Model-theoretic approach to multi-dimensional de Finetti theory

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Model theory

- We fix a complete countable first-order theory T in a language L.
- Let $\mathbb{M}$ be a monster model of T (i.e. κ^*-saturated and κ^*-homogeneous for some sufficiently large cardinal κ^*).
- Given a set $A \subseteq \mathbb{M}$, we let $S(A)$ denote the space of types over A (i.e. the Stone space of ultrafilters on the Boolean algebra of A-definable subsets of $\mathbb{M}$).

Stability

Definition

1. We say that T *encodes a linear order* if there is a formula $\phi(\bar{x}, \bar{y}) \in L$ and $(\bar{a}_i : i \in \omega)$ in $\mathbb{M}$ such that $\mathbb{M} \models \phi(\bar{a}_i, \bar{a}_j) \Leftrightarrow i < j$.
2. A theory T is *stable* if it cannot encode a linear order.
3. Equivalently, for some cardinal κ we have $\sup\{|S(M)| : M \models T, |M| = \kappa\} = \kappa$.

- Examples of stable first-order theories: equivalence relations, modules, algebraically closed fields, separably closed fields, free groups, planar graphs.

Stability: indiscernible sequences and sets

Definition

1. $(a_i : i \in \omega)$ is an *indiscernible sequence* over a set of parameters B if $\mathrm{tp}\,(a_{i_0} \ldots a_{i_n}/B) = \mathrm{tp}\,(a_{j_0} \ldots a_{j_n}/B)$ for any $i_0 < \ldots < i_n$ and $j_0 < \ldots < j_n$ from ω.
2. $(a_i : i \in \omega)$ is an *indiscernible set* over B if $\mathrm{tp}\,(a_{i_0} \ldots a_{i_n}/B) = \mathrm{tp}\left(a_{\sigma(i_0)} \ldots a_{\sigma(i_n)}/B\right)$ for any $\sigma \in S_\infty$.

Fact

The following are equivalent:

1. *T is stable.*
2. *Every indiscernible sequence is an indiscernible set.*

Stability: limit types

Fact

If T is stable and $(a_i : i \in \omega)$ is an indiscernible sequence, then for any formula $\phi(x) \in L(\mathbb{M})$, the set $\{i : \models \phi(a_i)\}$ is either finite or cofinite.

Definition

For an indiscernible sequence $\bar{a} = (a_i : i \in \omega)$ and a set of parameters B, we let $\lim(\bar{a}/B)$, the *limit type* of $\bar{a}$ over B, be the set $\{\phi(x) \in L(B) : \models \phi(a_i) \text{ for all but finitely many } i \in \omega\}$.

In view of the fact, this is a consistent complete type.

Stability: the independence relation

Fact

The following are equivalent:

1. *T is stable.*
2. *There is an independence relation $\mathop{\smile\!\!\!\!\!\!\vert}$ on small subsets of $\mathbb{M}$ (i.e. of cardinality $< \kappa^*$) satisfying certain natural axioms:* $\mathrm{Aut}(\mathbb{M})$*-invariance, finite character, symmetry, monotonicity, base monotonicity, transitivity, extension, local character, boundedness.*

- In fact, if such a relation exists, then it is unique and corresponds to Shelah's *non-forking* — a canonically defined way of producing "generic" extensions of types.
- Examples: linear independence in vector spaces, algebraic independence in algebraically closed fields.

Stability: Morley sequences

Definition

A sequence $(a_i)_{i \in \omega}$ in $\mathbb{M}$ is a *Morley sequence* in a type $p \in S(B)$ if it is a sequence of realizations of p indiscernible over B and such that moreover $a_i \mathop{\smile\!\!\!\!\!\!\mid}_B a_{<i}$ for all $i \in \omega$.

Fact

In a stable theory, every type admits a Morley sequence (Erdős-Rado + compactness + properties of forking independence).

- An important technical tool in the development of stability.
- Example: an infinite basis in a vector space is a Morley sequence over $\emptyset$.

Stability: Canonical basis

A type $p \in S(A)$ is *stationary* if it admits a unique global non-forking extension.

Definition

In a stable theory, every stationary type has a *canonical base* — a small set such that every automorphism of $\mathbb{M}$ fixing it fixes the global non-forking extension of p.

- In fact, such a set is unique up to bi-definability, so we can talk about the canonical base of a type, $\mathrm{Cb}(p)$.
- If we want every type to have a canonical base, we might have to add imaginary elements for classes of definable equivalence relations to the structure, i.e. working in $\mathbb{M}^{\mathrm{eq}}$, but this is a tame procedure.

- The *definable closure* of a set $A \subseteq \mathbb{M}$: $\operatorname{dcl}(A) = \{b \in \mathbb{M} : \exists \phi(x) \in L(A) \text{ s.t. } \models \phi(b) \wedge |\phi(x)| = 1\}$.
- The *algebraic closure* of a set $A \subseteq \mathbb{M}$: $\operatorname{acl}(A) = \{b \in \mathbb{M} : \exists \phi(x) \in L(A) \text{ s.t. } \models \phi(b) \wedge |\phi(x)| < \infty\}$.

