

Definition. A **vector space** V over a **field** F is a set with two operations:

(**vector addition**) For any x and y in V , there is a uniquely defined element $x+y$ in V .

(**scalar multiplication**) For any a in F and x in V , there is a uniquely defined element $a \cdot x$ in V .

Satisfying the following eight conditions (VS1) - (VS8):

(VS1) $x+y = y+x$ for any x, y in V . (commutativity of addition)

(VS2) $(x+y)+z = x+(y+z)$ for any x, y, z in V (associativity of addition)

(VS3) There is an element in V denoted by 0 such that $x+0 = x$ for all x in V .

(VS4) For each x in V there is some y in V such that $x+y = 0$.

(VS5) $1 \cdot x = x$ for all x in V (where 1 is the multiplicative identity of F).

(VS6) $a \cdot (b \cdot x) = (a \cdot b) \cdot x$ for all x in V and a, b in F . (associativity of scalar multiplication)

(VS7) $a \cdot (x+y) = a \cdot x + a \cdot y$ for all a in F and x, y in V

(VS8) $(a+b) \cdot x = a \cdot x + b \cdot x$ for all a, b in F and x in V . } distributive laws

• Elements of V are called **vectors**, and elements of F are called **scalars**.

• Usually F will be either the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} .

Example 1. Given a field F , consider the set

$$F^n = \{ (x_1, x_2, \dots, x_n) : x_i \in F \} \quad (\text{the set of all } n\text{-tuples of elements from } F).$$

We define the operations of vector addition and scalar multiplication in the following way:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$a \cdot (x_1, \dots, x_n) = (a \cdot x_1, \dots, a \cdot x_n)$$

where $(x_1, \dots, x_n), (y_1, \dots, y_n)$ are arbitrary elements of F^n and a is an arbitrary element of F .

(and $x_i + y_i$ and $a \cdot x_i$ are calculated in F).

With these operations F^n is a vector space over the field F . (One has to check that (VS1) - (VS8) hold. Do it!)

In particular, if $n=2$ and $F=\mathbb{R}$, we obtain the familiar space \mathbb{R}^2 of vectors on the plane.

Example 2. Let F be a field, and let $P(F)$ denote the set of all polynomials with coefficients in F . That is, $P(F)$ consists of all expressions of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

for some $n \geq 0$, with a_i in F for all $i=0, 1, \dots, n$. If $a_i=0$ for all $i=0, 1, \dots, n$ then $p(x)$ is called the **zero polynomial**.

The **degree** of a non-zero polynomial is the largest i such that $a_i \neq 0$. (The degree of the zero polynomial is defined to be -1 .)

Two polynomials $p(x) = a_n x^n + \dots + a_1 x + a_0$ and $q(x) = b_m x^m + \dots + b_1 x + b_0$ are equal if $m=n$ and $a_i = b_i$ for all $i=0, 1, \dots, n$.

We define addition and scalar multiplication on $P(F)$ in the usual way:

Addition: Given $p(x), q(x)$ as above, suppose that $m \leq n$. Then we can also write $q(x)$ as $b_n x^n + \dots + b_1 x + b_0$, where $b_{m+1} = b_{m+2} = \dots = b_n = 0$.

$$\text{Then we define } (p+q)(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \dots + (a_1 + b_1)x + (a_0 + b_0).$$

Scalar multiplication: For any c in F , define $c \cdot p(x) = ca_n x^n + ca_{n-1} x^{n-1} + \dots + ca_1 x + ca_0$.

(Equivalently, $p+q$ can be defined as the polynomial satisfying $(p+q)(x) = p(x) + q(x)$ for all x in F , and $c \cdot p$ as the polynomial satisfying $(c \cdot p)(x) = c \cdot p(x)$ for all x in F).

With these operations, $P(F)$ is a vector space over F . (Again, need to check that (VS1) - (VS8) hold.)

Example 3. Let $M_2(\mathbb{R})$ denote the set of all 2×2 matrices with entries from \mathbb{R} .

We define addition and scalar multiplication in the familiar way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \quad \text{and} \quad \lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \quad \text{for any } a, b, c, d, e, f, g, h, d \text{ in } \mathbb{R}.$$

With these operations, $M_2(\mathbb{R})$ is a vector space over \mathbb{R} .

