Fields and model-theoretic classification, 3

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Simple theories

Definition
[Shelah] A formula $\varphi(x; y)$ has the tree property (TP) if there is $k < \omega$ and a tree of tuples $(a_\eta)_{\eta \in \omega^k}$ in $\mathbb{M}$ such that:

- for all $\eta \in \omega^\omega$, $\{\varphi(x; a_\eta|\alpha) : \alpha < \omega\}$ is consistent,
- for all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_\eta \langle i \rangle) : i < \omega\}$ is $k$-inconsistent.

- $T$ is simple if no formula has TP.
- $T$ is supersimple if there is no such tree even if we allow to use a different formula $\phi_\alpha(x, y_\alpha)$ on each level $\alpha < \omega$.
- Simplicity of $T$ admits an alternative characterization via existence of a canonical independence relation on subsets of a saturated model of $T$ with properties generalizing those of algebraic independence (given by Shelah’s forking).
- All stable theories are simple.
Pseudofinite fields

Definition
An infinite field $K$ is pseudofinite if for every first-order sentence $\sigma \in \mathcal{L}_{\text{ring}}$ there is some finite field $K_0 \models \sigma$.

▶ Equivalently, $K$ is elementarily equivalent to a (non-principal) ultraprodut of finite fields.

▶ Ax developed model theory of pseudofinite fields, in particular giving the following algebraic characterization:

Fact
[Ax, 68] A field $K$ is pseudofinite if and only if:

1. $K$ is perfect,
2. $K$ has a unique extension of every finite degree,
3. $K$ is PAC.

These properties are first-order axiomatizable, and completions of the theory are described by fixing the isomorphism type of the algebraic closure of the prime field.
PAC fields

- A field $F$ is *pseudo-algebraically closed* (or *PAC*) if every absolutely irreducible variety defined over $F$ has an $F$-rational point.

- A field $F$ is *bounded* if for each $n \in \mathbb{N}$, there are only finitely many extensions of degree $n$.

- [Parigot] If $F$ is PAC and not separable, then $F$ is not NIP.

- [Beyarslan] In fact, every pseudofinite field interprets the random $n$-hypergraph, for all $n \in \mathbb{N}$ ($n = 2$ — Paley graphs).

- [Hrushovski], [Kim,Pillay] Every perfect bounded PAC field is supersimple.

- [Chatzidakis] A PAC field has a simple theory if and only if it is bounded.
Converse

- [Pillay, Poizat] Supersimple $\implies$ perfect and bounded.

**Question** [Pillay]. Is every supersimple field PAC?
- $F$ is PAC $\iff$ the set of the $F$-rational points of every absolutely irreducible variety over $F$ is Zariski-dense.
- [Geyer] Enough to show for curves over $F$ (i.e. one-dimensional absolutely irreducible varieties over $F$).
- [Pillay, Scanlon, Wagner] True for curves of genus 0.
- [Pillay, Martin-Pizarro] True for (hyper-)elliptic curves with generic moduli.
- [Martin-Pizarro, Wagner] True for all elliptic curves over $F$ with a unique extension of degree 2.
- [Kaplan, Scanlon, Wagner] An infinite field $K$ with $\text{Th}(K)$ simple has only finitely many Artin-Schreier extension (see below).
More PAC fields

- No apparent conjecture for general simple fields.
- In general, PAC fields can have wild behavior. However, there are some unbounded well-behaved PAC fields.

Definition
A field $F$ is called $\omega$-free if it has a countable elementary substructure $F_0$ with $\mathcal{G}(F_0) \cong \hat{F}_\omega$, the free profinite group on countably many generators.

- [Chatzidakis] Not simple. However, admits a notion of independence satisfying an amalgamation theorem.
- By [C., Ramsey], this implies that if $F$ is an $\omega$-free PAC field, then $\text{Th}(F)$ is $\text{NSOP}_1$. 
inp-patterns and NTP$_2$

- $T$ a complete theory, $\mathbb{M}$ a saturated model for $T$.

**Definition**

An inp-pattern of depth $\kappa$ consists of $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha), k_\alpha)_{\alpha \in \kappa}$ with $\bar{a}_\alpha = (a_\alpha, i)_{i \in \omega}$ from $\mathbb{M}$ and $k_\alpha \in \omega$ such that:

- $\{\varphi_\alpha(x, a_\alpha, i)\}_{i \in \omega}$ is $k_\alpha$-inconsistent for every $\alpha \in \kappa$,
- $\{\varphi_\alpha(x, a_\alpha, f(\alpha))\}_{\alpha \in \kappa}$ is consistent for every $f : \kappa \to \omega$.

