MATH 115A (CHERNIKOV), SPRING 2016

PROBLEM SET 2

DUE FRIDAY, APRIL 15

Problem 1. Show that the vectors $(1,1,0), (1,0,1)$ and $(0,1,1)$ generate $\mathbb{R}^3$.

Problem 2. Show that a subset W of a vector space V is a subspace if and only if $\operatorname{Span}(W) = W$.

Problem 3. Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$.

(1) Show that then $\operatorname{Span}(S_1) \subseteq \operatorname{Span}(S_2)$.
(2) If $\operatorname{Span}(S_1) = V$, deduce that $\operatorname{Span}(S_2) = V$.

Problem 4. Let $M_{m\times n}(\mathbb{R})$ be the vector space of all m-by-n matrices with real entries.

For an $m \times n$ matrix $A \in M_{m\times n}(\mathbb{R})$, its *transpose* A^t is the $n \times m$ matrix obtained from A by interchanging the rows with the columns. That is, $(A^t)_{ij} = A_{ji}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. So for example if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

A *symmetric* matrix is a matrix A such that $A^t = A$ (so it has to be a square matrix, that is $m = n$).

Let W be the set of all symmetric matrices in $M_{2\times 2}(\mathbb{R})$.

(1) Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$ (Hint: you will need to prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{2\times 2}(\mathbb{R})$ and $a, b \in \mathbb{R}$).
(2) Let
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Show that $\operatorname{Span}(\{A_1, A_2, A_3\}) = W$.

Problem 5. Consider the following sets of vectors.

(1) $\{(-1,1,2), (1,-2,1), (1,1,4)\}$ in $\mathbb{R}^3$,
(2) $\{(1,-1,2), (2,0,1), (-1,2,-1)\}$ in $\mathbb{R}^3$,
(3) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$.

Determine if they are linearly dependent or linearly independent (and justify).

Problem 6. Let $V = \mathbb{R}^3$. Find three vectors $w, v, z \in V$ with the following properties:

(1) $\operatorname{Span}(\{w,v\}) = \operatorname{Span}(\{v,z\}) = \operatorname{Span}(\{w,v,z\})$,
(2) $\operatorname{Span}(\{w,v,z\}) \neq \operatorname{Span}\{w,z\}$.

Suppose that w, v, z are any three vectors with the above listed properties. Prove or disprove the following statements:

(1) w, v are linearly independent.
(2) v, z are linearly independent.
(3) w, z are linearly independent.

Problem 7. Determine whether the vectors

$$f(x) = \sin^2 x, g(x) = \cos^2 x, h(x) = 1$$

in the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ are linearly independent.

Problem 8. Give three *different* bases for each of the following spaces:

(1) $\mathbb{R}^2$,
(2) $M_{2\times 2}(\mathbb{R})$,
(3) $P_2(\mathbb{R})$.