Notes for Juvitop, Fall 2016 on power operations

Disclaimer: These are my notes from the Juvitop student seminar at MIT. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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The \TeX{} source for these notes can be found at [http://math.mit.edu/~ebelmont/juvitop-fall-2016-notes.zip](http://math.mit.edu/~ebelmont/juvitop-fall-2016-notes.zip).

1. September 28, Hood Chatham

Let $E$ be an $E_{\infty}$ ring, and $X$ an $E_{\infty}$-algebra over $E$. Look at an element of $\pi_* X$, i.e. a map $S^i \to X$. Using the $E$ action on $X$, I can tensor up to get a map $S^i \wedge E \to X$. Then take the $p^{th}$ external power to get $S^p \wedge E \to \Lambda^p_E X$. I have a multiplication $\mu : X^\wedge p \to X$; if it were a commutative multiplication, that would lead to a factorization

$\begin{array}{ccc}
X^\wedge p & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow \\
X^\wedge p & \xrightarrow{\mu} & X
\end{array}$

(orbits of the cyclic group $\pi = C_p$), but it’s only $E_{\infty}$, so instead I get a factorization through the homotopy orbits:

$\begin{array}{ccc}
X^\wedge p & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow \\
D_\pi X := X_{h\pi}^\wedge p
\end{array}$

Given an element $b \in E_j(B\pi)$, I can put this together into a diagram:

$\begin{array}{ccc}
S^p \wedge E & \xrightarrow{\mu} & \Lambda^p_E X & \xrightarrow{\mu} & X \\
\downarrow & & \downarrow & & \downarrow \\
S^p \wedge E \wedge B\pi_+ & \xrightarrow{\mu} & D_\pi(S^i \wedge E) & \xrightarrow{b} & D_\pi X
\end{array}$

(This is maybe off by a sign, but I’ll specialize to $p = 2$.) The long composite gives an element in $\pi_{pi+j}(X)$. This whole discussion started with an element in $\pi_* X$, so altogether we have a map $\pi_i(X) \to \pi_{pi+j}(X)$. This is called a power operation.

In this talk, I’m going to look at $E = H\mathbb{F}_2$, and look at $\pi = C_2 \subset \Sigma_2$. 

In this case, $H\mathbb{F}_2\ast BC_2 = \mathbb{F}_2\{b_0, b_1, \ldots\}$. Define $P_\pi(x)$ to be the map $S^{p_i} \wedge B_{\pi_k} \wedge E \to D_\pi X$. Define $P_\pi(x) = \mu_* P_\pi(x)$.

**Definition 1.1.** Define $\overline{Q}^s(x)$ to be the piece that raises degree by exactly $s$. That is, $\overline{Q}^s(x) = b_{s-|x|}^* P_\pi(x)$. (If $s - |x| < 0$, then this is zero.) Define $Q^s(x) = \mu_* \overline{Q}^s(x)$.

**Remark 1.2.** For $1 \in E_0 B\pi$, $1^* P_\pi(x) = x^p$.

**Example 1.3.** Recall $\pi_* E^Z = E^*(Z)$. In this case, $Q^s(x) = \begin{cases} Sq^s & s > 0 \\ 1 & s = 0 \\ 0 & s < 0. \end{cases}$

So the Steenrod algebra is a quotient by $Q^0 - 1$.

The part generated by $Q^s$ (for $s \geq 0$) is called the Dyer-Lashof algebra.

This has the following properties:

- stable
- additive: $\overline{Q}^s(x + y) = \overline{Q}^s(x) + \overline{Q}^s(y)$
- Cartan formula: $Q^s(xy) = \sum_{i+j=s} Q^i(x)Q^j(y)$
- Adem relations
- Nishida relations

The idea is we want to show that some version of these relations holds in $\pi_*(D_2 X)$ for all $X$, by universal example. Then check some diagram holds for $D_2$. (You don’t want to reduce all the way to $H\mathbb{F}_2$, because that’s not universal.)

