Ideas for Bar and Michael’s second talks and Alex’s talk

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November 20, 2018

1 Bar’s talk

1.1 Hopf algebroids

Talk about the structure maps of \((E, E^*E)\) in general.

\(\eta_L\) is induced by \(E \otimes S^0 \rightarrow E \otimes E\). \(\eta_R\) is induced by \(S^0 \otimes E \rightarrow E \otimes E\).
\(\Delta\) is induced by \(E \otimes S^0 \otimes E \rightarrow E \otimes E \otimes E\).

1.2 Construction of \(BP\)

This depends on a prime \(p\).

1.3 The Hopf algebroid \((BP^*, BP^*BP)\)

Let \(H = H\mathbb{Z}\).

**Theorem.** \(BP_* \rightarrow (H \otimes BP)_*\) is injective and \((H \otimes BP)_* = H_*(BP) = \mathbb{Z}(m_1, m_2, \ldots)\) where \(m_1, m_2, m_3, \ldots\) are canonical generators coming from FGL stuff.

We have a map of AHSSs induced by \(BP_*BP \rightarrow (H \otimes BP)_*BP\). On \(E_2\)-pages it is
\[
H_*(BP; BP_*) \rightarrow H_*(BP; (H \otimes BP)_*).
\]

It degenerates to tell us about an injection \(BP_*[t_1, t_2, \ldots] \rightarrow (H \otimes BP)_*[t_1, t_2, \ldots]\).

So the Hopf algebroid structure of \((BP^*, BP^*BP)\) is determined by the Hopf algebroid \(((H \otimes BP)_*, (H \otimes BP)_*(BP))\).

**Theorem.** Let \(m_0 = 1\) and \(t_0 = 1\). \(t_1, t_2, t_3, \ldots\) may be chosen inductively so that

\[
\eta_L(m_n) = m_n, \eta_R(m_n) = \sum_{i+j=n} m_i t_j^p, \text{ and }
\]
\[
\Delta(t_n) = \sum_{i+j+k=n} m_i t_j^p \otimes t_k^{p^i+j} - \sum_{i=1}^n m_i (\Delta t_{n-i})^p.
\]

**Definition.** The Hazewinkel generators are defined by \(v_0 = p\) and

\[
v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^p.
\]

The Araki generators are defined by \(v_0 = p\) and the same formulae with \(n\) replacing \(n-1\).

It is a theorem that both are generators for the image of the injection \(BP_* \rightarrow (H \otimes BP)_*\).
2 Michael’s Talk

2.1 The algebraic Novikov spectral sequence

Recall the dual Steenrod algebra \( A = (H\mathbb{F}_p)_*(H\mathbb{F}_p) \) and the Hopf algebroid \((BP_*, BP_*BP)\).

**Notation.** When \( p \) is odd, write \( \zeta_n \) for \( \xi_n \). When \( p = 2 \), write \( \zeta_n \) for \( \xi_n^2 \) and \( \pi_{n-1} \) for \( \xi_n \).

**Remark.** We have a map \( BP \to H\mathbb{F}_p \). From Christian’s talk we should be able to deduce that

\[
(H\mathbb{F}_p)_*(BP) \to (H\mathbb{F}_p)_*(H\mathbb{F}_p)
\]

is an inclusion with image \( P = \mathbb{F}_p[\zeta_1, \zeta_2, \zeta_3, \ldots] \). Moreover, we have

\[
\begin{array}{cccc}
BP_*BP & \to & (H\mathbb{Z} \wedge BP)_*(BP) & \to (H\mathbb{F}_p)_*(BP) \\
\downarrow v_n & & \downarrow 0 & \to 0 \\
t_n & \to & \zeta_n & \to \zeta_n.
\end{array}
\]

- Letting \( I = (p, v_1, v_2, \ldots) \), we immediately see that \( \Delta(t_n) = \sum_{i+j=n} t_i \otimes t_j^p \) modulo \( I \).
- \( \eta_R : BP_* \to BP_* \otimes_{BP_*} BP_*BP \) determines a map \( \pi_R : I/I^2 \to I/I^2 \otimes_{BP_*} BP_*BP \).

