Forking and dividing in dependent theories

(and NTP_2 in there)

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“**NIP**” and “**dependent**” are synonyms during this talk

Tree property of the second kind (TP_2)

$\phi(x, y)$ has **TP$_2$** if there is an array $\{a_i^j\}_{i,j<\omega}$ and k s. t.

$$\begin{array}{cccc}
\phi(x, a_0^0) & \phi(x, a_1^0) & \phi(x, a_2^0) & \dots \\
\phi(x, a_0^1) & \phi(x, a_1^1) & \phi(x, a_2^1) & \dots \\
\phi(x, a_0^1) & \phi(x, a_1^1) & \phi(x, a_2^1) & \dots \\
\dots & \dots & \dots & \dots
\end{array}$$

rows are k-inconsistent:

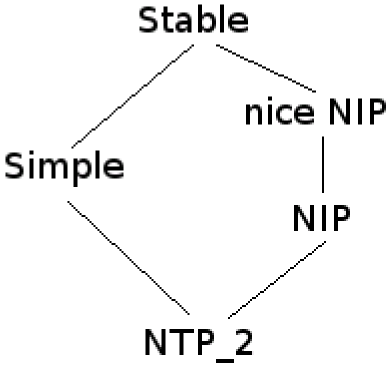
$\forall j < \omega \; \forall i_0 < i_1 < \dots < i_k < \omega$

$\phi(x, a_{i_0}^j) \wedge \phi(x, a_{i_1}^j) \wedge \dots \wedge \phi(x, a_{i_k}^j) = \emptyset$

vertical paths are consistent:

$\forall f : \omega \to \omega \; \exists c_f \models \bigwedge_{j<\omega} \phi(x, a_{f(j)}^j)$

Classification: How do all these classes of theories relate?



nice dependent: dependent + types don't fork over their domains
Examples: stable, *o*-minimal, *C*-minimal, (*D*-minimal?)

Forking / dividing / quasi-dividing

- $\phi(x, a)$ **divides** over A if exists an A-indiscerible sequence $\{a_i\}_{i<\omega}$ s. t. $\bigwedge_{i<\omega} \phi(x, a_i) = \emptyset$.
- $\phi(x, a)$ **forks** over A if $\phi(x, a) \vdash \bigvee_{i<n} \psi_i(x, a_i)$, where ψ_i divides over A.
- $\phi(x, a)$ **quasi-divides** (some people say weakly divides instead) over A if exist $a_0 \equiv_A a_1 \equiv_A ... \equiv_A a_n \equiv_A a$ s. t. $\bigwedge_{i<n} \phi(x, a_i) = \emptyset$

Note: dividing always implies both forking and quasi-dividing, but the converse in general is not true.

Sequences generated by types

Let p be a global type. We say that a sequence $\{a_i\}_{i<\omega}$ is **generated** by p over M if

$a_0 \models p|_M$

$a_1 \models p|_{Ma_0}$

$a_2 \models p|_{Ma_0a_1}$

...

Note:

p is M-invariant $\implies$ $\{a_i\}_{i<\omega}$ is M-indiscernible and $tp(\{a_i\}_{i<\omega}/M)$ depends only on p and M.

Shoulders of giants, with repetitions for the sake of chronology

Shelah: introduced NTP_2
Poizat/Shelah correspondence: classical theory of *NIP* theories
Kim: in simple theories forking = dividing
Dolich: forking = quasi-dividing in nice o-minimal theories (+goodness machinery)
Shelah, **Adler**, **Hrushovski/Peterzil/Pillay**, **Usvyatsov/Onshuus**: modern theory of dependent theories
Tressl: heirs and coheirs in o-minimal theories

Main theorem

- (NTP_2) $\phi(x,a)$ forks $/M \iff \phi(x,a)$ divides $/M$
- (NIP) in addition $\iff \phi(x,a)$ quasi-divides $/M$
- (T is nice dependent) $\phi(x,a)$ forks $/A \iff \phi(x,a)$ divides $/A$
- (T is nice dependent + Lascar strong type over A = types over A) in addition $\iff \phi(x,a)$ quasi-divides $/A$

NTP_2: From indiscernibles to coheir sequences

Observation: Suppose $\phi(x, a)$ divides over M. Then **exists a global coheir** of $tp(a/M)$, s.t. some (equivalentely any) sequence generated by it witnesses dividing

Proof(very imprecise sketch):
Suppose not, let I be an M-indiscernible witnessing dividing.
Take $N \supseteq M$, $|M|^+$-saturated, and I' is an indicsernible with the same EM type, very long w.r.t. N.
Take its type over M, expand to N - infinitely often it is the same coheir, so forget all other variables.
Generate sequence in it - this is our array.

Strict non-forking and non-forking heirs

Let $A \subseteq B$, $p \in S(B)$. We say that p **lifts indiscernibles** from A if for every A-indiscernible sequence $\{a_i\}_{i<\omega}$ s.t. $a_i \models p|_A$ there is some $\{a'_i\}_{i<\omega}$ satisfying:
1) $a'_i \models p$
2) $tp(\{a'_i\}_{i<\omega}/A) = tp(\{a_i\}_{i<\omega}/A)$

Definition (Shelah): Type p **strictly does not fork** if it does not fork and lifts indiscernibles.

Example: Global non-forking heir over M.

Analog of Kim's lemma in dependent context?

Lemma (T dependent): Let $\phi(x, a)$ divide over M. Let $p \in S(\mathbb{M})$ be **any** global type strictly non-forking $/M$, $tp(a/M) \subseteq p$. Then any sequence generated by it witnesses dividing.

But do non-forking heirs always exist?

Broom lemma: Let
$\alpha(x,e) \vdash \psi(x,c) \vee \bigvee_{i<n} \phi_i(x,a_i)$,
where
1) each $\phi_i(x,a_i)$ is k-dividing, witnessed by the sequence $I_i := \{a^i_j\}_{j<\omega}$, with $a^i_0 = a_i$
2) for each $i < n$ and $1 \leq j$ holds $a^i_j \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^u_A a^i_{<j} I_{<i}$
3) $c \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^u_M I_{<i}$

then for some $e_0 \equiv_M e_1 \equiv_M \dots \equiv_M e_m \equiv_M e$ we have
$\bigwedge_{l<m} \alpha(x,e_l) \vdash \psi(x,c)$

(so essentially if a formula is covered by finitely many formulas in "nice position", then we can throw away dividing ones, by passing to intersection of finitely many conjugates at worst)

Yes, they do

Corollary 1 (NTP_2): Forking implies quasi-dividing over models
Why? Can always arrange assumption of the broom lemma for a forking formula, using existence of global coheirs witnessing dividing.
Corrolary 2 (NTP_2): Every type over model has a global non-forking heir

Nice dependent theories

Why proofs work over arbitrary sets instead of models?
Hint: broom lemma works with non-forking instead of coheirs.

Corrolaries

- (T dependent) Forking is type-definable, so dependent theories are low
- (T is NTP_2) Non-forking satisfies left extension
- (T is NTP_2) T is dependent iff non-forking is bounded

Optimality of results?

Martin Ziegler (several days ago):

- T with TP_2 s.t. forking $\neq$ dividing **over model**
- T with TP_2 s.t. there is a type over model without any global non-forking heirs
- T with TP_2 s.t. forking = dividing always

First and second are delivered by a “dense independent” family of circular orderings, third is a “dense independent” family of linear ones.

Questions

- Does "T has TP_2" imply "forking is not type-definable"?
- Understand when types over models have global non-forking heirs
- Characterize dependence of a theory by behaviour of forking / strict non-forking / ...