Model-theoretic approach to multi-dimensional de Finetti theory

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Joint work with Itaï Ben Yaacov.
We fix a complete countable first-order theory $T$ in a language $L$.

Let $\mathbb{M}$ be a monster model of $T$ (i.e. $\kappa^*$-saturated and $\kappa^*$-homogeneous for some sufficiently large cardinal $\kappa^*$).

Given a set $A \subseteq \mathbb{M}$, we let $S(A)$ denote the space of types over $A$ (i.e. the Stone space of ultrafilters on the Boolean algebra of $A$-definable subsets of $\mathbb{M}$).
Definition

1. We say that $T$ encodes a linear order if there is a formula $\phi(\bar{x}, \bar{y}) \in L$ and $(\bar{a}_i : i \in \omega)$ in $M$ such that $M \models \phi(\bar{a}_i, \bar{a}_j) \iff i < j$.

2. A theory $T$ is stable if it cannot encode a linear order.

3. Equivalently, for some cardinal $\kappa$ we have

$$\sup \{|S(M)| : M \models T, |M| = \kappa\} = \kappa.$$  

Examples of stable first-order theories: equivalence relations, modules, algebraically closed fields, separably closed fields, free groups, planar graphs.
Stability: indiscernible sequences and sets

Definition

1. \((a_i : i \in \omega)\) is an indiscernible sequence over a set of parameters \(B\) if \(\text{tp}(a_{i_0} \ldots a_{i_n}/B) = \text{tp}(a_{j_0} \ldots a_{j_n}/B)\) for any \(i_0 < \ldots < i_n\) and \(j_0 < \ldots < j_n\) from \(\omega\).
2. \((a_i : i \in \omega)\) is an indiscernible set over \(B\) if \(\text{tp}(a_{i_0} \ldots a_{i_n}/B) = \text{tp}(a_{\sigma(i_0)} \ldots a_{\sigma(i_n)}/B)\) for any \(\sigma \in S_\infty\).

Fact

The following are equivalent:

1. \(T\) is stable.
2. Every indiscernible sequence is an indiscernible set.
Stability: limit types

Fact
If $T$ is stable and $(a_i : i \in \omega)$ is an indiscernible sequence, then for any formula $\phi(x) \in L(M)$, the set $\{ i : \models \phi(a_i) \}$ is either finite or cofinite.

Definition
For an indiscernible sequence $\bar{a} = (a_i : i \in \omega)$ and a set of parameters $B$, we let $\lim (\bar{a}/B)$, the limit type of $\bar{a}$ over $B$, be the set $\{ \phi(x) \in L(B) : \models \phi(a_i) \text{ for all but finitely many } i \in \omega \}$. In view of the fact, this is a consistent complete type.
Stability: the independence relation

Fact

The following are equivalent:

1. $T$ is stable.

2. There is an independence relation $\perp$ on small subsets of $\mathbb{M}$ (i.e. of cardinality $< \kappa^*$) satisfying certain natural axioms: Aut$(\mathbb{M})$-invariance, finite character, symmetry, monotonicity, base monotonicity, transitivity, extension, local character, boundedness.

- In fact, if such a relation exists, then it is unique and corresponds to Shelah’s non-forking — a canonically defined way of producing “generic” extensions of types.

- Examples: linear independence in vector spaces, algebraic independence in algebraically closed fields.
Stability: Morley sequences

Definition
A sequence \((a_i)_{i \in \omega}\) in \(M\) is a Morley sequence in a type \(p \in S(B)\) if it is a sequence of realizations of \(p\) indiscernible over \(B\) and such that moreover \(a_i \downarrow_B a_{<i}\) for all \(i \in \omega\).

Fact
In a stable theory, every type admits a Morley sequence (Erdős-Rado + compactness + properties of forking independence).

- An important technical tool in the development of stability.
- Example: an infinite basis in a vector space is a Morley sequence over \(\emptyset\).
A type $p \in S(A)$ is stationary if it admits a unique global non-forking extension.

**Definition**
In a stable theory, every stationary type has a canonical base — a small set such that every automorphism of $M$ fixing it fixes the global non-forking extension of $p$.

- In fact, such a set is unique up to bi-definability, so we can talk about the canonical base of a type, $Cb(p)$.
- If we want every type to have a canonical base, we might have to add imaginary elements for classes of definable equivalence relations to the structure, i.e. working in $M^{eq}$, but this is a tame procedure.
The **definable closure** of a set $A \subseteq M$: $\text{dcl}(A) = \{b \in M : \exists \phi(x) \in L(A) \text{ s.t. } \models \phi(b) \land |\phi(x)| = 1\}$.