- The burden of $T$ is the supremum of the depths of inp-patterns with $x$ a singleton, either a cardinal or $\infty$.
- $T$ is NTP$_2$ if burden of $T$ is $< \infty$. Equivalently, if there is no inp-pattern of infinite depth with the same formula and $k$ on each row.
- $T$ is strong if there is no infinite inp-pattern.
- $T$ is inp-minimal if there is no inp-pattern of depth 2, with $|x| = 1$.
- Retroactively, $T$ is dp-minimal if it is NIP and inp-minimal.
$\textbf{inp-patterns and NTP}_2$

- $T$ is simple or NIP $\iff$ $T$ is NTP$_2$ (exercise).
- [C., Kaplan], [Ben Yaacov, C.], etc. There is a theory of forking in NTP$_2$ theories (generalizing the simple case).
- There are many new algebraic examples in this class!
Examples of NTP$_2$ fields: ultraproducts of $p$-adics

- We saw that for every prime $p$, the field $\mathbb{Q}_p$ is NIP.
- However, consider the field $\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathcal{U}$ (where $\mathcal{U}$ is a non-principal ultrafilter on the set of prime numbers) — a central object in the applications of model theory, after [Ax-Kochen], [Denef-Loeser], ....
- The theory of $\mathcal{K}$ is not simple: because the value group is linearly ordered.
- The theory of $\mathcal{K}$ is not NIP: the residue field is pseudofinite.
- Both already in the pure ring language, as the valuation ring is definable uniformly in $p$ [e.g. Ax].
Ax-Kochen principle for NTP$_2$

- Delon’s transfer theorem for NIP has an analog for NTP$_2$ as well.

Theorem

[C.] Let $\mathcal{K} = (K, \Gamma, k, v, ac)$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that $k$ is NTP$_2$. Then $\mathcal{K}$ is NTP$_2$.

- Being strong is preserved as well.

Corollary

$\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p/U$ is NTP$_2$ because the residue field is pseudofinite, hence simple, hence NTP$_2$.

- More recently, [C., Simon]. $\mathcal{K}$ is inp-minimal in $\mathcal{L}_\text{ring}$ (but not in the language with ac, of course).
Valued difference fields, 1

- \((K, \Gamma, k, \nu, \sigma)\) is a valued difference field if \((K, \Gamma, k, \nu, \text{ac})\) is a valued field and \(\sigma\) is a field automorphism preserving the valuation ring.
- Note: \(\sigma\) induces natural automorphisms on \(k\) and on \(\Gamma\).
- Because of the order on the value group, by [Kikyo, Shelah] there is no model companion of the theory of valued difference fields.
- The automorphism \(\sigma\) is contractive if for all \(x \in K\) with \(\nu(x) > 0\) we have \(\nu(\sigma(x)) > n\nu(x)\) for all \(n \in \omega\).
- Example: Let \((K_p, \Gamma, k, \nu, \sigma)\) be an algebraically closed valued field of char \(p\) with \(\sigma\) interpreted as the Frobenius automorphism. Then \(\prod_p \text{prime } K_p/U\) is a contractive valued difference field.
[Hrushovski], [Durhan] Ax-Kochen-Ershov principle for $\sigma$-henselian contractive valued difference fields $(K, \Gamma, k, \nu, \sigma, \text{ac})$:

- Elimination of the field quantifier.
- $(K, \Gamma, k, \nu, \sigma) \equiv (K', \Gamma', k', \nu, \sigma)$ iff $(k, \sigma) \equiv (k', \sigma)$ and $(\Gamma, <, \sigma) \equiv (\Gamma', <, \sigma)$;

- There is a model companion VFA$_0$ and it is axiomatized by requiring that $(k, \sigma) \models \text{ACFA}_0$ and that $(\Gamma, +, <, \sigma)$ is a divisible ordered abelian group with an $\omega$-increasing automorphism.

- Nonstandard Frobenius is a model of VFA$_0$.

- The reduct to the field language is a model of ACFA$_0$, hence simple but not NIP. On the other hand this theory is not simple as the valuation group is definable.
Valued difference fields and NTP$_2$

Theorem
[C., Hils] Let $\bar{K} = (K, \Gamma, k, v, ac, \sigma)$ be a $\sigma$-Henselian contractive valued difference field of equicharacteristic 0. Assume that both $(K, \sigma)$ and $(\Gamma, \sigma)$, with the induced automorphisms, are NTP$_2$. Then $\bar{K}$ is NTP$_2$.