Analogously, the space $M_{m \times n}(\mathbb{R})$ of all $m \times n$ matrices is a vector space.

Example 4 The most boring vector space, aka **the zero vector space**.

• It consists of a single vector 0 (so $V = \{0\}$).

Operations of addition and scalar multiplication are defined by:

$$0 + 0 = 0$$

$\lambda \cdot 0 = 0$ for all λ in F .

With these operations, V is a vector space over F . (check it!)

Basic properties of vector spaces

Now, we will see some of the basic properties of vector spaces (we will deduce them as logical consequences of the axioms (VS1)-(VS8).)

Theorem 1.1 (Cancellation law)

Let V be a vector space, and x, y, z be arbitrary elements of V .

If $x+z = y+z$, then $x=y$.

Proof. As V is a vector space, it satisfies all of the properties (VS1)-(VS8).

By (VS4) there exists an element \tilde{z} in V such that $z + \tilde{z} = 0$. We have:

$$x = x+0 = x+(z+\tilde{z}) \stackrel{\text{by (VS2)}}{=} (x+z)+\tilde{z} \stackrel{\text{by assumption}}{=} (y+z)+\tilde{z} \stackrel{\text{by (VS2) again}}{=} y+(z+\tilde{z}) \stackrel{\text{by the choice of } \tilde{z}}{=} y+0 \stackrel{\text{by (VS3) again}}{=} y.$$

Thus $x=y$.

By commutativity of addition (VS1) we also have: if $z+x = z+y$, then $x=y$.

Corollary 1. The vector 0 described in (VS3) is unique. (and is called the zero vector of V).

Proof. Suppose that 0 and $0'$ are two elements in a vector space V that both satisfy (VS3).

Then for any x in V we have:

$$x+0 = x = x+0' \\ \text{by (VS3) for } 0 \quad \text{by (VS3) for } 0'$$

By the cancellation law it follows that $0=0'$.

For example, in F^n the zero vector is $(\underbrace{0, 0, \dots, 0}_{n \text{ times}})$, and in $M_2(\mathbb{R})$ the zero vector is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Corollary 2. For any x in V , the vector y described in (VS4) is unique.

(it is called the additive inverse of x and is denoted by $-x$).

Proof. Let x in V be arbitrary, and suppose that y_1 and y_2 are two elements in V both satisfying (VS4).

That is, $x+y_1 = 0 = x+y_2$.

By cancellation law this implies $y_1 = y_2$.

Finally, we state some further useful properties of vector spaces.

Theorem 1.2. Let V be a vector space over a field F .

For all x in V and a in F we have:

- 1) $0 \cdot x = 0$ ← the zero vector of V .
scalar in F
- 2) $(-a) \cdot x = -(ax) = a \cdot (-x)$.
the additive inverse of the vector ax .
- 3) $a \cdot 0 = 0$
the zero vector in V

Proof. Problem Set 1.

Subspaces

Definition. Let V be a vector space over a field F .

A subset W of V ($W \subseteq V$) is a subspace of V if W itself is a vector space over F , with respect to the addition and scalar multiplication defined on V .

That is, W satisfies all of the properties (VS1)-(VS8).

Example 1 Let $n \geq 1$ be an integer and F a field.

Recall that the vector space F^n is the set $\{(x_1, \dots, x_n) : x_i \in F\}$ with addition and scalar multiplication

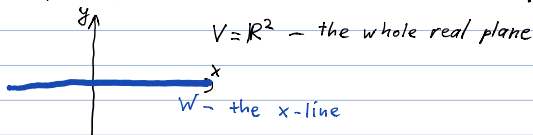
given by $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ and $a \cdot (x_1, \dots, x_n) = (ax_1, \dots, ax_n)$.

Consider the subset

$$W = \{(x_1, \dots, x_{n-1}, 0) : x_i \in F\}.$$

Then W is a subspace of F^n (we will prove it later).

