p\times r}(F)$. Then
$A(BC) = (AB)C$.
Proof.

We have (using Theorem 2.15(e) and associativity of the composition of functions)
$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B) L_C = L_{AB} L_C = L_{(AB)C}$.
By Theorem 2.15(b), $A(BC) = (AB)C$.

Invertibility

Definition. Let V and W be v.s. and $T: V \to W$ linear.
A function $U: W \to V$ is an inverse of T if $TU = I_W$ and $UT = I_V$.
If T has an inverse, then T is invertible.
If T is invertible, then the inverse of T is unique and is denoted by T^{-1}.

Basic facts about invertible functions.
1) Let T and U be invertible. Then the following holds:
a) $(TU)^{-1} = U^{-1}T^{-1}$
b) $(T^{-1})^{-1} = T$; in particular, T^{-1} is invertible.
2) T is invertible $\Leftrightarrow$ T is a bijection.

Proof. 2) "$\Rightarrow$" For any $y \in W$, $TT^{-1}(y) = I_W(y) = y$. Hence $y = T(\underbrace{T^{-1}(y)}_{\in V})$, so T is surjective.
Assume $T(x_1) = T(x_2)$, then $T^{-1}(T(x_1)) = T^{-1}(T(x_2))$, hence $x_1 = x_2$ — so T is injective.

Theorem 2.17. Let V, W be v.s., let $T: V \to W$ be lin. and invertible.
Then $T^{-1}: W \to V$ is also linear.

Proof.
Let $y_1, y_2 \in W$ and $c \in F$. Since T is both surjective and injective, there exist <u>unique</u> vectors $x_1, x_2 \in V$ s.t. $T(x_1) = y_1$ and $T(x_2) = y_2$.
Thus $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$. And so
$T^{-1}(cy_1 + y_2) = T^{-1}(cT(x_1) + T(x_2)) = T^{-1}(T(cx_1 + x_2)) = I_V(cx_1 + x_2) = cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2)$.

Example. Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be the lin. transf. defined by $T(a + bx) = (a, a+b)$.
Then $T^{-1}: \mathbb{R}^2 \to P_1(\mathbb{R})$ is defined by $T^{-1}(c,d) = c + (d-c)x$ — also linear, as Theorem 2.17 predicts.

• Recall the analogy between linear transformations and matrices.

Definition. Let $A \in M_{n\times n}(F)$. Then A is invertible if there exists $B \in M_{n\times n}(F)$ s.t. $AB = BA = I$.

Note. If A is invertible, then the matrix B such that $AB = BA = I$ is unique, called the inverse of A and denoted A^{-1}.
(If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$).

Example. The inverse of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Indeed, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Lemma. Let $T: V \to W$ be lin. and invertible, and $\dim(V) < \infty$. Then $\dim(V) = \dim(W)$.

Proof. Let $\beta = \{x_1, \ldots, x_n\}$ be a basis for V.
By Theorem 2.2, $\mathrm{Span}(T(\beta)) = R(T) = W$.
Next, T is a bijection, so:
$\dim(N(T)) = 0$ (as $N(T) = \{0\}$ as T is injective).
$\dim(R(T)) = \dim(W)$ (as $R(T) = W$).
Hence, by the dimension theorem, $\dim(V) = \dim(N(T)) + \dim(R(T)) = \dim(W)$.

Theorem 2.18 Let V, W be fin. dim. v.s. with ordered bases β and γ, resp.
Let $T: V \to W$ be lin.
Then T is invertible $\iff$ $[T]_\beta^\gamma$ is invertible.
Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Proof.

"$\Rightarrow$" Suppose T is invertible.
By the Lemma, $\dim(V) = \dim(W) = n$. So $[T]_\beta^\gamma \in M_{n\times n}(F)$.
By definition, $T^{-1}: W \to V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$. Thus
$$I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma.$$
Similarly,
$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n.$$
So $[T]_\beta^\gamma$ is invertible and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

"$\Leftarrow$" Suppose $A = [T]_\beta^\gamma$ is invertible. Then there exists $B \in M_{n\times n}(F)$ s.t. $AB = BA = I_n$.
By Theorem 2.6, there exists $U \in \mathcal{L}(W, V)$ s.t.
$$U(w_j) = \sum_{i=1}^n B_{ij} v_i \quad \text{for } j = 1, \dots, n,$$
where $\gamma = \{w_1, \dots, w_n\}$, $\beta = \{v_1, \dots, v_n\}$.
It follows that $[U]_\gamma^\beta = B$.
To show that $U = T^{-1}$, notice that
$$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta \quad \text{— by Theorem 2.11.}$$
So $UT = I_V$, and similarly, $TU = I_W$.

Example. Let β and γ be the standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$, resp.
For T given by $T(a+bx) = (a, a+b)$ from the previous example, we have
$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } [T^{-1}]_\gamma^\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$
We have already checked that each of these matrices is the inverse of the other.

Corollary. Let $A \in M_{n\times n}(F)$. Then A is invertible $\iff$ L_A is invertible. Moreover, $(L_A)^{-1} = L_{A^{-1}}$.