Fields with $\text{NTP}_2$

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Outline

Shelah’s classification theory and NTP₂

Examples of fields with NTP₂

Implications of NTP₂ for properties of definable groups and fields

Quantitative refinements of NTP₂ — burden, strongness, inp-minimality
Some history

- We consider complete first-order theories in a countable language, $\mathcal{M}$ denotes a monster model.
- Shelah’s philosophy of dividing lines — classify complete first-order theories by their ability to encode certain combinatorial configurations. He identified several very concrete configurations (e.g. linear order in the case of stability) such that:
  - when the theory cannot encode them, the category of definable sets and types admits a coherent theory (forking, ranks, weight, analyzability, etc leading to a classification of models);
  - when it can, one can prove a non-structure result (many models in the case of stability).
- In algebraic situations such as groups or fields, these model-theoretic properties turn out to be closely related to algebraic properties of the structure.
- Later work of Zilber, Hrushovski and others on geometric stability theory produced deep applications to purely algebraic questions.
Some history

- Unfortunately, most structures studied in mathematics are not stable.
- Simple theories: developed by Shelah, Hrushovski, Kim, Pillay, Chatzidakis, Wagner and others. Applications in algebraic dynamics, etc.
- Various minimality settings: o-minimality, c-minimality, p-minimality, etc — concentrated on definable sets rather than types, not quite in the spirit of stability theory.
- Common context to treat these settings — NIP: Pillay’s conjecture on groups in o-minimal theories, work of Haskell, Hrushovski and Macpherson on algebraically closed valued fields and stable domination.
Shelah’s classification theory and generalizations of stability

Simple theories
- Random graph
- Pseudofinite fields
- ACFA

Stable theories
- ACF
- Free groups
- Planar graphs

NIP theories
- Linear orders
- Trees
- Ordered abelian groups
- O-minimal theories
- ACVF
- $\mathbb{Q}_p$

NTP$_2$

PA, ZFC
NTP₂

**Definition**

[Shelah]

1. A formula $\phi(x, y)$, where $x$ and $y$ are tuples of variables, has TP₂ (*Tree Property of the 2nd kind*) if there is an array $(a_{i,j})_{i,j \in \omega}$ of tuples from $\mathbb{M}$ and $k \in \omega$ such that:
   - $\{\phi(x, a_{i,j})\}_{j \in \omega}$ is $k$-inconsistent for every $i \in \omega$.
   - $\{\phi(x, a_{i,f(i)})\}_{i \in \omega}$ is consistent for every $f : \omega \to \omega$.

2. A theory is NTP₂ if it implies that no formula has TP₂.

**Fact**

[Ch.] *Enough to check formulas with $|x| = 1$.*

**Fact**

*Every simple or NIP theory is NTP₂.*
In [Ch., Kaplan] and later [Ben Yaacov, Ch.] a reasonable theory of forking over extension bases in NTP$_2$ theories was developed:

- incorporates the theory of forking in simple theories due to Kim, Pillay, Hrushovski and others as a special case;
- provides answers to some questions of Pillay and Adler around forking and dividing in the case of NIP.

Guiding principle (rather naive) — NTP$_2$ is a combination of simple and NIP (e.g. densely ordered random graph, the model companion of the theory of ordered graphs, is neither simple nor NIP; but it is NTP$_2$).
Examples of NTP$_2$ fields: ultraproducts of $p$-adics

- For every prime $p$, the valued field $(\mathbb{Q}_p, +, \times, 0, 1)$ is NIP.
- However, consider the valued field $\mathcal{K} = \prod_{\text{prime}} \mathbb{Q}_p / \mathcal{U}$ (where $\mathcal{U}$ is a non-principal ultrafilter on the set of prime numbers) — a central object in the model theoretic applications to valued fields after the work of Ax and Kochen.
- The theory of $\mathcal{K}$ is not simple: because the value group is linearly ordered.
- The theory of $\mathcal{K}$ is not NIP: the residue field is pseudofinite, thus has the independence property by a result of Duret.
- Both even in the pure ring language: as the valuation ring is definable uniformly in $p$ (Ax).
- Canonical models: Hahn fields of the form $k((t^\mathbb{Z}))$, where $k$ is a pseudofinite field.
Ax-Kochen principle for $\text{NTP}_2$

