NTP_2

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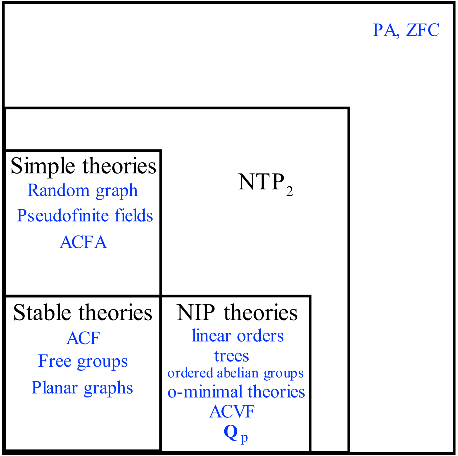
Generalizations of stability



NTP_2: Definition

Definition

(Shelah) A formula $\phi(x, y)$ has TP_2 if there are $(a_{i,j})_{i,j\in\omega}$ and $k \in \omega$ such that:

- $\{\phi(x, a_{i,j})\}_{j\in\omega}$ is k-inconsistent for every $i \in \omega$,
- $\left\{\phi(x, a_{i,f(i)})\right\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$.

T is called NTP_2 if no formula has TP_2.

NTP_2: Examples

- Every simple or NIP theory is NTP_2.
- Let T be a model complete geometric theory in the language L (i.e. it eliminates $\exists^{\infty}$ and the model theoretic algebraic closure satisfies exchange). Let L' be an expansion of L by a new unary predicate $P(x)$. Then T has a model companion T' in L' and this model companion is NTP_2 (generalizing Chatzidakis-Pillay).
- E.g. fusing a dense linear order with a random graph gives an NTP_2 theory.

TP_2 is witnessed in one variable

The following is quite useful for checking that a particular structure is NTP_2.

Theorem

(Ch.) T is NTP_2 if and only if every formula $\varphi(x, y)$ with x singleton is NTP_2.

(In fact, this follows from a more general result on sub-multiplicativity of burden in arbitrary theories and answers a question of Shelah).

NTP$_2$: Valued fields

- Consider the valued field $\mathbf{K} = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathfrak{U}$, where $\mathfrak{U}$ is a non-principal ultrafilter.
- The theory of $\mathbf{K}$ is not simple: because the value group is linearly ordered.
- The theory of $\mathbf{K}$ is not NIP: the residue field is pseudo-finite, thus has the independence property by a result of Duret.
- Even in the pure field language, as the valuation ring is definable uniformly in p (Ax).

NTP_2: Valued fields

However, $\mathbf{K}$ is NTP_2 (and even strong, of finite burden) by the following:

Theorem

(Ch.) Let $\mathbf{K} = (K, k, \Gamma)$ *be a henselian valued field of equicharacteristic* 0*, in the Denef-Pas language. Assume that* k *is* NTP_2*. Then* $\mathbf{K}$ *is* NTP_2*.*

Analogous to the theorem of Delon for NIP.

NTP$_2$: Valued difference fields

- We consider valued difference fields $\mathbf{K} = (K, k, \Gamma, \sigma)$ of equicharacteristic 0 (i.e. σ is an automorphism of K preserving the valuation ring).

NTP_2: Valued difference fields

- We consider valued difference fields $\mathbf{K} = (K, k, \Gamma, \sigma)$ of equicharacteristic 0 (i.e. σ is an automorphism of K preserving the valuation ring).
- Kikyo-Shelah: It T has the Strict Order Property (which is the case with valued fields), then the model companion of $T \cup \{\sigma \text{ is an automorphism}\}$ does not exist.
- Hrushovski/Azgin:
  - However, if we impose in addition that σ is contractive (i.e. $v(\sigma(x)) > n \cdot v(x)$ for all $n \in \omega$), then the model companion VFA_0 exists. It is axiomatized by saying that (k, σ) is a model of ACFA_0, (Γ, σ) is a divisible ordered $\mathbb{Z}[\sigma]$ module and $\mathbf{K}$ is σ-henselian.
  - A natural model of VFA_0: a non-standard Frobenius acting on an algebraically closed valued field of char 0.
- Again neither simple nor NIP.

