The adjoint of a lin. operator

If V is an inner prod. space, then for any $y \in V$ the function $g: V \to F$ defined by $g(x) = \langle x, y \rangle$ is linear. If V is finite dimensional, then every lin. transformation from V to F is of this form:

Thm 6.8 Let V be a fin. dim. inn. prod. space over F, and let $g: V \to F$ be a lin. transf. Then there exists a unique vector $y \in V$ s.t. $g(x) = \langle x, y \rangle$ for all $x \in V$.

Proof

If $\beta = \{v_1, \dots, v_n\}$ an orthonorm. basis for V, can take $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$. (see Textbook, p. 357 for a proof.)

Ex

Define $g: \mathbb{R}^2 \to \mathbb{R}$ by $g(a_1, a_2) = 2a_1 + a_2$, g is lin.

Let $\beta = \{e_1, e_2\}$ – an orthonormal basis, and let $y = g(e_1)e_1 + g(e_2)e_2 = 2e_1 + e_2 = (2, 1)$.

Then $g(a_1, a_2) = \langle (a_1, a_2), (2,1) \rangle = 2a_1 + a_2$.

Thm 6.9 Let V be a fin. dim. inn. prod. space and $T \in \mathcal{L}(V)$. Then there exists a unique lin. operator $T^* \in \mathcal{L}(V)$ s.t.

$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$.

(See Textbook, p. 358 for a proof.)

T^* is called the adjoint of T.

Remark 1) We also have $\langle x, T(y) \rangle = \overline{\langle T(y), x \rangle} = \overline{\langle y, T^*(x) \rangle} = \langle T^*(x), y \rangle$ for all $x, y \in V$.

2) If V is inf. dim, then the existence of the adjoint is not guaranteed (See Ex. 24, Sec. 6.4).

Recall that for a matrix $A \in M_{m \times n}(F)$, the adjoint of A is defined as the matrix $A^* \in M_{n \times m}(F)$ s.t. $A^*_{ij} = \overline{A_{ji}}$ for all i, j. So if $F = \mathbb{R}$, then simply $A^* = A^t$.

Thm 6.10 Let V be a fin. dim. inn. prod. space and β an orthonorm. basis for V. If $T \in \mathcal{L}(V)$, then $[T^*]_\beta = [T]_\beta^*$.

Proof

Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{v_1, \dots, v_n\}$. From the corollary to Thm 6.5 we have:

$$B_{ij} = \langle T^*(v_j), v_i \rangle \underset{\substack{\uparrow \\ \text{"symmetry" of} \\ \text{inner product}}}{=} \overline{\langle v_i, T^*(v_j) \rangle} \underset{\substack{\uparrow \\ \text{def. of} \\ \text{adjoint}}}{=} \overline{\langle T(v_i), v_j \rangle} = \overline{A_{ji}} = (A^*)_{ij}. \quad \text{Hence } B = A^*.$$

Cor Let $A \in M_{n \times n}(F)$. Then $L_{A^*} = (L_A)^*$.

Proof.

If β is the standard ordered basis for F^n, then $[L_A]_\beta = A$. Hence $[(L_A)^*]_\beta = [L_A]_\beta^* = A^* = [L_{A^*}]_\beta$, so $(L_A)^* = L_{A^*}$.

Ex

Let $T \in \mathcal{L}(\mathbb{C}^2)$ be defined by $T(a_1, a_2) = (2i a_1 + 3a_2, a_1 - a_2)$.

If β is the standard ordered basis for $V = \mathbb{C}^2$, $\beta = \{e_1, e_2\}$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$[T]_\beta = \begin{pmatrix} | & | \\ [T(e_1)]_\beta & [T(e_2)]_\beta \\ | & | \end{pmatrix} = \begin{pmatrix} 2i & 3 \\ 1 & -1 \end{pmatrix}. \qquad \text{So } [T^*]_\beta = [T]_\beta^* = \begin{pmatrix} -2i & 1 \\ 3 & -1 \end{pmatrix}.$$

Hence $T^*(a_1, a_2) = [T]_\beta^* \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (-2i a_1 + a_2, 3a_1 - a_2)$.

There are many algebraic analogies between the conjugates of complex numbers and the adjoints of lin. operators, e.g. $(T+U)^* = T^* + U^*$, $(cT)^* = \bar{c}T^*$, etc. (see Thm 6.11 in the book).

Normal and self-adjoint operators

In a vector space V:

$T \in \mathcal{L}(V)$ is diag'z $\iff$ $[T]_\beta$ is diagonal for some basis β for V $\iff$ V has a basis of e.vects for T

In an inn. prod. space V:

??? $\iff$ V has an orthonormal basis of e.vects for T.

If such an orthonorm. basis β exists, then $[T]_\beta$ is a diagonal matrix. Hence $[T^*]_\beta = [T]_\beta^*$ is also diagonal. Because diagonal matrices commute, T and T^* commute. This motivates:

Def 1) Let V be an inn. prod. space, let $T \in \mathcal{L}(V)$.
T is normal if $TT^* = T^*T$.
2) $A \in M_{n\times n}(F)$, for $F = \mathbb{R}, \mathbb{C}$, is normal if $AA^* = A^*A$.

By Thm 6.10, T is normal $\iff$ $[T]_\beta$ is normal.

Thm 6.16 Let $T \in \mathcal{L}(V)$ for V a fin. dim. complex inn. prod. space (so $F = \mathbb{C}$)
Then T is normal $\iff$ exists an orthonorm. basis for V of e.vects. for T.

This solves our problem when $F = \mathbb{C}$, but not for $F = \mathbb{R}$:

Ex Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by θ, $0 < \theta < \pi$. If β - stand. ord. basis,

$$[T]_\beta = A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note that $AA^* = I = A^*A$. So A (and T) are normal.

However, T has no e.vects at all! (exercise).

For real inn. prod. spaces, need a stronger condition.

Def 1) $T \in \mathcal{L}(V)$ is self-adjoint (Hermitian) if $T = T^*$.
2) $A \in M_{n\times n}(F)$ is self-adjoint (Hermitian) if $A = A^*$.

Lemma Let $T \in \mathcal{L}(V)$ be self-adjoint. Then:
1) Every e.val. of T is real. (even when $F = \mathbb{C}$!)
2) If $F = \mathbb{R}$, then the char. poly. of T splits.

Thm 6.17
Let $T \in \mathcal{L}(V)$, V - fin. dim. real inn. prod. space. Then:
T is self-adjoint $\iff$ exists an orthonorm. basis β for V of e.vects. for T.