**Fact**

[Delon + Gurevich, Schmitt] Let $\mathcal{K} = (K, \Gamma, k, \nu, \text{ac})$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that $k$ is $\text{NIP}$. Then $\mathcal{K}$ is $\text{NIP}$.

**Theorem**

[Ch.] Let $\mathcal{K} = (K, \Gamma, k, \nu, \text{ac})$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that $k$ is $\text{NTP}_2$. Then $\mathcal{K}$ is $\text{NTP}_2$.

**Corollary**

$\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p/U$ is $\text{NTP}_2$ because the residue field is pseudofinite, so simple, so $\text{NTP}_2$.

**Problem**: Show an analogue for positive characteristic (Belair for $\text{NIP}$).
Valued difference fields

- $(K, \Gamma, k, \nu, \sigma)$ is a valued difference field if $(K, \Gamma, k, \nu, ac)$ is a valued field and $\sigma$ is a field automorphism preserving the valuation ring.
- Note that $\sigma$ induces natural automorphisms on $k$ and on $\Gamma$.
- Because of the order on the value group, it follows by [Kikyo, Shelah] that there is no model companion of the theory of valued difference fields.
- The automorphism $\sigma$ is contractive if for all $x \in K$ with $\nu(x) > 0$ we have $\nu(\sigma(x)) > n\nu(x)$ for all $n \in \omega$.
- **Example:** Let $(F_p, \Gamma, k, \nu, \sigma)$ be an algebraically closed valued field of char $p$ with $\sigma$ interpreted as the Frobenius automorphism. Then $\prod_{p \text{ prime}} F_p/U$ is a contractive valued difference field.
Valued difference fields

[Harushovski], [Durhan] Ax-Kochen principle for $\sigma$-henselian contractive valued difference fields $(K, \Gamma, k, v, \sigma, ac)$:

- Elimination of the field quantifier;
- $(K, \Gamma, k, v, \sigma) \equiv (K', \Gamma', k', v, \sigma)$ iff $(k, \sigma) \equiv (k', \sigma)$ and $(\Gamma, <, \sigma) \equiv (\Gamma', <, \sigma)$;
- There is a model companion $VFA_0$ and it is axiomatized by requiring that $(k, \sigma) \models ACFA_0$ and that $(\Gamma, +, <, \sigma)$ is a divisible ordered abelian group with an $\omega$-increasing automorphism.
- Nonstandard Frobenius is a model of $VFA_0$.
- The reduct to the field language is a model of $ACFA_0$, hence simple but not NIP. On the other hand this theory is not simple as the valuation group is definable.
Valued difference fields and $\text{NTP}_2$

Theorem

[Ch.-Hils] Let $\bar{K} = (K, \Gamma, k, \nu, ac, \sigma)$ be a $\sigma$-Henselian contractive valued difference field of equicharacteristic 0. Assume that both $(K, \sigma)$ and $(\Gamma, \sigma)$, with the induced automorphisms, are $\text{NTP}_2$. Then $\bar{K}$ is $\text{NTP}_2$.

Corollary

$\text{VFA}_0$ is $\text{NTP}_2$ (as $\text{ACFA}_0$ is simple and $(\Gamma, +, <, \sigma)$ is NIP).