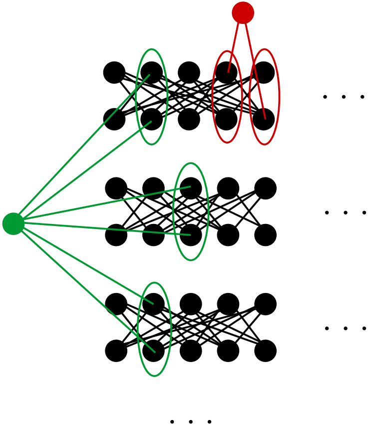
NTP$_2$: Valued difference fields

Theorem

(Ch., Hils) Let $\mathbf{K} = (K, k, \Gamma, \sigma)$ *be a* σ*-henselian contractive valued difference field of equicharacteristic* 0*. Assume that both* (k, σ) *and* (Γ, σ) *are* NTP_2*. Then* $\mathbf{K}$ *is* NTP_2*.*

A non-example

Let T be the theory of a triangle-free random graph. Let $\varphi(x, y_1 y_2) = xRy_1 \wedge xRy_2$. Then it has TP_2:



Forking

Let $\phi(x, y)$ be a formula and A a set.

- We say that $\phi(x, a)$ *divides* over A if there is $k \in \omega$ and $(a_i)_{i\in\omega}$ such that $\operatorname{tp}(a_i/A) = \operatorname{tp}(a/A)$ and $\{\phi(x, a_i)\}_{i\in\omega}$ is k-inconsistent.
- We say that $\phi(x, a)$ *forks* over A if there are $\phi_0(x, a_0), \ldots, \phi_n(x, a_n)$ such that $\phi(x, a) \vdash \bigvee_{i\leq n} \phi_i(x, a_i)$ and $\phi_i(x, a_i)$ divides over A for each $i \leq n$.
- We say that a (partial) type $p(x)$ does not divide (fork) over A if it does not imply any formula which divides (forks) over A.

Note that formulas forking over A form an ideal in $\operatorname{Def}(\mathbb{M})$ generated by the formulas dividing over A.

Example

If μ is an A-invariant finitely additive probability measure on $\operatorname{Def}(\mathbb{M})$ and $\mu(\phi(x, a)) > 0$ then $\phi(x, a)$ does not fork over A.

Forking in NTP_2 theories

Recall the picture for simple theories:

1. Nice combinatorial structure of the forking ideal: forking equals dividing, every Morley sequence witnesses dividing, chain condition, ...
2. Let $a \mathop{\smile\!\!\!\!\!\!\,|}_c b$ denote that $\mathrm{tp}(a/bc)$ does not fork over c. Then $\mathop{\smile\!\!\!\!\!\!\,|}$ is a nice independence relation: invariant under automorphisms of $\mathbb{M}$, symmetric, transitive, finite character, ...
3. Amalgamation of types (the "Independence theorem" of Kim and Pillay, over models): Assume that $a_1 \mathop{\smile\!\!\!\!\!\!\,|}_M b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_M b_2$ and $\mathrm{tp}(a_1/M) = \mathrm{tp}(a_2/M)$. Then there is $a \mathop{\smile\!\!\!\!\!\!\,|}_M b_1 b_2$ and s.t. $\mathrm{tp}(ab_i/M) = \mathrm{tp}(a_i b_i/M)$ for $i = 1, 2$.

The rest of the talk in one sentence: 1 (completely) and 3 (essentially) survive in NTP_2, as long as one is working over an *extension base*.