1.1. **Additivity.**

$$D_2(X \vee Y) = ((X \wedge X) \vee (X \wedge Y) \vee (Y \wedge X) \vee (Y \wedge Y))_{hC_2}$$

$$= D_2 X \vee (X \wedge Y) \vee D_2 Y$$

In particular,

$$\Sigma^d(H\mathbb{F}_2 \vee H\mathbb{F}_2) = \Sigma^{2d} BC_{2+} \wedge H\mathbb{F}_2 \vee H\mathbb{F}_2 \vee \Sigma^{2d} BC_{2+} \wedge H\mathbb{F}_2$$

Watch out: some of these $\wedge$’s are over $H\mathbb{F}_2$!

$$\overline{Q}^d(x + y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y$$

But $x \otimes y = y \otimes x$ because we’re working in homotopy orbits, and we’re working mod 2 so they cancel. So this is $\overline{Q}^d(x) + \overline{Q}^d(y)$. If $s > d$, we get

$$
\begin{array}{ccc}
\pi_d(\Sigma^d H\mathbb{F}_2 \vee H\mathbb{F}_2) & \xrightarrow{Q^s} & \pi_d(\Sigma^d H\mathbb{F}_2) \oplus \pi_d(\Sigma^d H\mathbb{F}_2) \\
\pi_{d+s} D_2(\Sigma^d H\mathbb{F}_2 \vee H\mathbb{F}_2) & \xrightarrow{Q^s} & \pi_{d+s}(D_2(\Sigma^d H\mathbb{F}_2) \oplus D_2(\Sigma^d H\mathbb{F}_2))
\end{array}
$$
The map in the bottom is an isomorphism for degree reasons, because the extra factor \( \Sigma^{2d} H\mathbb{F}_2 \) is gone.

1.2. Multiplicativity (Cartan formula).

\[
\begin{array}{ccc}
D_2(X \wedge Y) & \xrightarrow{\psi} & (X \wedge X) \wedge (Y \wedge Y)_{h\Sigma_2} \\
\downarrow & & \downarrow \\
D_2(X) \wedge D_2(Y) & \xrightarrow{=} & (X \wedge X) \wedge (Y \wedge Y)_{h(\Sigma_2 \times \Sigma_2)}
\end{array}
\]

Claim 1.4. \( \psi(\bar{Q}(x \otimes y)) = \sum_{i+j=s} Q_i(x) \otimes Q_j(y) \)

This immediately descends to the formula without the bars, and to get the internal formula, take \( x = y \) and multiply internally.

I will demonstrate this via universal example again. Set \( X = \Sigma^m H\mathbb{F}_2 \), \( Y = \Sigma^n H\mathbb{F}_2 \), and \( s = m + n + r \). In the diagram,

\[
\begin{array}{ccc}
D_2(X \wedge_{H\mathbb{F}_2} Y) & \xrightarrow{=} & \Sigma^{2m+2n} BC_{2+} \wedge H\mathbb{F}_2 \\
\downarrow & & \downarrow \Delta \\
D_2(X) \wedge_{H\mathbb{F}_2} D_2(Y) & \xrightarrow{=} & \Sigma^{2m+2n} BC_{2+} \wedge BC_{2+} \wedge H\mathbb{F}_2
\end{array}
\]

On elements, on the top, \( \sigma_{2m+2n} b_i \mapsto \sum_{j+k=i} \sigma_{2m} b_j \otimes \sigma_{2n} b_k \) (here \( \sigma \) means suspension). And \( \bar{Q}^{n+n+i}(1) \mapsto \sum \bar{Q}^{m+j}(1) \otimes \bar{Q}^{n+k}(1) \).