- **Conjecture**: One can omit the requirement on the value group.
- Besides, our argument also covers the case of $\sigma$-henselian valued difference fields with a value-preserving automorphism of [Belair, Macintyre, Scanlon] and the multiplicative generalizations of Kushik.
Some conjectural examples

- A field is pseudo algebraically closed (PAC) if every absolutely irreducible variety defined over it has a point in it.
- It is well-known that the theory of a PAC field is simple if and only if it is bounded (i.e. for any integer \( n \) it has only finitely many Galois extensions of degree \( n \)). Moreover, if a PAC field is unbounded, then it has TP\(_2\) [Chatzidakis].
- On the other hand, the following fields were studied extensively:
  1. Pseudo real closed (or PRC) fields: a field \( F \) is PRC if every absolutely irreducible variety defined over \( F \) that has a rational point in every real closure of \( F \), has an \( F \)-rational point.
  2. Pseudo \( p \)-adically closed (or PpC) fields: a field \( F \) is PpC if every absolutely irreducible variety defined over \( F \) that has a rational point in every \( p \)-adic closure of \( F \), has an \( F \)-rational point.
- **Conjecture:** A PRC field is NTP\(_2\) if and only if it is bounded. Similarly, a PpC field is NTP\(_2\) if and only if it is bounded.
Algebraic properties from tameness assumptions

- [Macintyre] Every $\omega$-stable field is algebraically closed.
- [Cherlin-Shelah] Every superstable field is algebraically closed.
- **Conjecture:** Every stable field is separably closed.
- Many further results: every $o$-minimal field is real-closed, every $C$-minimal valued field is algebraically closed, etc...
Recall that given a field $K$ of characteristic $p > 0$, an extension $L/K$ is Artin-Schreier if $L = K(\alpha)$ for some $\alpha \in L \setminus K$ such that $\alpha^p - \alpha \in K$.

- [Kaplan, Scanlon, Wagner]:
  1. Let $K$ be an NIP field. Then it is Artin-Schreier closed.
  2. Let $K$ be a (type-definable) simple field. Then it has only finitely many Artin-Schreier extensions.

- Remember our guiding principle: $\text{NTP}_2 \sim \text{NIP} + \text{simple}$. 
NTP\textsubscript{2} fields have finitely many Artin-Schreier extensions

**Theorem**

[Ch., Kaplan, Simon] Let $K$ be a field definable in an NTP\textsubscript{2} structure. Then it has only finitely many Artin-Schreier extensions.

- Type-definable case is open even for NIP theories.
Ingredients of the proof

1. [Kaplan-Scanlon-Wagner] For a perfect field $K$ of characteristic $p$, given a tuple of algebraically independent elements $\bar{a} = (a_1, \ldots, a_n)$ from $K$ and some large algebraically closed extension $\bar{K}$, the group $G_{\bar{a}} = \{(t, x_1, \ldots, x_n) \in \bar{K}^{n+1} : t = a_i (x_i^p - x_i) \text{ for } 1 \leq i \leq n\}$ is algebraically isomorphic over $K$ to $(\bar{K}, +)$.

2. Chain condition for uniformly definable normal subgroups: Let $G$ be NTP$_2$ and $\{\varphi(x, a) : a \in C\}$ be a family of normal subgroups of $G$. Then there is some $k \in \omega$ (depending only on $\varphi$) such that for every finite $C' \subseteq C$ there is some $C_0 \subseteq C'$ with $|C_0| \leq k$ and such that

$$\left[ \bigcap_{a \in C_0} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a) \right] < \infty.$$  

3. Combine.
Quantitative measure of NTP$_2$: burden

Definition

1. An inp-pattern of depth $\kappa$ consists of $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha), k_\alpha)_{\alpha \in \kappa}$ with $\bar{a}_\alpha = (a_\alpha, i)_{i \in \omega}$ and $k_\alpha \in \omega$ such that:
   - $\{\varphi_\alpha(x, a_\alpha, i)\}_{i \in \omega}$ is $k_\alpha$-inconsistent for every $\alpha \in \kappa$,
   - $\{\varphi_\alpha(x, a_\alpha, f(\alpha))\}_{\alpha \in \kappa}$ is consistent for every $f : \kappa \to \omega$.