Extension bases

- A set A is called an extension base if every type in $p(x) \in S(A)$ has a global non-forking extension.
- Examples of extension bases:
  - any model in any theory,
  - every set in an o-minimal, c-minimal or ordered dp-minimal theory.
- A non-example: $\emptyset$ in the theory of dense circular order.

Forking = dividing

Question (Pillay). Is forking = dividing over models in NIP theories?

Theorem

(Ch., Kaplan) Let A be an extension base in an NTP_2 theory T. Then $\phi(x, a)$ divides over A if and only if it forks over A.

Forking = dividing: why?

- The reason: existence of strictly invariant types.
- A type $p(x) \in S(\mathbb{M})$ is called *strictly invariant* over A if it is invariant (i.e. $\phi(x,a) \in p$ and $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ implies $\phi(x,b) \in p$) and for every small $A \subseteq B \subseteq \mathbb{M}$, if $c \models p|_B$ then $\mathrm{tp}(B/cA)$ does not fork over A.
- E.g. every generically stable type or every invariant type in a simple theory are strictly invariant.
- The crucial step of the proof is to show that in NTP_2 theories every type $p(x)$ over a model M has a global strictly invariant extension $q(x)$ (using the so called Broom lemma).
- Then one can show that TFAE:
  - $\varphi(x,a)$ divides over M
  - For any $q(x) \in S(\mathbb{M})$, a strictly invariant extension of $\mathrm{tp}(a/M)$, and $(a_i)_{i\in\omega}$ a Morley sequence in q (i.e. $a_i \models q|_{a_{<i}M}$) we have that $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent.

Dividing = array-dividing

- We say that $(a_{ij})_{i,j\in\omega}$ is a (2-dimensional) indiscernible array over A if both the sequence of rows and the sequence of columns are indiscernible over A.
- $\varphi(x,a)$ *array-divides* over A if there is an indiscernible array over A such that $a = a_{00}$ and $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is inconsistent.
- **Theorem** (Ben Yaacov, Ch.) Let T be NTP$_2$. Then $\varphi(x,a)$ array-divides over A if and only if it divides over A.
- Generalizes to κ-dimensional arrays for any ordinal κ.

Chain condition

- We say that forking satisfies the *chain condition* over A if whenever $(a_i)_{i\in\omega}$ is an indiscernible sequence and $\varphi(x, a_0)$ does not fork over A then $\varphi(x, a_0) \wedge \varphi(x, a_1)$ does not fork over A.
- **Problem** (Adler/Hrushovski) What is the relationship between NTP_2 and the chain condition of non-forking?

Theorem

(Ben Yaacov, Ch.) Let T be NTP_2 and A an extension base. Then forking satisfies the chain condition over A.

Example

(Ch., Kaplan, Shelah) There is a theory with TP_2 in which forking satisfies the chain condition over arbitrary sets.

Weak independence theorem

- Recall the amalgamation of types in simple theories.
- Of course, fails in the presence of a linear order.
- However, we prove a weak independence theorem over an extension base:

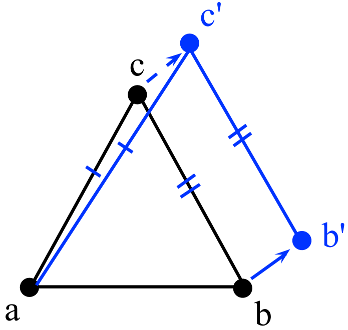
Theorem

(Ben Yaacov, Ch.) Let T be NTP_2 and A an extension base. Assume that $c \mathop{\smile\!\!\!\!\!\mid}_A ab$, $a \mathop{\smile\!\!\!\!\!\mid}_A bb'$ and $b \equiv_A^{\mathrm{Lstp}} b'$. Then there is c' such that $c' \mathop{\smile\!\!\!\!\!\mid}_A ab'$, $c'a \equiv_A ca$, $c'b' \equiv_A cb$.