1.3. Compositions (Adem relations). We’re now interested in \( D_2(D_2 X) \). We’re supposed to have a diagram

\[
\begin{array}{ccc}
D_2(D_2 X) & \xrightarrow{D_2\mu} & D_2 X \\
\downarrow & & \downarrow \mu \\
D_4 X & \xrightarrow{=} & X
\end{array}
\]

My claim is that \( D_2(D_2 X) = D_{(\Sigma_2 \times \Sigma_2) \times \Sigma_2} X \), where \( \rtimes \) is the wreath product. I had

\[
((X \wedge X) \wedge (X \wedge X))_{h(\Sigma_2 \times \Sigma_2)}
\]

but you can try to combine these as

\[
((X \wedge X) \wedge (X \wedge X))_{h((\Sigma_2 \times \Sigma_2) \times \Sigma_2)}.
\]

I have

\[
B(\Sigma_2 \times \Sigma_2) \xrightarrow{\Delta \times 1} B((\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2)
\]

If I take \((\tau, 1) \in B(\Sigma_2 \times \Sigma_2)\), this maps to \((\tau, \tau, 1) \in B((\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2)\), and that maps to \((1 2)(3 4)\). The other element \((1, \tau) \in B(\Sigma_2 \times \Sigma_2)\) goes to \((1, 1, \tau)\) which goes to \((1 3)(2 4)\). These have the same cycle type, so they are conjugate. In general, \( a \otimes b \) and \( b \otimes a \) go to conjugate things, and conjugation is trivial in group cohomology. This allows you to figure out
\[
HF_2^*(BC_2 \times BC_2) \to HF_2^*(B((C_2 \times C_2) \times C_2)).
\]
So you have to look at the actual formulas for the image of \(a \otimes b\) and \(b \otimes a\). This gives the Adem relations.

2. Jeremy Hahn, October 5

**Theorem 2.1** (Nishida, 1973). Suppose \(x \in \pi_*(S)\) such that \(* > 0\). Then \(x\) is nilpotent.

This was the first in a long line of nilpotence results culminating in Devinatz-Hopkins-Smith. We’ll prove a special case of this, though it’s most of the work:

**Theorem 2.2.** If \(x \in \pi_*(S)\) satisfies \(2x = 0\), then \(x\) is nilpotent.

**Remark 2.3.** If \(x \in \pi_{2k+1}(S)\), then \(x^2 = -x^2\) since \(\pi_* S\) forms a graded commutative ring. So \(x\) is nilpotent. For even degree \(x\), you need a slick argument involving the Kan-Priddy theorem.

**Theorem 2.4** (Mahowald). The free \(E_2\)-algebra with \(2 = 0\) is \(HF_2\).

The main ingredient in this proof is the Nishida relations. This requires more or less the same tools to prove as Nishida’s theorem.

How do you go from Mahowald’s theorem to Nishida’s theorem?

**Theorem 2.5.** Suppose \(A\) is an \(E_2\)-algebra (e.g. \(A = S\)), and suppose \(x \in \pi_n(A)\) is simple 2-torsion. Then \(x : S^n \to A\) is nilpotent iff the composite \(S^n \to A \wedge S \to A \wedge HF_2\) is nilpotent.

**Proof.** I need to show \(A[x^{-1}] = 0\) iff \(A[x^{-1}] \wedge HF_2 = 0\). \(A[x^{-1}]\) is an \(E_2\)-ring (there is a good theory of inverting elements in any \(E_2\) ring). Since \(A[x^{-1}]\) satisfies \(2 = 0\), there is a canonical map from the free \(E_2\)-algebra with \(2 = 0\) (i.e. \(HF_2\)) to \(A[x^{-1}]\). So \(A[x^{-1}]\) is an \(HF_2\)-module, and any such is free – a wedge of \(HF_2\)'s. Since \(\pi_*(HF_2 \wedge HF_2) \neq 0\), and \(HF_2\)-module is zero iff it is zero after smashing with \(HF_2\). \(\square\)

