Generalizations of stability and $NTP_2$

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Outline

Classification of first-order theories

Simple theories

NIP theories

$\text{NTP}_2$
Let $T$ be a complete countable first-order theory, and we fix some very large saturated model $\mathbb{M}$ (a “universal domain”).

For a model $M \models T$, we let $\text{Def}(M)$ be the Boolean algebra of definable subsets of $M$ (with parameters).

Let $S(M)$, the space of types over $M$, be the Stone dual of $\text{Def}(M)$. I.e. the set of ultrafilters on $\text{Def}(M)$ with the clopen basis consisting of sets of the form $[\phi] = \{ p \in S(M) : \phi \in p \}$. It is a totally disconnected compact Hausdorff space.

We abuse the notation slightly by not distinguishing between tuples of elements and singletons unless it matters.
General philosophy

▶ Shelah’s philosophy of dividing lines: characterize complete first-order theories by their ability to encode certain combinatorial configurations.
▶ Analysis of definable sets (and types) vs analysis of models.
▶ Looking at algebraic structures such as groups or fields, the model-theoretic properties are usually closely related to algebraic properties.
Let $s_T(\kappa) = \sup \{|S(M)| : M \models T, |M| = \kappa\}$. Note that always $s_T(\kappa) \geq \kappa$.

$T$ is called *stable* if any of the following equivalent properties hold:

- For every cardinal $\kappa$, $s_T(\kappa) \leq \kappa^{\aleph_0}$.
- There is some cardinal $\kappa$ such that $s_T(\kappa) = \kappa$.
- There is no formula $\phi(x, y)$ and $(a_i)_{i \in \omega}$ (in some model) such that $\phi(a_i, a_j) \Leftrightarrow i < j$.  

Examples

- Modules
- Algebraically closed fields
- Separably closed fields (C. Wood)
- Differentially closed fields
- Free groups (Z. Sela)
- Planar graphs (K. Podewski and M. Ziegler)
Dividing and Forking

Let $\phi(x, y)$ be a formula and $A$ a set.

- We say that $\phi(x, a)$ divides over $A$ if there is $k \in \omega$ and $(a_i)_{i \in \omega}$ such that $\text{tp}(a_i/A) = \text{tp}(a/A)$ and \{ $\phi(x, a_i)$ $\}_{i \in \omega}$ is $k$-inconsistent.
- Note that if $a \in A$ then $\phi(x, a)$ does not divide over $A$.
- We say that $\phi(x, a)$ forks over $A$ if there are $\phi_0(x, a_0), \ldots, \phi_n(x, a_n)$ such that $\phi(x, a) \vdash \bigvee_{i \leq n} \phi_i(x, a_i)$ and $\phi_i(x, a_i)$ divides over $A$ for each $i \leq n$.
- We say that a (partial) type $p(x)$ does not divide (fork) over $A$ if it does not imply any formula which divides (forks) over $A$.

Note that the formulas forking over $A$ form an ideal in $\text{Def}(\mathcal{M})$ generated by the formulas dividing over $A$.

Example

If $\mu$ is an $A$-invariant finitely additive probability measure on $\text{Def}(\mathcal{M})$ and $\mu(\phi(x, a)) > 0$ then $\phi(x, a)$ does not fork over $A$. 
Forking in stable theories

Assume that $T$ is stable.

1. Forking equals dividing: $\phi(x, a)$ forks over $A$ if and only if it divides over $A$.

2. Let’s write $a \downarrow_c b$ when $\text{tp}(a/bc)$ does not fork over $c$. Then $\downarrow$ is a nice notion of independence (i.e. invariant under automorphisms of $\mathbb{M}$, symmetric, transitive, satisfies finite character, ...)

3. Assume that $A$ is algebraically closed, in $M^{\text{eq}}$. Every $p \in S(A)$ has a unique non-forking extension $p' \in S(M)$ (i.e. $p \subseteq p'$ and that $p'$ does not fork over $A$).
Use of forking

- Shelah’s original purpose: to count the number of models a first-order theory may have. Essentially amounted to isolating the conditions for models to be classifiable by cardinal invariants.

