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Proposition 0.1. Let $\mathscr{M}$ be κ-saturated, $A \subset M^n$ with $|A| < \kappa$. $P : M^n \to [0,1]$ is definable in $\mathscr{M}$ over A if and only if, for every $r \in [0,1]$, the sets

$$\{a \in M^n : P(a) \geq r\}, \quad \{a \in M^n : P(a) \leq r\}$$

are type definable in $\mathscr{M}$ over A.

Proof. ($\Rightarrow$) Let $\Phi : S_n(A) \to [0,1]$ be continuous such that, for all $a \in M^n$,

$$\Phi(\mathrm{tp}_{\mathscr{M}}(a/A)) = P(a).$$

For any $r \in [0,1]$, $\Phi^{-1}[0,r]$ is closed, hence of the form

$$C_\Gamma = \{p \in S_n(A) : \Gamma \subset p\}$$

for some $\Gamma \subset L(A)$. Thus, Γ type defines the set $\{a \in M^n : P(a) \leq r\}$. For the set $\{a \in M^n : P(a) \geq r$, look at $\Phi^{-1}[r,1]$.

($\Leftarrow$) We show that there is a continuous $\Phi : S_n(A) \to [0,1]$ such that, for all $a \in M^n$,

$$\Phi(\mathrm{tp}_{\mathscr{M}}(a/A)) = P(a).$$

Define Φ on $S_n(A)$ by

$$\Phi(p) = P(a), \quad \text{where } a \models_{\mathscr{M}} p.$$

Note that Φ is well defined; indeed, such an $a \in M$ exists by saturation, and by our hypothesis any $a, b \in M$ with the same type over A have $P(a) = P(b)$. Next, we show that Φ is continuous. Fix $p \in S_n(A)$, $\epsilon > 0$. Let $r = \Phi(p)$.

Claim. $\exists \varphi$ with $(\varphi = 0) \in p$, $\exists \delta > 0$ such that $\forall a \in M^n$,

$$\varphi^{\mathscr{M}}(a) < \delta \implies P(a) \in (r - \epsilon, 1].$$

To prove the claim, suppose it fails. Then $\forall \varphi$ with $(\varphi = 0) \in p$, $\forall \delta > 0$, there exists $a \in M^n$ such that $\varphi^{\mathscr{M}}(a) < \delta$ and $P(a) \leq r - \epsilon$. If Γ type defines the set $\{a \in M^n : P(a) \leq r - \epsilon\}$, then $p^+ \cup \Gamma$ is finitely satisfiable in $\mathscr{M}$. By saturation, there is $a \in M^n$ with $a \models_{\mathscr{M}} p \cup \Gamma$. But this yield the contradiciton

$$r = \Phi(p) = P(a) \leq r - \epsilon.$$

The claim gives us a neighborhood $[\varphi < \delta]$ of p such that

$$q \in [\varphi < \delta] \implies P(q) \in (r - \epsilon, 1].$$

Similarly, we can show there is a neighborhood $[\varphi' < \delta']$ of p such that

$$q \in [\varphi' < \delta'] \implies P(q) \in [0, r + \epsilon).$$

Thus, $p \in [\varphi < \delta] \cap [\varphi' < \delta'] \subset \Phi^{-1}(r - \epsilon, r + \epsilon)$. This completes the poof that Φ is continuous.

$\square$