**Definition 2.6.** If \(X\) is a spectrum,
\[
F_{E_2}(X) = S \vee X \vee (X^\wedge 2 \wedge \Sigma_2 \text{Config}_2(\mathbb{R}^2)) \vee (X^\wedge 3 \wedge \Sigma_3 \text{Config}_3(\mathbb{R}^2)) \vee \ldots.
\]
If \(X\) is a connected pointed space,
\[
F_{E_2}(\Sigma^\infty X) = \Sigma^\infty_+ \Omega^2 \Sigma^2 X.
\]
For any spectrum \(X\), the free \(E_2\)-\(HF_2\)-algebra is just \(HF_2 \wedge F_{E_2}(X)\).
**Definition 2.7.** The free $E_2$-algebra with $2 = 0$ (which I will denote by $R$) is the pushout

$$
\begin{array}{ccc}
F_{E_2}(S^0) & \overset{\pi}{\longrightarrow} & S = F_{E_2}(*) \\
\downarrow{\pi} & & \downarrow{S} \\
S & \longrightarrow & R
\end{array}
$$

where $\pi : F_{E_2}(S^0) \to S^0$ is the adjoint to $m : S^0 \to S^0$ (using the free-forgetful adjunction between $E_2$-algebras and spectra).

You could try to compute the pushout by decomposing things using the fact that the forgetful functor is monadic, so it preserves sifted colimits. But this is still tricky.

**Proposition 2.8.**

$$HF_2 \wedge R \simeq HF_2 \wedge F_{E_2}(S^1) \simeq HF_2 \wedge \Sigma_{\infty}^\infty \Omega^2 S^3.$$  
(the last $\simeq$ is by Snaith’s theorem).

This can be seen as a consequence of $R$ being a Thom spectrum. But you don’t need that.

**Proof.** Smash the previous pushout square with $HF_2$:

$$
\begin{array}{ccc}
HF_2 \wedge F_{E_2}(S^0) & \overset{HF_2 \wedge \pi}{\longrightarrow} & HF_2 \\
\downarrow{HF_2 \wedge \pi} & & \downarrow{\pi} \\
HF_2 & \longrightarrow & R \wedge HF_2
\end{array}
$$

In $HF_2$-modules, twice the identity is null, so these two maps $HF_2 \wedge F_{E_2}(S^0) \to HF_2$ are the same. This helps, because the map $\pi$ is just $F_{E_2}$ applied to a map of spaces, namely $0 : S^0 \to \ast$, but $\pi$ does not arise this way. This means that the entire pushout square above can be viewed as $HF_2 \wedge F_{E_2}$ applied to

$$
\begin{array}{ccc}
S^0 & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & S^1
\end{array}
$$

\[\square\]

Our next goal is to compute $H_\ast(\Omega^2 S^3)$. More generally, one might want to compute the homology of iterated loop spaces of spheres.

There is a general sense in which this is saying something about power operations and the Dyer-Lashof algebra.

Suppose $A$ is an $E_n$-algebra in spectra, and $x \in H_i(A)$. Then one obtains an element $Q_x \in H_j(A)$ for every $Q \in H_j(\Omega^n S^{n+i})$. So knowing $H_j(\Omega^n S^{n+i})$ constructs natural operations on $H_\ast(A)$.  

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So $x$ is a map $S^i \to H\mathbb{F}_2 \wedge A$, which induces a map $H\mathbb{F}_2 \wedge F_{E_n}(S^i) \to H\mathbb{F}_2 \wedge A$, and we have a map $Q : S^i \to H\mathbb{F}_2 \wedge \Omega^n S^{n+i}$. But $H\mathbb{F}_2 \wedge \Omega^n S^{n+i} = H\mathbb{F}_2 \wedge F_{E_n}(S^i)$, so you can compose those things.

**Theorem 2.9.** If $A$ is an $E_n$-algebra in $H\mathbb{F}_2$-modules, there are operations $Q^r : \pi_\ast A \to \pi_{\ast+r}(A)$ satisfying

1. $Q^2(x_i) = x_i^2$ for $x_i \in \pi_i(A)$
2. $Q^r(x_i) = 0$ for $r < i$ or $r > i + n - 1$.
3. The $Q^r$ satisfy Adem relations and the Cartan formula.