Weak independence theorem

Theorem

(Ben Yaacov, Ch.) Let T be NTP_2 and A an extension base. Assume that $c \mathop{\smile\!\!\!\!\!\!\;|}_A ab$, $a \mathop{\smile\!\!\!\!\!\!\;|}_A bb'$ and $b \equiv_A^{\mathrm{Lstp}} b'$. Then there is c' such that $c' \mathop{\smile\!\!\!\!\!\!\;|}_A ab'$, $c'a \equiv_A ca$, $c'b' \equiv_A cb$.



Applications of the WIT

- Let T be NTP$_2$ and A an extension base. Then Lascar strong type over A equals Kim-Pillay strong type over A (we show that $a \equiv_A^{\mathrm{Lstp}} b$ implies $d_A(a, b) \leq 3$).
- The sufficient conditions of the *stabilizer theorem* of Hrushovski are satisfied in NTP$_2$ theories as the chain condition of non-forking means precisely that the forking ideal is *S1*.

NIP types in NTP_2 theories

- A (partial) type $p(x)$ is NIP if there are no $(a_i)_{i\in\omega}$ with $a_i \models p(x)$, $(b_s)_{s\subseteq\omega}$ and $\varphi(x,y)$ such that $\varphi(a_i,b_s) \Leftrightarrow i \in s$.
- Whole NIP theory can be done locally with respect to an NIP type (e.g. dp-rank of an NIP type in an arbitrary theory is always witness by mutually indiscernible sequences of its realizations, Kaplan-Simon, Ch.).

Theorem

(Ch., Kaplan) Let T be NTP_2. Then $p(x)$ is NIP if and only if every $q(x) \supseteq p(x)$ has only boundedly many global non-forking extensions (compare to stable types).

It is not true without the NTP_2 assumption, by the same example from [Ch., Kaplan and Shelah].

Simple types in NTP_2 theories

- A (partial) type $p(x)$ is simple if there are no $(a_\eta)_{\eta \in \omega^{<\omega}}$, $\varphi(x, y)$ and $k \in \omega$ such that:
  - $p(x) \cup \{\varphi(x, a_{\eta|i})\}_{i \in \omega}$ is consistent for every $\eta \in \omega^\omega$,
  - $\{\varphi(x, a_{\eta i})\}_{i \in \omega}$ is k-inconsistent for every $\eta \in \omega^{<\omega}$.

Theorem

(Ch.) Let T be NTP_2*. TFAE:*

1. *$p(x)$ is simple*
2. *Every $q(x) \supseteq p(x)$, satisfies the independence theorem over models.*
3. *For every $A \supseteq dom(p)$, $a \models p$ and b: $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\mid}_A a$.*

Dependent dividing

Definition

We will say that T has *dependent dividing* if whenever $p(x) \in S(N)$ divides over $M \preceq N$, there is some $\varphi(x, a) \in p$ dividing over M and such that $\varphi(x, y)$ is NIP.

- Of course, every NIP theory has dependent dividing.
- If T is a simple theory, then it has dependent dividing if and only if it has stable forking. I.e., all known simple theories have dependent dividing.

Theorem

(Ch.) Assume that T has dependent dividing. Then it is NTP_2.

The dependent dividing conjecture: Every NTP_2 theory has dependent dividing.

References

1. *Forking and dividing in NTP_2 theories*
   Artem Chernikov and Itay Kaplan, J. Symbolic Logic, 77(1):1–20, 2012
2. *Theories without the tree property of the second kind*
   Artem Chernikov, preprint (arXiv:1204.0832)
3. *On non-forking spectra*
   Artem Chernikov, Itay Kaplan and Saharon Shelah, submitted (arXiv:1205.3101)
4. *A weak independence theorem for NTP_2 theories*
   Itaï Ben Yaacov and Artem Chernikov, in preparation
5. *Valued difference fields and NTP_2*
   Artem Chernikov and Martin Hils, in preparation