The **algebraic closure** of a set $A \subseteq M$: $\text{acl}(A) = \{b \in M : \exists \phi(x) \in L(A) \text{ s.t. } \models \phi(b) \land |\phi(x)| < \infty\}$.

**Fact**

Every indiscernible sequence $(a_i)_{i \in \omega}$ is a Morley sequence over the canonical base of its limit type, and this canonical base is equal to $\bigcap_{n \in \omega} \text{dcl}^{eq}(a_{\geq n})$. 
Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

Let $\bar{X} = (X_i)_{i \in \omega}$ be a sequence of $[0, 1]$-valued random variables on $\Omega$ (i.e. $X_i : \Omega \to [0, 1]$ is a measurable function).

The sequence $\bar{X}$ is *exchangeable* if
\[(X_{i_0}, \ldots, X_{i_n}) \overset{d}{=} (X_0, \ldots, X_n) \text{ for any } i_0 \neq \ldots \neq i_n \text{ and } n \in \omega.\]

Example: A sequence of i.i.d. (independent, identically distributed) random variables.

Is the converse true? Yes, *up to a “mixing”*. 
Classical de Finetti’s theorem

Definition
If $A$ is a collection of random variables, let $\sigma(A) \subseteq \mathcal{F}$ denote the minimal $\sigma$-subalgebra with respect to which every $X \in A$ is measurable.

Fact
[de Finetti] A sequence of random variables $(X_i)_{i \in \omega}$ is exchangeable if and only if it is i.i.d. over its tail $\sigma$-algebra $T = \bigcap_{n \in \omega} \sigma(X_{\geq n})$.

- It is a special case of the model-theoretic result above, but in the sense of continuous logic.
Continuous logic

- Reference: Ben Yaacov, Berenstein, Henson, Usvyatsov “Model theory for metric structures”.

- Every structure $M$ is a complete metric space of bounded diameter, with metric $d$.

- Signature:
  - function symbols with given moduli of uniform continuity (correspond to uniformly continuous functions from $M^n$ to $M$),
  - predicate symbols with given moduli of uniform continuity (uniformly continuous functions from $M$ to $[0, 1]$).

- Connectives: the set of all continuous functions from $[0, 1] \rightarrow [0, 1]$, or any subfamily which generates a dense subset (e.g. $\{\neg, \frac{x}{2}, \cdot\}$).

- Quantifiers: sup for $\forall$, inf for $\exists$.

- This logic admits a compactness theorem, etc.
Stability in continuous logic

- Summary: everything is essentially the same as in the classical case (Ben Yaacov, Usvyatsov “Continuous first-order logic and local stability”).

- Of course, modulo some natural changes: cardinality is replaced by the density character, in acl “finite” is replaced by “compact”, some equivalences are replaced by the ability to approximate uniformly, etc.

- Examples of stable continuous theories: (unit balls in) infinite-dimensional Hilbert space, atomless probability algebras, (atomless) random variables, Keisler randomization of an arbitrary stable theory.
The theory of random variables

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $L^1((\Omega, \mathcal{F}; \mu), [0, 1])$ be the space of $[0, 1]$-valued random variables on it.

We consider it as a continuous structure in the language $L_{RV} = \{0, \neg, \frac{x}{2}, \cdot\}$ with the natural interpretation of the connectives (e.g. $(X \cdot Y)(\omega) = X(\omega) \cdot Y(\omega)$) and the distance $d(X, Y) = E[|X - Y|] = \int_{\Omega} |X - Y| \, d\mu$. 
The theory of random variables

Consider the following continuous theory RV in the language $L_{RV}$, we write 1 as an abbreviation for $\neg 0$, $E(x)$ for $d(0, x)$ and $x \land y$ for $x - \left(x - y\right)$:

- $E(x) = E\left(x - y\right) + E(y \land x)$
- $E(1) = 1$
- $d(x, y) = E\left(x - y\right) + E\left(y - x\right)$
- $\tau = 0$ for every term $\tau$ which can be deduced in the propositional continuous logic.

The theory ARV is defined by adding:

- Atomlessness: $\inf_y \left( E\left(y \land \neg y\right) \lor \left| E\left(y \land x\right) - \frac{E(x)}{2}\right| \right) = 0$. 
The theory of random variables: basic properties

Fact
[Ben Yaacov, “On theories of random variables”]

1. $M \models \text{RV} \iff$ it is isomorphic to $L^1(\Omega, [0, 1])$ for some probability space $(\Omega, \mathcal{F}, \mu)$.

2. $M \models \text{ARV} \iff$ it is isomorphic $L^1(\Omega, [0, 1])$ for some atomless probability space $(\Omega, \mathcal{F}, \mu)$.