Condition (2) becomes vacuous as $n > 0$, and you get exactly the structure that Hood talked about last week.

We have $H_\ast(R) \cong H_\ast(\Omega^2 S^3) \cong H_\ast(F_{E_2}(S^1))$. Since $R$ is a ring, this is a ring. These have $\alpha$ operations:

There is a natural map $f : R \wedge H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The computation of $H_\ast(\Omega^2 S^3)$ shows that $\pi_n$ of both sides of this arrow are equal for every $n$. So we have to show that $\pi_\ast f$ has no kernel.

If $X$ is any spectrum, then $H_\ast X$ has a degree-lowering action of the Steenrod algebra on the right. This structure does not exist on the homotopy of every $H\mathbb{F}_2$-module. If $X$ is an $E_n$-algebra, then $H_\ast X$ has Dyer-Lashof operations. This gives two strategies to finish Mahowald’s theorem.

1. Compute the action of the Steenrod algebra on $H_\ast(R, \mathbb{F}_2)$. This relies on knowing $R$ as a spectrum, and we just know that as a pushout.
2. Compute the Dyer-Lashof action on $H_\ast(H\mathbb{F}_2, \mathbb{F}_2)$.

My plan is to tell you how (1) works, and go through the proof of (2). (1) is complicated, but maybe more useful in other contexts.

If you pursue the project of understanding Steenrod operations on $E_n$-algebras, you come to:
Theorem 2.10 (Nishida relations). For $A$ an $E_n$-algebra, $t \geq k$, and $y \in H_\ast A$,

$$(Q^t y) \text{Sq}^k = \sum_i \binom{t-k}{k-2i} Q^{t-k+i} (y \text{Sq}^i).$$

Look up Haynes Miller’s notes on the Nishida relations.

From the picture, almost all the classes are a product of classes or involves a $Q$. The only thing that isn’t is $\alpha$. So if we understand the action on $\alpha$ (which is what distinguishes $R$ from $\Omega^2 S^3$, where there is no Steenrod operation on $\alpha$), we understand everything using the Cartan formula. You can also show that there is a Sq$^1$ from 1 to $\alpha$.

Now talk about the Dyer-Lashof action on $H_\ast(HF_2, F_2)$. As a ring, Milnor proved that $H_\ast(HF_2) \cong F_2[\xi_1, \xi_2, \ldots]$ where $|\xi_i| = 2^i - 1$.

**Theorem 2.11.** For each $i \geq 1$, $Q^{2i}(\xi_i) = \xi_{i+1}$.

**Proof.** Consider the map $x : \mathbb{RP}^\infty \to \Sigma HF_2$ picking out the generator. In homology, $x_* (b_{2i}) = \xi_i$. (This might be the definition of $\xi_i$.) There is a sequence of $C_2$-equivariant maps

$$\mathbb{RP}^\infty \xrightarrow{\Delta} \mathbb{RP}^\infty \times \mathbb{RP}^\infty \xrightarrow{(x,x)} (S^1 \wedge HF_2) \wedge (S^1 \wedge HF_2) \cong S^{1+i} \wedge HF_2 \wedge HF_2.$$

Apply homotopy orbits to this diagram:

$$\mathbb{RP}^\infty \times \mathbb{RP}^\infty \xrightarrow{\Delta(hC_2)} (\mathbb{RP}^\infty \times \mathbb{RP}^\infty)_{hC_2} \cong BD_8 \xrightarrow{(x,x)} (S^{1+i} \wedge HF_2 \wedge HF_2)_{hC_2} \cong \Sigma \mathbb{RP}^\infty \wedge (HF_2 \wedge HF_2)_{hC_2}$$

Apply $m \circ (x,m)$ (where $m$ is the multiplication) to map this to $\Sigma^2 HF_2$.

