Model-theoretic weight and algebraic examples

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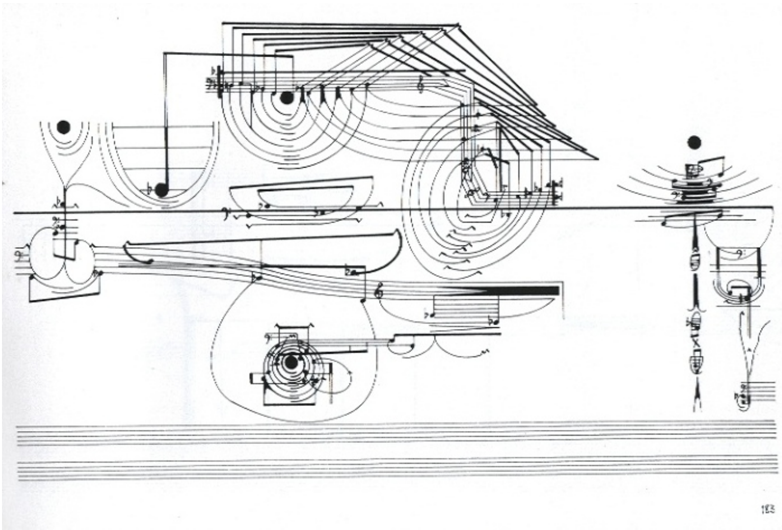
Weight: History

- Introduced by Shelah for the classification program in stable theories.
- Generalized to simple theories by Wagner, Pillay.
- Generalized to *NIP* by Shelah, Usvyatsov, Onshuus.
- Indiscernible arrays were considered by Kim, Ben Yaacov.
- Adler introduced a general definition.

Idiscernible sequences, due to Hodges

William Byrd, Non vos relinquam.

Idiscernible arrays



Cornelius Cardew, Treatise, pg. 183

Burden

Work in an arbitrary theory T. Let $p(x)$ be a partial type.

An *inp-pattern* in $p(x)$ of depth κ consists of $(\phi_\alpha(x, y_\alpha))_{\alpha<\kappa}$, $(a_{\alpha,i})_{\alpha<\kappa, i<\omega}$ and $k_\alpha < \omega$ such that:

1. $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent for each $\alpha < \kappa$.
2. $\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\}_{\alpha<\kappa} \cup p(x)$ is consistent for any $f : \kappa \to \omega$.

Adler: The *burden* of $p(x)$, denoted $bdn(p)$, is the supremum of the depths of all *inp*-patterns in $p(x)$. By $bdn(a/C)$ we mean $bdn(tp(a/C))$.

$\kappa_{inp}(T)$ and NTP_2

For a complete first-order theory T, we let $\kappa_{inp}(T)$ be the smallest infinite cardinal such that no finitary type has an *inp*-pattern of depth κ in it.
Define $\kappa^n_{inp}(T)$ similarly, but only looking at types in at most n variables.

T is called NTP_2 (*No Tree Property of the second kind*) if $\kappa_{inp}(T) < \infty$ (equivalently, $\kappa_{inp}(T) < |T|^+$).

Examples

1. Picture.
2. If T is simple then it is NTP_2.
3. If T is NIP then it is NTP_2.
4. Assume that T eliminates $\exists^\infty$. Chatzidakis and Pillay show that the expansion of T by a new unary predicate has a model companion T_P. If T is NTP_2, then T_P is NTP_2. For example, fusion of DLO with the random graph is NTP_2.

However, e.g. triangle-free random graph has TP_2.

One variable is enough

Shelah: Is $\kappa_{inp}(T) = \kappa^n_{inp}(T) = \kappa^1_{inp}(T)$?

Theorem: Burden is sub-multiplicative, that is if $bdn(a_i/C) < k_i$, finite, then $bdn(a_0 ... a_n/C) < k_0 \times ... \times k_n$.

Corollary: Yes. In particular, if T has TP_2, there is a formula $\phi(x, y)$ witnessing it, with $|x| = 1$.

Burden in special cases

1. Adler: In a simple theory, burden of a type is the supremum of the weights of its complete extensions.
2. In an NIP theory, burden corresponds to *dp*-rank. In particular, NIP theories with $\kappa^1_{inp}(T) = 1$ are precisely *dp*-minimal theories.

Hereditarily finite vs finite

Let's say that T has *hereditarily finite* burden if there is no *inp*-pattern of infinite depth.

Is it true that hereditarily finite burden implies finite burden? In NIP?

Positive answer for simple theories follows from Hyttinen / Wagner.

Issue: Unless T is simple, types of finite burden need not exist, as well as types of burden 1 need not exist in a theory of finite burden. Example: Model companion of infinitely many linear orders and model companion of two linear orders, respectively.

Dividing and forking

Recall:

1. $\phi(x, b)$ *divides* over C if there is a C-indiscernible sequence $(b_i)_{i<\omega}$ such that $b_0 = b$ and $\{\phi(x, b_i)\}_{i<\omega}$ is inconsistent.
2. $\phi(x, b)$ *forks* over C if $\phi(x, b) \vdash \bigvee_{i<n} \phi_i(x, b_i)$ and each of $\phi_i(x, b_i)$ divides over C.

Kim: Let T be simple. Then $\phi(x, b)$ divides over C if and only if it forks over C.

Not true in NIP: in circular order "$x = x$" forks over $\emptyset$.

Lets say that C is an *extension base* if every $p(x) \in S(C)$ does not fork over C. Pillay: does forking = dividing over extension bases in NIP?

Dividing and forking in NTP_2

Not every indiscernible sequence witnesses dividing.

Kim: In a simple theory, if $\phi(x, b)$ divides over C, then some/every Morley sequence in $tp(b/C)$ witnesses dividing.