2. The burden of $T$ is the supremum of the depths of inp-patterns with $x$ a singleton, computed in Card*.
Quantitative measure of NTP$_2$: burden

Possible values of the burden of a theory in a countable language:

1. $n \in \omega \setminus \{0\}$ — there is no inp-pattern of depth $\geq n$;

2. $\aleph_0^-$ — there are patterns of arbitrary finite depth, but not of infinite depth. Theories with this burden are called strong;

3. $\aleph_0$ — there is an inp-pattern of infinite depth, but not of arbitrary large depth. This means that a theory is NTP$_2$, but not strong;

4. $\infty$ — there are inp-patterns of depth $\kappa$ for any cardinal $\kappa$. This is equivalent to TP$_2$ by compactness.
Burden of pseudo-local valued fields

Definition
Theories of burden 1 are called inp-minimal.

Theorem
[Ch., finer version] Let $\mathcal{K} = (K, \Gamma, k, \nu, ac)$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that $k$ and $\Gamma$ are strong (of finite burden). Then $\mathcal{K}$ is strong (resp. of finite burden).

▶ But the bound is given by some Ramsey number!

Theorem
[Ch., Simon] All ultraproducts of $p$-adics are inp-minimal.

Fact
[Simon] Let $G$ be inp-minimal. Then there is a definable normal abelian subgroup $H$ such that $G/H$ is of finite exponent.

▶ Question: What happens in higher dimensions? Is burden subadditive, at least in this example?
Burden of $\text{VFA}_0$

- What is the burden of $\text{VFA}_0$? We know that it is bounded.
- **Observation:** [Ch., Hils] Burden of $\text{VFA}_0$ is $\geq n$ for all $n \in \omega$ (as every completion of ACFA has a 1-type of weight $n$).
- **Problem:** Is $\text{VFA}_0$ strong?
Results about definable objects can be now proved about type-definable objects.

Proposition [Ch., Kaplan, Simon], a slight generalization of the argument of [Krupinski, Pillay] for the stable case: Any infinite strong field is perfect.

A valued field \((K, v)\) of characteristic \(p > 0\) is Kaplansky if it satisfies:

1. The valuation group \(\Gamma\) is \(p\)-divisible.
2. The residue field \(k\) is perfect, and does not admit a finite separable extension whose degree is divisible by \(p\).

Corollary

[Ch., Kaplan, Simon] Every strongly dependent (i.e. strong and dependent) valued field is Kaplansky.
Conjecture about definable envelopes of groups

1. [Shelah], [Aldama] If $G$ is a group definable in an NIP theory and $H$ is a subgroup which is abelian (nilpotent of class $n$; normal and soluble of derived length $n$) then there is a definable group containing $H$ which is also abelian (resp. nilpotent of class $n$; normal and soluble of derived length $n$).

2. [Milliet] Let $G$ be a group definable in a simple theory and let $H$ be a subgroup of $G$.
   
   2.1 If $H$ is nilpotent of class $n$, then there is a definable (with parameters from $H$) nilpotent group of class at most $2n$ finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal nilpotent group of class at most $3n$ containing $H$.

   2.2 If $H$ is a soluble of class $n$, then there is a definable (with parameters from $H$) soluble group of derived length at most $2n$ finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal soluble group of derived length at most $3n$ containing $H$. 
**Conjecture about definable envelopes of groups**

**Conjecture:** Let $G$ be an NTP$_2$ group and assume that $H$ is a subgroup. If $H$ is nilpotent (soluble), then there is a definable nilpotent (resp. soluble) group finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal nilpotent (resp. soluble) group containing $H$. 
References

1. Saharon Shelah, “Classification theory and the Number of Non-Isomorphic Models”
2. Artem Chernikov and Itay Kaplan, “Forking and dividing in NTP$_2$ theories”, JSL