This fits into a commutative triangle

$$\begin{array}{ccc}
\mathbb{RP}^\infty \times \mathbb{RP}^\infty & \xrightarrow{m \circ (x,x)} & BD_8 \\
\downarrow & & \downarrow \\
& \Sigma^2 HF_2 & 
\end{array}$$

This sends $b_1 \otimes b_{2i+1} \in \mathbb{RP}^\infty \times \mathbb{RP}^\infty \mapsto b_1 \otimes b_{2i} \otimes b_{2i} \in H_\ast BD_8 \mapsto Q^{2i}(\xi_i) \in H_\ast (\Sigma^2 HF_2)$, but the other way around it sends $b_1 \otimes b_{2i+1} \mapsto \xi_{i+1} \in H_\ast (\Sigma^2 HF_2)$, so $\xi_{i+1} = Q^{2i}(\xi_i)$. $\square$

3. Robert Burkland, October 12: Power operations in $K$-theory

Today we will be working with $K$-theory and $p$-adic $K$-theory $K_p$ (all $K$-theory today will be complex $K$-theory). We will be computing

$$K_0(\text{Free}_{E_\infty} S^0) \cong \mathbb{Z}[\tau_1, \ldots]$$

$$K_{p_0}(\text{Free}_{E_\infty} S^0) \cong \mathbb{Z}_p [x, \theta x, \theta^2 x, \ldots]$$

where Free$_{E_\infty} S^0$ is the free $E_\infty$ algebra on the sphere. The $p^\text{th}$ Adams operation is $\psi x = x^p + p\theta x$.  

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We will use the fact that
\[ \text{Free}_{E_\infty} S^0 \simeq \bigvee_{n \geq 0} B\Sigma_n \]
By the Atiyah-Segal completion theorem,
\[ K^0 B\Sigma_n \cong R\Sigma_n \]
(where \( R(\cdot) \) is the representation ring) so it suffices to prove
\[ \bigoplus_{n \geq 0} R\Sigma_n \cong \mathbb{Z}[\tau_1, \tau_2, \ldots] \]
This has a product given by \( i : \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m} \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be a partition of \((1, \ldots, k)\). This gives a map \( \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \ldots \times \Sigma_{\lambda_m} \hookrightarrow \Sigma_k \). This takes the product of trivial representations \( 1 \boxtimes 1 \boxtimes \ldots \boxtimes 1 \) to \( \rho_\lambda \). (Note: this is an exterior product: as a vector space it’s just the tensor product of the vector spaces, but the action comes from outside.)

Fix \( k \), so we’re working with \( \Sigma_k \). Let \( V \) be a complex vector space with chosen isomorphism \( V \cong \mathbb{C}^n \). \( V \otimes^k \) has an action of \( \Sigma_k \) and \( GL_n \), and we can think of it as an element of \( R(\Sigma_k \times GL_n) \cong R(\Sigma_k) \otimes R(GL_n) \). This can be written as a sum of irreducible representations:
\[ \Delta = \bigoplus_{\pi} \pi \boxtimes V_\pi. \]

Pick a diagonal matrix \( \begin{pmatrix} t_1 & \cdots & t_n \end{pmatrix} \in GL_n \). This gives rise to a map \( D : R\Sigma_k \otimes RGL_n \to R\Sigma_k \otimes \text{Sym}(t_1, \ldots, t_n) \). Abuse notation and also let \( \Delta \) denote the image of \( \Delta \) under this map. By freeness, this gives rise to a map \( D : R\Sigma_k^\vee \to \text{Sym}(t_1, \ldots, t_n) \). For example, \( D(1) \) is the coefficient of \( x^k \) in \( \frac{1}{(1-t_1x)(1-t_2x)\ldots(1-t_mx)} \). (This isn’t just the elementary symmetric functions: you have \( x_1^2 + x_1 x_2 + x_2^2 \), not just \( x_1 x_2 \).)

From group theory, the rank of \( R\Sigma_k^\vee \) is the number of conjugacy classes, which is the number of partitions of \( k \). This is also the dimension of \( \text{Sym}^k(t_1, \ldots, t_n) \) (space of symmetric polynomials).

