Equivalence of Axiom of Choice and Zorn’s Lemma

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**AC \Rightarrow** Zorn’s Lemma. Let \((P, \leq)\) be any partially ordered set satisfying the hypothesis of Zorn’s Lemma (any chain has an upper bound in \((P,\leq)\)). Then we construct a maximal element using the well-ordering principle, an equivalent of choice.\(^1\)

Let \(b \notin P\) be an element not in the poset. This will be the default value of the functions we are building. Let \((P,\preceq)\) be a well-ordering of \(P\). Note that there is no relationship between \(\leq\) and \(\preceq\).

Now define \(H(x) = \) the \(\preceq\)-least\(^2\) member of \(P\) which is strictly \(<\)-greater than every element of \(x\), if \(x \subset P\) and such exists, and \(b\) otherwise. Define \(F(\alpha) = H(F[\alpha])\). So \(H(x)\) just picks something greater than anything in \(x\), and \(F\) is the construction of a chain using \(H\) to pick elements one at a time.

Now we do the usual argument to show that there is an \(\alpha\) with \(F(\alpha) = b\).\(^3\)

So since the ordering of ordinals is a well-order, we can pick \(\beta\) to be least such that \(F(\beta) = b\). Define \(C = F[\beta]\), so \(C \subset P\) is a chain.

Now let \(q\) be an upper bound of this chain. I claim \(q\) is a maximal element, because if there were some \(q' > q\), then \(q'\) is greater than every element of \(F[\beta]\) so we would have picked \(F(\beta)\) to be some member of \(P\), not \(b\). This completes the proof.

**Zorn’s Lemma \Rightarrow** AC. Suppose \(\mathcal{C}\) is a set of nonempty sets. Let \(P\) be the subset of functions \(F : A \rightarrow \bigcup \mathcal{C}\), where \(A \subset \mathcal{C}\) and \(F(X) \in X\) for all \(X \in \mathcal{C}\). In other words, \(P\) consists of “choice functions” whose domains need not be the whole \(\mathcal{C}\). Make \(P\) into a poset using the ordering of extension, so \(F < G\) if domain\((F) \subset \text{domain}(G)\) and \(G \upharpoonright \text{domain}(F) = F\). This satisfies the hypothesis of Zorn’s Lemma, since the union of any chain is an upper bound for the chain. So there is a maximal element, call it \(H\). Now I claim \(H\) is a

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\(^1\)Conceptually it is the same as using the axiom of choice, but notationally easier. It gives us the choice function that we need, namely picking the least thing each time.

\(^2\)this is the only time we use the \(\preceq\) order

\(^3\)Otherwise consider the operation \(Q\) defined on a subset of \(P\) (namely, the “range” of \(F\) in \(P\), which is a set by separation) with \(Q(p) = \alpha\) if \(F(\alpha) = p\). This is the operation obtained by “inverting” \(F\). It is well-defined since if \(F(\alpha) = F(\beta) = p\) with \(\alpha < \beta\), we get a contradiction since \(F(\alpha) \in F[\beta]\) so \(F(\beta)\) was supposed to be strictly \(<\)-greater than \(F(\alpha)\). Now we use replacement to show that \(\{\alpha : F(\alpha) \in P\}\), the range of \(Q\), is a set. But we assumed it was the whole \(\text{ON}\), contradiction.
full choice function, since otherwise there is some $X \in \mathcal{C}$ not in the domain of $H$, and I can form a function $H' = H \cup \{(X, x)\}$ where $x$ is any element of $X$, contradicting maximality of $H$. 