Corollary
VFA$_0$ is NTP$_2$ (as ACFA$_0$ is simple and $(\Gamma, +, <, \sigma)$ is NIP).

- The argument also covers the case of $\sigma$-henselian valued difference fields with a value-preserving automorphism of [Belair, Macintyre, Scanlon] and the multiplicative generalizations of Kushik.
- Open problem: is VFA$_0$ strong?
PRC fields, 1

- $F$ is PAC $\iff M$ is existentially closed (in the language of rings) in each regular field extension of $F$.

**Definition**

[Basarab, Prestel] A field $F$ is *Pseudo Real Closed* (or PRC) if $F$ is existentially closed (in the ring language) in each regular field extension $F'$ to which all orderings of $F$ extend.

- Equivalently, for every absolutely irreducible variety $V$ defined over $F$, if $V$ has a simple rational point in every real closure of $F$, then $V$ has an $F$-rational point.
- E.g. PAC (has no orderings) and real closed fields are PRC (no proper real closures).
- The class of PRC fields is elementary.
- Were studied by Prestel, Jarden, Basarab, McKenna, van den Dries and others.
If $K$ is a bounded field, then it has only finitely many orders (bounded by the number of extensions of degree 2).

[Chatzidakis] If a PAC field is not bounded, then it has TP$_2$. Easily generalizes to PRC.

**Conjecture** [C., Kaplan, Simon]. A PRC field is NTP$_2$ if and only if it is bounded (and the same for P$pC$ fields).

**Fact**

[Montenegro, 2015] A PRC field $K$ is bounded if and only if $\text{Th}(K)$ is NTP$_2$. Moreover, the burden of $K$ is equal to the number of the orderings.
\textbf{PpC fields}

- A valuation \((F, \nu)\) is \(p\)-adic if the residue field is \(\mathbb{F}_p\) and \(\nu(p)\) is the smallest positive element of the value group.

\textbf{Definition}

[Grob, Jarden and Haran] \(F\) is pseudo \(p\)-adically closed (PpC) if \(F\) is existentially closed (in \(\mathcal{L}_{\text{ring}}\)) in each regular extension \(F'\) such that all the \(p\)-adic valuations of \(M\) can be extended by \(p\)-adic valuations on \(F'\).

\textbf{Fact}

[Montenegro, 2015] All bounded PpC fields are NTP\(_2\).

- The converse is still open.
What do we know about general $\text{NTP}_2$ fields?

Generalizing the simple case, we have:

**Theorem**

[C., Kaplan, Simon] Let $K$ be an infinite $\text{NTP}_2$ field. Then it has only finitely many Artin-Schreier extensions.

**Corollary**

$\mathbb{F}_p((t))$ has $\text{TP}_2$. 
Ingredients of the proof

1. The proof generalizes the arguments in [Kaplan-Scanlon-Wagner] for the NIP case, using a new chain condition for NTP\(_2\) groups.

2. Let \(G\) be NTP\(_2\) and \(\{\varphi(x, a) : a \in C\}\) be a family of normal subgroups of \(G\). Then there is some \(k \in \omega\) (depending only on \(\varphi\)) such that for every finite \(C' \subseteq C\) there is some \(C_0 \subseteq C'\) with \(|C_0| \leq k\) and such that

\[
\left[ \bigcap_{a \in C_0} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a) \right] < \infty.
\]

3. Open problem: does it hold without the normality assumption?
Definable envelopes of groups in NTP$_2$

- A group $G$ is finite-by-abelian if there exists a finite normal subgroup $F$ of $G$ such that $G/F$ is abelian.
- If $H, K \leq G$, $H$ is *almost contained in* $K$ if $[H : H \cap K]$ is finite.
- Generalizing the results of Poizat, Shelah, de Aldama, Milliet from stable, simple and NIP cases:

**Fact**

[Hempel, Onshuus] Let $G$ be a group definable in an NTP$_2$ theory, $H$ a subgroup of $G$ (not necessarily definable!) and

- If $H$ is abelian (nilpotent of class $n$), then there exists a *definable* finite-by-abelian (resp. nilpotent of class $\leq 2n$) subgroup $H'$ of $G$ which contains (resp. almost contains) $H$. If $H$ was normal, can choose $H'$ normal as well.
- If $H$ is a normal solvable subgroup of class $n$, there exists a definable normal solvable subgroup $H'$ of $G$ of class at most $2n$ which almost contains $H$. 