- Geometric stability. Complexity of forking should be interrelated with the complexity of algebraic structures interpretable in the theory: trichotomy, group configuration, ...
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Simple theories

- A combinatorial definition: “not being able to encode a tree by some formula”.
- Equivalently, every $p \in S(M)$ does not fork over some countable subset $A \subset M$.
- Introduced by Shelah for purely model-theoretic reasons trying to characterize existence of certain limit models.
- Later work of Hrushovski and Hrushovski-Cherlin in the special case rank 1.
- Kim and Pillay carried out the analysis in the general case.
Examples

- The theory of the random Rado graph.
- Pseudo-finite fields.
- ACFA (and in general stable theories with some random “noise”).
Forking: Simple theories

1. Forking equals dividing: $\phi(x, a)$ forks over $A$ if and only if it divides over $A$.

2. $\downdash$ is still a nice notion of independence (symmetric, transitive, ...)

3. Stationarity and definability of types fail, types may have unboundedly many non-forking extensions.

(1) and (2) are due to Kim. Does anything of (3) survive?
Independence theorem

Turns out that the uniqueness of non-forking extensions can be replaced by an amalgamation statement.

Fact

*Independence theorem over models (Hrushovski in the finite rank case, Kim and Pillay in full generality):*

Assume that $a_1 \fork_M b_1$, $a_2 \fork_M b_2$ and $tp(a_1/M) = tp(a_2/M)$. Then there is a $\downarrow_M b_1 b_2$ and s.t. $tp(ab_i/M) = tp(a_i b_i/M)$ for $i = 1, 2$.

In fact, existence of a relation satisfying (2) and the independence theorem implies that the theory is simple and that this relation is given by non-forking.
2. Independence is given by: $a \perp_C b$ if and only if $\text{acl}_\sigma(ac)$ is algebraically independent from $\text{acl}_\sigma(bc)$ over $\text{acl}_\sigma(c)$.
3. Trichotomy for sets of rank 1 holds.
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A theory is NIP (No independence property) if it cannot “encode the random bipartite graph by a formula”.

NIP is equivalent to the finite Vapnik-Chervonenkis dimension of the families of $\varphi$-definable sets for all $\varphi$.

We remark that if a theory is both simple and NIP, then it is stable.
Examples

- linear orders and trees
- ordered abelian groups (Gurevich-Schmitt)
- any o-minimal theory
- algebraically closed valued fields (and in fact any c-minimal theory)
- $\mathbb{Q}_p$
Forking in NIP

- Symmetry of \( \downarrow \) fails badly – linear order.
- Some weaker replacements of stationarity:
  - A type \( p \in S(M) \) does not fork over \( M \) if and only if it is invariant over \( M \), i.e. \( \varphi(x, a) \in p \) and \( \text{tp}(a/M) = \text{tp}(b/M) \) implies \( \varphi(x, b) \in p \). It follows that every type has boundedly many non-forking extensions.
  - Some forms of definability of types remain (uniform definability of types over finite sets, joint work with P. Simon).
- What about forking vs dividing? May fail over some sets.
- However, Pillay posed the problem whether forking equals dividing over models in NIP.
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Definition
We say that $\phi(x, y)$ has TP$_2$ if there are $(a_{i,j})_{i,j \in \omega}$ and $k \in \omega$ such that:

- $\{\phi(x, a_{i,j})\}_{j \in \omega}$ is $k$-inconsistent for every $i \in \omega$,
- $\{\phi(x, a_{i,f(i)})\}_{i \in \omega}$ is consistent for every $f : \omega \to \omega$.

$T$ is called NTP$_2$ if no formula has TP$_2$.

- Every simple or NIP theory is NTP$_2$, but there is much more.
- To make sure that $T$ is NTP$_2$ it is enough to check it for all formulas $\varphi(x, y)$ in which $x$ is a singleton.
Example 1: Ultraproducts of p-adics

- Consider the valued field $K = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathcal{U}$, where $\mathcal{U}$ is a non-principal ultrafilter.
- The theory of $K$ is not simple: because the value group is linearly ordered.
- The theory of $K$ is not NIP: the residue field is pseudo-finite, thus has the independence property by a result of J.L. Duret.
- Even in the pure field language, as the valuation ring is definable uniformly in $p$ (J. Ax).
Ax-Kochen for NTP$_2$

However, $K$ is NTP$_2$ by the following:

**Theorem**

*Let* $K = (K, k, \Gamma)$ *be a henselian valued field of equicharacteristic* $0$, *in the Denef-Pas language. Assume that* $k$ *is NTP$_2$. Then* $K$ *is NTP$_2$.*