It suffices to show that \( D \) hits things of degree \( k \) and is surjective on \( \text{Sym}^k \); by dimension counting, this will show it’s an isomorphism. (Note that \( \text{Sym}^k(t_1, \ldots, t_n) \to \text{Sym}^k(t_1, \ldots, t_{n+1}) \) is an isomorphism if \( n > k \).)

Let \( R_* = \bigoplus R\Sigma_k^\vee \). We want to show that \( R_* \cong \text{Sym}(t_1, \ldots) \); it suffices to show that this is a surjection.
\[ V^\otimes n \otimes V^\otimes m \cong \bigoplus_{\pi, \tau} (\pi \boxtimes V_\pi) \otimes (\tau \boxtimes V_\tau) \]
\[ = \bigoplus_{\tau} (\pi \otimes \tau) \boxtimes (V_\pi \otimes V_\tau) \]
This shows that \( \pi \cdot \tau \mapsto D(\pi) \cdot D(\tau) \).

\( \tau_i \) corresponds to the trivial representation of \( \Sigma_i \). From that, we get all of the representations that we wanted. By the Atiyah-Segal completion theorem, we have computed all the power operations that we set out to compute in the beginning.
3.1. $p$-completed $K$-theory. We know

$$K_{p,*} B\Sigma_n \cong \text{Hom}(K^*_{p} B\Sigma_n, \mathbb{Z}_p)$$

(these are continuous homs). By the UCT, we know that the RHS $\subset \text{Hom}(K^0 B\Sigma_n, \mathbb{Z})$. All we have to do is calculate which Homomorphisms in $\text{Hom}(K^0 B\Sigma_n, \mathbb{Z})$ are continuous w.r.t. the $p$-adic topology.

From a result of Atiyah, $\text{rank}(K^0_{p}BG) = \text{the number of conjugacy classes of } p$-power order. We want to show:

1. $\tau_p^k$ is continuous for the $p$-adic topology;
2. anything that includes $\tau_j$ where $j$ is not a $p$-power is not continuous.

So we’re looking at $R\Sigma_m^\vee \otimes \mathbb{Z}/p^r$. Now we’re working in the dual, so $\tau_i$ means the dual basis element to the $\tau_i$ above. So we want to show that $x \in R\Sigma_m^\vee \otimes \mathbb{Z}/p^r$ is continuous iff it sends $I^n$ to 0 for $n \gg 0$.

We’re working with a Hopf algebra. Let $\Delta$ denote the coproduct map $\Delta : (R\Sigma_m^\vee \otimes \mathbb{Z}/p^r) \rightarrow (R\Sigma_m^\vee \otimes \mathbb{Z}/p^r) \otimes (R\Sigma_m^\vee \otimes \mathbb{Z}/p^r)$. The condition to be continuous is the same as asking $\Delta^n(x)$ to annihilate $I^\otimes n$ for $n \gg 0$.

I claim the annihilator of $I$ is just $\tau_1^n$. The $m^{th}$ power of $\tau_1$ is evaluation at the unit for the monoidal thing (you induce up from the 1-dimensional thing for an empty group); so it’s the permutation representation, and evaluating at that yields zero for something that’s zero-dimensional. (If you take the regular representation and tensor up over any other representation, you get a permutation representation for a free set.)

Let $\eta$ be the map $(R\Sigma_m^\vee \otimes \mathbb{Z}/p^r) \rightarrow (R\Sigma_m^\vee \otimes \mathbb{Z}/p^r)$ sending $\tau_1^n \mapsto 0$ (and is the identity on other basis elements). Then $x$ is continuous if $(\eta^\otimes n)(\Delta^n x) = 0$ for $n \gg 0$.

Then $\Delta \sigma_m = \sigma_m \otimes \sigma_m$ ($m^{th}$ powers). We have $\sigma_m = \sum_{d|n} d \cdot \tau_1^{n/d}$. You can show that $\sigma_p^k = 1 \cdot \tau_1^{p^k} + \cdots + p^k \cdot \tau_1^1$. Now use an inductive argument.