Write \( \pi_n \) for the class of \( v_n \) in \( I/I^2 \). Recall \( \eta_R(m_n) = \sum_{i+j=n} m_i t_j^p \) and

\[
v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^p.
\]

These formulae hint that \( \eta_R(\pi_n) = \sum_{i+j=n} \pi_i t_j^p \).

Let’s expand on this remark. We have a short exact sequence of Hopf algebras:

\[
1 \to P \to A \to E \to 1.
\]

Thus, the \( E_2 \)-page of the ASS for \( BP_* = \pi_*(BP) \) is \( H^*(A; P) \cong H^*(E) \cong Q = \mathbb{F}_p[q_0, q_1, q_2, \ldots] \).

The ASS degenerates at the \( E_2 \)-page so \( q_n \) must detect \( v_n \). This shows that the Adams filtration on \( BP_* \) is the same as the I-adic filtration.

Moreover, the \( E_2 \)-page of the ASS for \( BP_*BP \) is \( H^*(A; P \otimes P) = Q \otimes P \). The ASS degenerates at the \( E_2 \)-page. By the remark, \( \zeta_n \) detects \( t_n \). The left and right unit induce maps of SSs. The left unit is easy inducing \( Q \cong Q \otimes \mathbb{F}_p \to Q \otimes P \), and this shows that \( q_n \) detects \( v_n \) in this SS too. On homology, the right unit induces \( (H\mathbb{F}_p)_*(S^0 \wedge BP) \to (H\mathbb{F}_p)_*(BP \wedge BP) \), which is \( \Delta : P \to P \otimes P \).

We analyze the induced map of spectral sequences...
Chasing some isomorphisms, the map of SSs shows that the associated graded of $\Omega^*(BP_*BP;BP_*)$ is $\Omega^*(P;Q)$, where $Q = \mathbb{F}_p[q_0,q_1,q_2,\ldots]$ is an algebra in right $P$-comodules with coaction

$$q_n \mapsto \sum_{i+j=n} q_i \otimes \zeta_j^p.$$  

The spectral sequence obtained by filtering $\Omega^*(BP_*BP;BP_*)$ using the $I$-adic filtration is called the algebraic Novikov spectral sequence.

$$E_{1}^{s,t,u} = H^{s,u}(P;Q^t) \Rightarrow H^{s,t,u}(BP_*BP), \quad d_r : E_{r}^{s,t,u} \rightarrow E_{r+1}^{s+1,t+r,u}.$$

### 2.2 The Cartan-Eilenberg Spectral Sequence

The SES $1 \rightarrow P \rightarrow A \rightarrow E \rightarrow 1$ also gives rise to the Cartan-Eilenberg spectral sequence

$$E_{2}^{s,t,u} = H^{s,u}(P;Q^t) \Rightarrow H^{s+t,u+t}(A), \quad d_r : E_{r}^{s,t,u} \rightarrow E_{r+r}^{s+r,t-r+1,u+r-1}.$$  

When $p$ is odd, the SES splits and so the SS is degenerate.

### 2.3 Miller’s square

We’ve now constructed Miller’s square:

$$\begin{array}{ccc}
H^*(P;Q) & \xrightarrow{f} & H^*(A) \\
\downarrow & & \downarrow \\
H^*(BP_*BP) & \xrightarrow{g} & \pi_*(S^0).
\end{array}$$

### 2.4 Inverting self-maps versus elements in a ring

Suppose $R$ is a ring spectrum and that $f \in \pi_d(R)$. We can define a self-map $\overline{f} : \Sigma^d R \rightarrow R$ by

$$\Sigma^d R = S^d \wedge R \xrightarrow{f \wedge R} R \wedge R \xrightarrow{\mu} R.$$  