3. ARV is the model completion of the universal theory RV (so every probability space embeds into a model of ARV).

4. ARV eliminates quantifiers, and two tuples have the same type over a set $A \subseteq M$ if and only if they have the same joint conditional distribution as random variables over $\sigma(A)$. 
Fact

[Ben Yaacov, “On theories of random variables”]

1. ARV is \( \aleph_0 \)-categorical (i.e., there is a unique separable model) and complete.

2. ARV is stable (and in fact \( \aleph_0 \)-stable).

3. ARV eliminates imaginaries.

4. If \( M \models ARV \) and \( A \subseteq M \), then
   \[ \text{dcl}(A) = \text{acl}(A) = L^1(\sigma(A), [0, 1]) \subseteq M. \]

5. Model-theoretic independence coincides with probabilistic independence:
   \( A \indep_B C \iff \mathbb{P}[X|\sigma(BC)] = \mathbb{P}[X|\sigma(B)] \)
   for every \( X \in \sigma(A) \). Moreover, every type is stationary.
Back to de Finetti

- As every model of RV embeds into a model of ARV, wlog our sequence of random variables is from $\mathbb{M} \models \text{ARV}$.
- Recall: In a stable theory, every indiscernible sequence is an indiscernible set.

**Corollary**

[Ryll-Nardzewski] A sequence of random variables is exchangeable iff it is contractable (i.e. $X_{i_0} \ldots X_{i_n} \overset{d}{=} X_0 \ldots X_n$ for all $i_0 < \ldots < i_n$).

- Recall: In a stable theory, every indiscernible sequence is a Morley sequence over the definable tail closure.

**Corollary**

*De Finetti’s theorem.*
Multi-dimensional de Finetti

A reformulation of de Finetti’s theorem:

Fact

$(X_i)_{i \in \omega}$ is exchangeable iff there is a measurable function $f : [0, 1]^2 \rightarrow \Omega$ and some i.i.d. $[0, 1]$-random variables $\alpha$ and $(\xi_i)_{i \in \omega}$ such that a.s. $X_n = f(\alpha, \xi_i)$.

$f$ is not unique here, and we might have to extend the basic probability space.
Multi-dimensional de Finetti

- So, 1-dimensional case was already folklore in stability theory.
- There is a multi-dimensional theory of exchangeable arrays in probability.

**Fact**

[Aldous, Hoover] An array of random variables $X = (X_{i,j})$ is exchangeable iff there exist a measurable function $f : [0, 1]^4 \to \Omega$ and some i.i.d. random variables $\alpha, \xi_i, \eta_j, \zeta_{i,j}$ such that a.s. $X_{i,j} = f(\alpha, \xi_i, \eta_j, \zeta_{i,j})$.

- [Kallenberg] for $n$-dimensional case.
- Can also be reformulated in terms of independence over certain “tail algebras”. We give a model-theoretic generalization for arbitrary stable theories.
Definition
A (2-dimensional) array \((a_{i,j} : i, j \in \omega)\) is indiscernible if both the sequence of rows and the sequence of columns are indiscernible. Appeared in [Hrushovski, Zilber, “Zariski geometries”] for recovering groups and fields, and in the study of forking and dividing in simple and \(NTP_2\) theories.
Theorem
Let $T$ be stable, and let $(a_{i,j} : i,j \in \omega)$ be an indiscernible array. Let:

- $r_i = \bigcap_{n \in \omega} \text{dcl}^{\text{eq}} (a_{i,>n})$ and $c_j = \bigcap_{n \in \omega} \text{dcl}^{\text{eq}} (a_{>n,j})$ be the tail closures of the $i$'s row and the $j$'s column, respectively.
- Let also $r'_i = \bigcap_{n \in \omega} \text{dcl}^{\text{eq}} (a_{i,>n} a_{>n,>n})$ and $c'_j = \bigcap_{n \in \omega} \text{dcl}^{\text{eq}} (a_{>n,j} a_{>n,>n})$, i.e. we add the limit corner closure as well.

Then, for any $i,j \in \omega$ we have $a_{i,j} \downarrow_{r_i c'_j} a \neq (i,j)$, as well as $a_{i,j} \downarrow_{r'_i c_j} a \neq (i,j)$.

- Also an appropriate generalization to $n$-dimensional array.
Some questions remain:

- whether \( Cb (a_{i,j} / a \neq (i,j)) \in dcl^{eq} (r'_i c'_j) \) (as opposed to \( acl^{eq} \), true in probability algebras, unlikely in general),
- whether it is enough to take \( c_i r_j d \) in the base, where \( d \) is the diagonal corner closure \( \bigcap_{n \in \omega} dcl^{eq} (a_{>n,>n}) \),
- some connections to lovely pairs of lovely pairs.

Non-commutative probability theory: no longer stable, no model complete theory and no quantifier elimination, but there is an appropriate notion of independence on quantifier-free types.