No longer true in NTP_2 (and even NIP).

Theorem [Ch., Kaplan]. In NTP_2 theories, if $\phi(x, b)$ divides over $M \models T$, some/every *strict* Morley sequence in $tp(b/M)$ witnesses it.

In fact, this property is equivalent to T being NTP_2.

Dividing and forking in NTP_2

Corollary: In NTP_2 theories, forking $=$ dividing over any extension base C.

Remark: Any model in any theory is an extension base. If T is simple, o-minimal, C-minimal or ordered dp-minimal, then every set C is an extension base. So, in particular, this generalizes work of Kim on simple theories and of Dolich on o-minimal theories.

Non-forking spectrum of T

Let T be fixed. For $M \preceq N$, let
$S^{nf}(N, M) = \{p(x) \in S(N) : p \text{ does not fork over } M\}$.

For $\kappa \leq \lambda$, we let the *non-forking spectrum* of T be
$f_T(\kappa, \lambda) = \sup\{|S^{nf}(N, M)| : |M| = \kappa, |N| = \lambda\}$.
In particular, $f_T(\kappa, \kappa)$ is the usual stability function.

We say that T has *bounded non-forking* if $f_T(\kappa, \lambda) \leq g(\kappa)$ for some function $g : Card \to Card$.

Bounded non-forking and NIP

Fact: If T is NIP then it has bounded non-forking (bounded by 2^{κ}).

Adler: If non-forking is bounded, then it is bounded by $2^{2^{\kappa}}$. Is bounded non-forking equivalent to NIP?

Theorem [Ch., Kaplan]. T is NIP $\Leftrightarrow$ T is NTP_2 + non-forking is bounded. In fact, works locally with respect to a fixed type.

False in general, example of Itay.

Work in progress, joint with Kaplan and Shelah: classify all non-forking spectra.

Simple types

A (partial) type $p(x)$ is called *simple* if $D(p, \Delta, k) < \infty$ for every finite Δ and k. Equivalently, no $\phi(x, y)$ has tree property with x ranging over $p(\mathbb{M})$.

Obervation: If $p(x) \in S(C)$ is simple, then for any $a \models p(x)$ and b, if $a \mathop{\smile\!\!\!\!\!\!|}_C b$, then $b \mathop{\smile\!\!\!\!\!\!|}_C a$.

Issue: for a formula $\phi(x, y)$, having tree property is not preserved by flipping x and y. So, in general there is no reason for it to be true exchanging the roles of a and b.

Theorem (answering a question of Casanovas): Let $p(x) \in S(C)$ be a simple type in an NTP_2 theory, and C an extension base. Then for any $a \models p(x)$ and b, $a \mathop{\smile\!\!\!\!\!\!|}_C b \Leftrightarrow b \mathop{\smile\!\!\!\!\!\!|}_C a$.

Stronger than Lascar strong

We say that a and b have the same very strong type over C if they are in the transitive closure of being connected by a Morley sequence over C.

- Over a model, very strong type is determined by type.
- Kim: In simple theories, very strong type is determined by Lascar strong type.
- Hrushovski-Pillay: In NIP theories, if C is an extension base, then very strong type is determined by Lascar strong type.

Independence theorem for simple types

Theorem: Let $p(x) \in S(C)$ be a simple type in an NTP_2 theory, and C an extension base. Let $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_C b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_C b_2$, $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_C b_2$ and a_1, a_2 have the same very strong type over C. Then there is $a \mathop{\smile\!\!\!\!\!\!\mid}_C b_1 b_2$ such that $a \equiv_{Cb_1} a_1$ and $a \equiv_{Cb_2} a_2$.

Application: T is simple $\Leftrightarrow$ T is NTP_2 and satisfies the independence theorem over models.

(This also follows from a result of Kim, assuming existence of a measurable cardinal.)

Decomposition?

We have two extreme classes of types in NTP_2 theories:

- NIP types: set of non-forking extensions is bounded.
- Simple types: set of non-forking extensions satisfies amalgamation.
- And, of course, if a type is both *NIP* and simple, then it is stable.

Big questions: is it possible to analize arbitrary types in terms of something like these?

Examples: Burden in valued fields

Let F be a valued field in the Denef-Pas language, that is $F = (F, k, \Gamma, v, ac)$, where k is the residue field, Γ is the value group, $v : F \to \Gamma$ is the valuation map and $ac : F \to k$ is the angular component.
Assume that F eliminates the field quantifier.

- Delon: If k is NIP, then F is NIP.
- Shelah: If k and Γ are strongly dependent, then F is strongly dependent.

Theorem: There is a function f such that $\kappa_{inp}(F) \leq f(\kappa_{inp}(k), \kappa_{inp}(\Gamma))$. In particular, finiteness of burden and NTP_2 is preserved.

Examples: Ultraproduct of p-adics

Dolich, Goodrick, Lippel: $\mathbb{Q}_p$ in the pure field language has dp-rank 1.

Now let $F = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathfrak{U}$ for some non-principal ultrafilter $\mathfrak{U}$.

It has IP (as k is pseudo-finite) and strict order property, both in the pure field language (as valuation is uniformly definable).

However, by the theorem, burden of F is finite. What is it exactly?

Examples: Mekler's construction

Let T be a complete theory in a finite relational language.

Mekler: There is a complete theory T' in the pure group language (in fact, nilpotent of class 2 and exponent $p > 2$), interpreting T and preserving the number of types over models ($+|T|$).

Facts:

1. Mekler: If T is (super-)stable, then T' is (super-)stable.
2. If T is NIP, then T' is NIP.
3. Baudisch: If T is simple, then T' is simple.

Theorem: If T is NTP_2, then T' is NTP_2.