Notice that given $g \in \pi_n(R) = \pi_{d+n}(\Sigma^d R)$ we have $\overline{f} \circ g = f \cdot g$ because:

$$\begin{array}{ccc}
S^{d+n} & \xrightarrow{g} & \Sigma^d R \\
\downarrow & & \downarrow \overline{f} \\
S^d \wedge S^n & \xrightarrow{S^d \wedge g} & S^d \wedge R \\
\downarrow & & \downarrow f \wedge R \\
S^d \wedge S^n & \xrightarrow{f \wedge g} & R \wedge R.
\end{array}$$
2.5 \( v_1 \)-periodic elements

I’ll describe Miller’s calculation of

\[
H^*(P; q_1^{-1}Q/q_0) \longrightarrow q_1^{-1}H^*(A; H_*(S/p))
\]

for odd primes using the isomorphism \( H^*(P; q_1^{-1}Q/q_0) \longrightarrow H^*(P(1); \mathbb{F}_p[q_1, q_1^{-1}]) \).

We’ll get a subgroup \( \mathbb{F}_p[v_1] \otimes E[\phi] \subset \pi_*(S/p) \). In order to understand the \( v_1 \)-periodic elements of \( \pi_*(S^0) \), we should analyze what happens to these elements in the Bockstein spectral sequence:

\[
\pi_*(S/p)[p] = \pi_*(S^0).
\]

**Remark 2.5.1.** The square in the following diagram does not commute.

\[
\begin{array}{ccc}
S^q \wedge S^{kq} & \xrightarrow{v_1 \wedge v_1^k} & S/p \wedge S/p \\
| & & | \\
| & & | \\
| & & | \\
S/p & \xrightarrow{\beta \wedge S/p} & \Sigma S/p \wedge S/p \\
\end{array}
\]

So \((\beta \circ v_1) \cdot v_1^k \neq \beta \circ v_1^{k+1}\) in general.

\( \varphi = \beta \circ v_1 \) but because of the fact just observed labelling it as such is confusing.

We have \( d_1(v_1) = \varphi \). Using the fact that differentials are derivations, we find that

\[ d_1(v_1^{p^n}) = 0. \]

The following diagram proves this very explicitly, and provides representatives with as high Adams filtration as possible. I’ll describe the ASS picture.

\[
\begin{array}{cccccc}
S^{p^n q - 1} & \longrightarrow & B_{p^n}^{p^n - n} & \longrightarrow & B_1^{p^n + 1} & \longrightarrow & S^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{p^n q - 1} & \longrightarrow & B_{p^n}^{p^n} & \longrightarrow & B_1^{p^n + 1} & \longrightarrow & S^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{p^n q - 1} & \longrightarrow & \Sigma^{p^n q - 1} S/p & \longrightarrow & \Sigma^{p^n q - 1} S/p & \longrightarrow & S^0 \\
\end{array}
\]

To see \( d_{n+1}(v_1^{p^n}) \) is non-zero and thus equal to \( v_1^{p^n - 1} \varphi \), you can either believe me or wait until Alex computes the 1-line of the ANSS.

I’ll then describe the corresponding elements for \( p = 2 \) and show them in the ASS after giving a quick construction of \( v_1^4: \Sigma^8 S/2 \longrightarrow S/2. \)
3 Alex’s talk

3.1 Assume \( p \) is odd throughout

Makes things easier!

3.2 Set up chromatic spectral sequence

This is fairly easy if you ignore details about the involved algebra. Of course, you’ll probably want to describe what on earth things like \( v_2^{-1}BP,/(p^{\infty},v_1^{\infty}) \) mean. That’s some details. Easy for you.

3.3 The 1-line of the \( E_2 \)-page of ANSS

You can use the CSS to calculate the 1-line. This is not too difficult. However, it will easily take up a lecture. I can find some notes I made for a talk about this. I might have even already passed them onto Christian.