Fact

Every indiscernible sequence $(a_i)_{i\in\omega}$ is a Morley sequence over the canonical base of its limit type, and this canonical base is equal to $\bigcap_{n\in\omega} \operatorname{dcl}^{\mathrm{eq}}(a_{\geq n})$.

Exchangeable sequences of random variables

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.
- Let $\bar{X} = (X_i)_{i \in \omega}$ be a sequence of $[0,1]$-valued random variables on Ω (i.e. $X_i : \Omega \to [0,1]$ is a measurable function).
- The sequence $\bar{X}$ is *exchangeable* if $(X_{i_0}, \ldots, X_{i_n}) \stackrel{d}{=} (X_0, \ldots, X_n)$ for any $i_0 \neq \ldots \neq i_n$ and $n \in \omega$.
- Example: A sequence of i.i.d. (independent, identically distributed) random variables.
- Is the converse true? Yes, *up to a "mixing"*.

Classical de Finetti's theorem

Definition

If A is a collection of random variables, let $\sigma(A) \subseteq \mathcal{F}$ denote the minimal σ-subalgebra with respect to which every $X \in A$ is measurable.

Fact

[de Finetti] A sequence of random variables $(X_i)_{i \in \omega}$ is exchangeable if and only if it is i.i.d. over its tail σ-algebra $T = \bigcap_{n \in \omega} \sigma(X_{\geq n})$.

- It is a special case of the model-theoretic result above, but in the sense of *continuous logic*.

Continuous logic

- Reference: Ben Yaacov, Berenstein, Henson, Usvyatsov "Model theory for metric structures".
- Every structure M is a complete metric space of bounded diameter, with metric d.
- Signature:
  - function symbols with given moduli of uniform continuity (correspond to uniformly continuous functions from M^n to M),
  - predicate symbols with given moduli of uniform continuity (uniformly continuous functions from M to $[0, 1]$).
- Connectives: the set of all continuous functions from $[0, 1] \to [0, 1]$, or any subfamily which generates a dense subset (e.g. $\left\{\neg, \frac{x}{2}, \dot{-}\right\}$).
- Quantifiers: sup for $\forall$, inf for $\exists$.
- This logic admits a compactness theorem, etc.

Stability in continuous logic

- Summary: everything is essentially the same as in the classical case (Ben Yaacov, Usvyatsov "Continuous first-order logic and local stability").
- Of course, modulo some natural changes: cardinality is replaced by the density character, in acl "finite" is replaced by "compact", some equivalences are replaced by the ability to approximate uniformly, etc.
- Examples of stable continuous theories: (unit balls in) infinite-dimensional Hilbert space, atomless probability algebras, *(atomless) random variables*, Keisler randomization of an arbitrary stable theory.

The theory of random variables

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $L^1((\Omega, \mathcal{F}; \mu), [0,1])$ be the space of $[0,1]$-valued random variables on it.
- We consider it as a continuous structure in the language $L_{\mathrm{RV}} = \left\{0, \neg, \frac{x}{2}, \dot{-}\right\}$ with the natural interpretation of the connectives (e.g. $\left(X \dot{-} Y\right)(\omega) = X(\omega) \dot{-} Y(\omega)$) and the distance $d(X, Y) = \mathbf{E}[|X - Y|] = \int_{\Omega} |X - Y| \, d\mu$.

The theory of random variables

- Consider the following continuous theory RV in the language L_{RV}, we write 1 as an abbreviation for $\neg 0$, $E(x)$ for $d(0,x)$ and $x \wedge y$ for $x \dot{-} \left(x \dot{-} y\right)$:
  - $E(x) = E\left(x \dot{-} y\right) + E(y \wedge x)$
  - $E(1) = 1$
  - $d(x,y) = E\left(x \dot{-} y\right) + E\left(y \dot{-} x\right)$
  - $\tau = 0$ for every term τ which can be deduced in the propositional continuous logic.
- The theory ARV is defined by adding:
  - Atomlessness: $\inf_y \left(E(y \wedge \neg y) \vee \left|E(y \wedge x) - \frac{E(x)}{2}\right|\right) = 0.$

The theory of random variables: basic properties

Fact

[Ben Yaacov, "On theories of random variables"]

1. $M \models \mathrm{RV} \Leftrightarrow$ *it is isomorphic to* $L^1(\Omega, [0,1])$ *for some probability space* $(\Omega, \mathcal{F}, \mu)$.
2. $M \models \mathrm{ARV} \Leftrightarrow$ *it is isomorphic* $L^1(\Omega, [0,1])$ *for some atomless probability space* $(\Omega, \mathcal{F}, \mu)$.
3. ARV *is the model completion of the universal theory* RV *(so every probability space embeds into a model of* ARV*).*
4. ARV *eliminates quantifiers, and two tuples have the same type over a set* $A \subseteq M$ *if and only if they have the same joint conditional distribution as random variables over* $\sigma(A)$.