For example, for $F = \mathbb{R}$ and $n = 2$ we have $V = \mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$, $W = \{(x_1, 0) : x_1 \in \mathbb{R}\}$:



Example 2. Let F be a field, and recall that $P(F)$ is the vector space of all polynomials with coefficients from F .

For an integer $n \geq 0$, consider the subset $P_n(F) \subseteq P(F)$ consisting of all polynomials of degree $\leq n$.

Then $P_n(F)$ is a subspace of $P(F)$.

Example 3. For any vector space V , V itself and $\{0\}$ are both subspaces of V .

Example 4. Let S be a non-empty set and F a field.

Let $\mathcal{F}(S, F)$ denote the set of all maps from S to F .

(A map $f: S \rightarrow F$ takes x in S as an input and returns $f(x)$ in F as an output.)

Synonym: function.)

Two maps f and g in $\mathcal{F}(S, F)$ are equal if $f(x) = g(x)$ for all x in S .

For any f and g in $\mathcal{F}(S, F)$ and c in F we define $f+g$ and $c \cdot f$ in $\mathcal{F}(S, F)$ by taking

$(f+g)(x) = f(x) + g(x)$ for each x in S (where "+" and "." on the right side are calculated in F).

$(c \cdot f)(x) = c \cdot f(x)$

With these operations $\mathcal{F}(S, F)$ is a vector space (check!)

In particular, $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consists of all real-valued functions defined on real numbers.

Let $C(\mathbb{R})$ denote the set of all continuous real functions.

Then $C(\mathbb{R})$ is a subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, and we will see that it is a subspace as well.

Let now V be a vector space, and let $W \subseteq V$ be a subset of V . According to the definition, in order to show that W is a subspace, we have to check that all of the properties (VS1)-(VS8) hold in W , with respect to addition and scalar multiplication defined on V .

Turns out that we actually need to check less in this situation.

Theorem 1.3. Let V be a vector space over F , and let $W \subseteq V$ be a subset of V . (need not be a subspace!)

Then W is a subspace of V if and only if the following holds for W :

(a) $0 \in W$ (that is, the zero vector of V is in W).

(b) If $x, y \in W$ then $x+y \in W$ (that is, W is closed under addition)

(c) If $x \in W$ and $c \in F$ then $c \cdot x \in W$ (that is, W is closed under scalar multiplication)

Proof.

1) First we assume that W is a subspace of V and show that (a), (b), (c) hold

" W is a subspace of V " means that it is a vector space under the operations of addition and scalar multiplication defined on V .

In particular:

• for any x, y in W and c in F , $x+y$ and $c \cdot x$ must also be in W . So (b) and (c) hold.

• (VS3) holds in W , that is there is some $0' \in W$ such that $x + 0' = x$ for all $x \in W$.

In particular, $0' + 0' = 0'$.

But as $0' \in V$, also $0' + 0 = 0'$ (since $0 \in V$ satisfies (VS3) in V .)

By cancellation law in the vector space V , this implies that $0' = 0$. In particular, $0 \in W$ and (a) holds.

2) Conversely, suppose (a), (b), (c) holds, and we will show that then W is not just a subset of V , but also a subspace. For this we have to check that W satisfies (VS1)-(VS8).

• As V is a vector space, (VS1), (VS2), (VS5), (VS6), (VS7), (VS8) hold for all elements of V . As W is a subset of V ,

they hold in particular for all elements of W . (so we get all these properties in W for free).

• Remains to check that (VS3) and (VS4) hold in W .

(VS3): holds by (a).

(VS4): Let $x \in W$ be arbitrary. We need to find some $y \in W$ such that $x+y=0$. We know that in V there is such an element, namely the additive inverse $-x$. We show that actually $-x \in W$ as well.

As -1 is a scalar in F , by (c) we have $(-1) \cdot x = -x$.

As V is a vector space, it satisfies $-x = (-1) \cdot x$, by Theorem 1.2.

Thus $-x \in W$, as wanted.

Now we return to the examples and verify our claims using Theorem 1.3.

Example 1. $V = F^n$, $W = \{ (x_1, \dots, x_{n-1}, 0) : x_i \in F \} \subseteq V$.

We check that W satisfies (a), (b), (c), and so indeed W is a subspace of V by Theorem 1.3.