Analogous to the theorem of F. Delon for NIP.*
Example 2: Valued difference fields

- We consider valued difference fields $K = (K, k, \Gamma, \sigma)$ of equicharacteristic 0.
- Kikyo-Shelah: If $T$ has the Strict Order Property (which is the case with valued fields), then the model companion of $T \cup \{\sigma \text{ is an automorphism}\}$ does not exist.
- However, if we impose in addition that $\sigma$ is contractive (i.e. $\nu(\sigma(x)) > n \cdot \nu(x)$ for all $n \in \omega$), then the model companion $VFA_0$ exists. It is axiomatized by saying that $(k, \sigma)$ is a model of ACFA$_0$, $(\Gamma, \sigma)$ is a divisible $\mathbb{Z}[\sigma]$ module and $K$ is $\sigma$-henselian.
- A natural model of $VFA_0$: non-standard Frobenius acting on an algebraically closed valued field of char 0.
- Again neither simple nor NIP.
Example 2: Valued difference fields

Theorem

(Ch., M. Hils) Let $K = (K, k, \Gamma, \sigma)$ be a $\sigma$-henselian contractive valued difference field of equicharacteristic 0. Assume that both $(k, \sigma)$ and $(\Gamma, \sigma)$ are NTP$_2$. Then $K$ is NTP$_2$.

The proof utilizes the analysis of S. Azgin and properties of indiscernible arrays to reduce the situation to the previous example.
Back to Pillay’s question: is forking = dividing over models in NIP theories?

NTP$_2$ turned out to be the right context for clarifying this.

We say that a set $A$ is an extension base if every $p \in S(A)$ does not fork over $A$. E.g. every model is an extension base, in any theory. In simple theories, o-minimal theories or c-minimal theories, every set is an extension base.

**Theorem**

(Ch., I. Kaplan) Let $A$ be an extension base in an NTP$_2$ theory $T$. Then $\phi(x, a)$ divides over $A$ if and only if it forks over $A$. 
Forking in NTP2

- The reason: existence of strictly invariant types.
- A type $p(x) \in S(\mathbb{M})$ is called strictly invariant over $A$ if it is invariant (i.e. $\phi(x, a) \in p$ and $tp(a/A) = tp(b/A)$ implies $\phi(x, b) \in p$) and for every small $A \subseteq B \subseteq \mathbb{M}$, if $c \models p|_B$ then $tp(B/cA)$ does not fork over $A$.
- E.g. every generically stable type or every invariant type in a simple theory are strictly invariant.
- The crucial step of the proof is to show that in NTP$_2$ theories every type $p(x)$ over a model $M$ has a global strictly invariant extension $q(x)$ (the so called Broom lemma).
- Then one can show that if $\varphi(x, a)$ divides over $M$, $p(x) \in S(\mathbb{M})$ is a strictly invariant extension and $(a_i)_{i \in \omega}$ is a Morley sequence in $q$ (i.e. $a_i \models q|_{a_{<i}M}$) then $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent.
Weak independence theorem

- Recall the amalgamation of types in simple theories.
- Of course, fails in the presence of a linear order.
- In his work on approximate subgroups, Hrushovski found a reformulation of the independence theorem which makes sense in the context where $\vdash$ is not symmetric.
- Combining it with some new results on forking in $\text{NTP}_2$ (specifically that the forking ideal is $\text{S}1$) we get:
Weak independence theorem

Theorem
(I. Ben Yaacov, Ch.) Let $T$ be NTP\textsubscript{2} and $A$ an extension base.
Assume that $c \models M_{ab}$, $a \models M_{bb'}$ and $b \equiv M_{b'}$. Then there is $c'$ such that $c' \models M_{ab'}$, $c'a \equiv M_{ca}$, $c'b' \equiv M_{cb}$.

Remains valid over extension bases, but with Lascar-strong type in the place of type. In fact, can be used to deduce that Lascar strong type equals Kim-Pillay strong type over extension bases in NTP\textsubscript{2} theories. Gives rise to some results on stabilizers.
So why should one care about NTP$_2$?

- **Empirical argument:** every dividing line for first-order theories introduced by Shelah eventually becomes important.
- **Methodical argument:** allows for uniform proofs of results in simple and NIP theories, but also arises naturally trying to understand some special cases.
- **Forking works.**
- **Important examples.**