The theory of random variables: stability

Fact

[Ben Yaacov, "On theories of random variables"]

1. ARV *is* $\aleph_0$*-categorical (i.e., there is a unique separable model) and complete.*
2. ARV *is stable (and in fact* $\aleph_0$*-stable).*
3. ARV *eliminates imaginaries.*
4. *If* $M \models \mathrm{ARV}$ *and* $A \subseteq M$*, then* $\mathrm{dcl}(A) = \mathrm{acl}(A) = L^1(\sigma(A), [0,1]) \subseteq M$.
5. *Model-theoretic independence coincides with probabilistic independence:* $A \mathop{\smile\!\!\!\!\!\!\mid}_B C \Leftrightarrow \mathbb{P}[X|\sigma(BC)] = \mathbb{P}[X|\sigma(B)]$ *for every* $X \in \sigma(A)$*. Moreover, every type is stationary.*

Back to de Finetti

- As every model of RV embeds into a model of ARV, wlog our sequence of random variables is from $\mathbb{M} \models$ ARV.
- Recall: In a stable theory, every indiscernible sequence is an indiscernible set.

Corollary

[Ryll-Nardzewski] A sequence of random variables is exchangeable iff it is contractable (i.e. $X_{i_0} \ldots X_{i_n} \stackrel{d}{=} X_0 \ldots X_n$ for all $i_0 < \ldots < i_n$).

- Recall: In a stable theory, every indiscernible sequence is a Morley sequence over the definable tail closure.

Corollary

De Finetti's theorem.

Multi-dimensional de Finetti

- A reformulation of de Finetti's theorem:

Fact

$(X_i)_{i \in \omega}$ *is exchangeable iff there is a measurable function* $f : [0,1]^2 \to \Omega$ *and some i.i.d.* $[0,1]$*-random variables* α *and* $(\xi_i)_{i \in \omega}$ *such that a.s.* $X_n = f(\alpha, \xi_i)$.

- f is not unique here, and we might have to extend the basic probability space.

Multi-dimensional de Finetti

- So, 1-dimensional case was already folklore in stability theory.
- There is a multi-dimensional theory of exchangeable arrays in probability.

Fact

[Aldous, Hoover] An array of random variables $X = (X_{i,j})$ is exchangeable iff there exist a measurable function $f : [0,1]^4 \to \Omega$ and some i.i.d. random variables $\alpha, \xi_i, \eta_j, \zeta_{i,j}$ such that a.s. $X_{i,j} = f(\alpha, \xi_i, \eta_j, \zeta_{i,j})$.

- [Kallenberg] for *n*-dimensional case.
- Can also be reformulated in terms of independence over certain "tail algebras". We give a model-theoretic generalization for arbitrary stable theories.

Indiscernible arrays

Definition

A (2-dimensional) array $(a_{i,j} : i, j \in \omega)$ is *indiscernible* if both the sequence of rows and the sequence of columns are indiscernible.

Appear in [Hrushovski, Zilber, "Zariski geometries"] for recovering groups and fields, and in the study of forking and dividing in simple and NTP_2 theories.

Model-theoretic multi-dimensional de Finetti

Theorem

Let T be stable, and let $(a_{i,j} : i, j \in \omega)$ be an indiscernible array. Let:

- $r_i = \bigcap_{n\in\omega} \mathrm{dcl}^{\mathrm{eq}}(a_{i,>n})$ *and* $c_j = \bigcap_{n\in\omega} \mathrm{dcl}^{\mathrm{eq}}(a_{>n,j})$ *be the tail closures of the i's row and the j's column, respectively.*
- *Let also* $r'_i = \bigcap_{n\in\omega} \mathrm{dcl}^{\mathrm{eq}}(a_{i,>n} a_{>n,>n})$ *and* $c'_j = \bigcap_{n\in\omega} \mathrm{dcl}^{\mathrm{eq}}(a_{>n,j} a_{>n,>n})$*, i.e. we add the limit corner closure as well.*

***Then**, for any $i, j \in \omega$ we have $a_{i,j} \mathop{\smile\!\!\!\!\!\!\mid}_{r_i c'_j} a_{\neq(i,j)}$, as well as $a_{i,j} \mathop{\smile\!\!\!\!\!\!\mid}_{r'_i c_j} a_{\neq(i,j)}$.*

- Also an appropriate generalization to n-dimensional array.

Directions

- Some questions remain:
  - whether $\mathrm{Cb}\left(a_{i,j}/a_{\neq(i,j)}\right) \in \mathrm{dcl}^{\mathrm{eq}}\left(r_i' c_j'\right)$ (as opposed to $\mathrm{acl}^{\mathrm{eq}}$, true in probability algebras, unlikely in general),
  - whether it is enough to take $c_i r_j d$ in the base, where d is the diagonal corner closure $\bigcap_{n\in\omega} \mathrm{dcl}^{\mathrm{eq}}\left(a_{>n,>n}\right)$.
  - some connections to lovely pairs of lovely pairs.
- Non-commutative probability theory: no longer stable, no model complete theory and no quantifier elimination, but there is an appropriate notion of independence on quantifier-free types.