(a) The zero vector of V is $(0, \dots, 0)$, it is in W (taking $x_1 = \dots = x_{n-1} = 0$).

(b), (c) W is closed under addition and scalar multiplication.

Given any $(x_1, \dots, x_{n-1}, 0), (y_1, \dots, y_{n-1}, 0)$ in W and a in F we have:

$$(x_1, \dots, x_{n-1}, 0) + (y_1, \dots, y_{n-1}, 0) = (x_1 + y_1, \dots, x_{n-1} + y_{n-1}, \underbrace{0+0}_{=0}) \text{ — also in } W.$$

$$a \cdot (x_1, \dots, x_{n-1}, 0) = (a \cdot x_1, \dots, a \cdot x_{n-1}, \underbrace{a \cdot 0}_{=0})$$

Similarly, $W = \{ (0, x_1, \dots, x_{n-1}) : x_i \in F \}$ is also a subspace of F^n .

Example 2. $V = P(F)$, $W = P_n(F)$ — the set of all polynomials of degree at most n .

(a) The zero vector in $P(F)$ is the zero polynomial $p_0(x) = a_n x^n + \dots + a_1 x + a_0$ with $a_n = \dots = a_1 = a_0 = 0$. Its degree is by definition -1 , so p_0 is in $P_n(F)$.

(b), (c) If both $p(x)$ and $q(x)$ are in $P_n(F)$, that is they have degree at most n , then $p(x)+q(x)$ also has degree at most n , and thus $p+q$ is in $P_n(F)$ as well.

If $p(x)$ has degree $\leq n$ and $a \in F$, then $a \cdot p(x)$ has degree $\leq n$ as well, and so $a \cdot p \in P_n(F)$.

Example 4. $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ — all real-valued functions, $W = C(\mathbb{R})$ — continuous real-valued functions.

The zero vector in \mathcal{F} is the function given by $f(x) = 0$ for all x in \mathbb{R} .

By basic calculus we know that all constant functions are continuous, that sum of any two continuous functions is continuous, and that a product of a constant function and a continuous function is also a continuous.

Thus $C(\mathbb{R})$ satisfies the conditions (a), (b), (c) in Theorem 1.3.

We can form new subspaces from the old ones.

Theorem 1.4. Let V be a vector space over F .

If W_1, \dots, W_n are subspaces of V , then the set $W = W_1 \cap W_2 \cap \dots \cap W_n$ is also a subspace of V .

Proof. We check that W satisfies (a), (b), (c) and apply Theorem 1.3.

By assumption each of W_i , $i=1, \dots, n$, is a subspace, and so satisfies (a), (b), (c).

(a) As each of W_i satisfies (a), we have $0 \in W_i$ for all $i=1, \dots, n$.

But this means that $0 \in W_1 \cap \dots \cap W_n = W$ as well.

(b) Let $x, y \in W$, which means precisely that $x, y \in W_i$ for all $i=1, \dots, n$.

As each of W_i satisfies (b), $x+y \in W_i$ for each $i=1, \dots, n$.

Hence $x+y \in W_1 \cap \dots \cap W_n$.

(c) Let $x \in W$, then $x \in W_i$ for each $i=1, \dots, n$.

Then for any $c \in F$, $c \cdot x \in W_i$ for each $i=1, \dots, n$ (as W_i satisfies (c)).

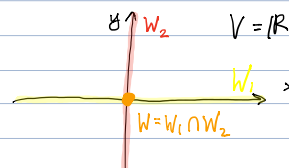
Hence $c \cdot x \in W_1 \cap \dots \cap W_n = W$.

Example. Let $V = \mathbb{R}^2$.

Let $W_1 = \{ (x, 0) : x \in \mathbb{R} \}$ and let $W_2 = \{ (0, x_2) : x_2 \in \mathbb{R} \}$.

We already know that both W_1 and W_2 are subspaces of V .

Then $W_1 \cap W_2 = \{ (0, 0) \}$ — the zero subspace of \mathbb{R}^2 .



However, the union $W = W_1 \cup W_2$ of two subspaces W_1, W_2 of V need not be a subspace of V in general!
(see Problem Set 1).