MATH 115A (CHERNIKOV), SPRING 2016

PROBLEM SET 3

DUE FRIDAY, APRIL 22

Problem 1. Do Exercise 1, Section 1.6. Justify each answer!

Problem 2. Determine which of the following sets are bases for $\mathbb{R}^3$:

(1) $\{(1,0,-1),(2,5,1),(0,4,-3)\}$.
(2) $\{(2,-4,1),(0,3,-1),(6,0,-1),(17,3,-3)\}$.
(3) $\{(1,2,-1),(1,0,2),(2,1,1)\}$.
(4) $\{(0,1,1),(0,2,0),(1,0,0)\}$.

Problem 3. The set of solutions to the system of linear equations

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Problem 4. Suppose that V is a vector space of dimension n, and let W be a subspace of V of dimension m (so $m \leq n$). Show that for every integer k such that $m \leq k \leq n$ there is a subspace U of V such that $W \subseteq U \subseteq V$ and $\dim(U) = k$.

Problem 5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map with
$T(1,0,0,0) = (0,1,2)$,
$T(0,1,0,0) = (0,1,2)$,
$T(0,0,1,0) = (1,0,2)$,
$T(0,0,0,1) = (1,1,4)$.

Determine a basis for the range $R(T)$ of T, a basis for the null space $N(T)$ of T, and compute the dimension of $N(T)$.

Problem 6. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,-2) = (1,1)$ and $T(-2,0,4) = (2,6)$?

Problem 7. Let $v_1, \ldots, v_n$ be vectors in a vector space V over a field F. Consider the map

$$T : F^n \to V, (a_1, \ldots, a_n) \mapsto a_1 v_1 + \ldots + a_n v_n.$$

Show that T is linear, and moreover:

(1) T is injective if and only if $\{v_1, \ldots, v_n\}$ is linearly independent.
(2) T is surjective if and only if $\{v_1, \ldots, v_n\}$ generates V.
(3) T is bijective if and only if $\{v_1, \ldots, v_n\}$ is a basis for V.

Problem 8. We consider $V = M_{n\times n}(\mathbb{R})$. The *trace* of an $n \times n$ matrix $A \in M_{n\times n}(\mathbb{R})$, denoted by $\mathrm{tr}(A)$, is the sum of the diagonal entries of A:

$$\mathrm{tr}(A) = A_{11} + A_{22} + \ldots + A_{nn}.$$

Consider the map $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \mathrm{tr}(A)$.

Show that it is a linear transformation, and determine the null space $N(T)$ of T and the dimension of $N(T)$.