Now we show that no other things are continuous. There are maps $RG \rightarrow RG^\wedge = \mathbb{Z} \oplus \mathbb{I}^\wedge \rightarrow \mathbb{Z} \oplus \mathbb{I}_p^\wedge$. There is a theorem of Atiyah that gives a factorization

$$RG \rightarrow RG^\wedge \rightarrow \mathbb{Z} \oplus \mathbb{I}_p^\wedge$$

Now analyze what the kernel is. See Hodgkins paper “Character of well-known spaces I” (there is no II).

We don’t yet know that there’s one operation $\theta$ that generates all of the polynomial generators. To do that, look at $B\Sigma_p$ (because that’s where $\theta$ lives), and do a quick calculation with the AHSS to show that everything you can hit comes from $\theta$.  


We have a map $B\Sigma_{p-1} \to B\Sigma_p$, and $K(1)$-locally, $B\Sigma_{p-1} \simeq S^0$. There’s a transfer map $B\Sigma_{p+} \to B\Sigma_{p+1}$ and a map $B\Sigma_{p+} \to S^0$. $K(1)$-locally, I claim $B\Sigma_{p+} \simeq S^0 \vee S^0$. K\(1\)-locally, there’s a transfer map $B\Sigma_{p+} \to B\Sigma_{p+1}$ and a map $B\Sigma_{p+} \to S^0$. K\(1\)-locally, I claim $B\Sigma_{p+} \simeq S^0 \vee S^0$.

![Diagram](image)

Maybe the transfer is kind of nontrivial. We have $\theta(1) = 0$, $\theta(V_p) = -1$, $\psi(1) = 1$, $\psi(V_p) = 0$. Tensor up $p$ times to get $\psi(1) = 1 = 1 + p \cdot \theta(1)$, $p \cdot p = 0$, $v \otimes p \cdot p \theta(v)$.

To get that the $\theta$’s generate everything, look at the AHSS. Suppose $X$ is a spectrum, and $K_{p,*}(X)$ is even, free, and finite rank. Then the AHSS (or Atiyah-Hirzebruch Serre spectral sequence) is

$$H_*(B\Sigma_p, K_{p,*}(X \otimes^p)) \implies K_{p,*}(X_{h\Sigma_p})$$

By evenness, this collapses. Because it’s free, there’s a Künneth formula. So we know that $K_{p,*}(X \otimes^p) \cong (K_{p,*}(X)) \otimes^p$. Everything in $K_{p,*}(X_{h\Sigma_p})$ is a sum of trivial things, and things which become free when we restrict to $G_p$. In particular, we get that $K_{p,*}(X_{h\Sigma_p}) \cong \text{Sym}^p(K_{p,*}(X)) \oplus \theta(K_{p-1}(X))$. Now plug in $X = S^0$. You can do the same thing where you replace $p$ by $p^n$, so this yields the fact that $\theta$ really is enough to generate everything. More generally, power operations are generated by degree $p$.

4. October 19, Denis Nardin: $p$-adic homotopy theory

We need a couple of facts about $E_\infty$-algebras. These can all be found in EKMM.

1. For every $E_\infty$ algebra $E$, there exists a symmetric monoidal category $(\text{Mod}_E, \otimes_E)$.
2. For every commutative ring $R$ we have an $E_\infty$-algebra $HR$, where $\text{Mod}_{HR} \cong D(R)$ (the derived category of the ring $R$).
3. For all $M, N \in \text{Mod}_R$, there is a Künneth spectral sequence
   $$\text{Tor}_{p,q}^{\pi_* R}(\pi_* M, \pi_* N) \implies \pi_{p+q}(M \otimes_R N)$$
   and a universal coefficient spectral sequence
   $$\text{Ext}_{p,q}^{\pi_* R}(\pi_* M, \pi_* N) \implies \pi_{p+q}F_R(M, N).$$
   These follow formally from having a nice category of modules (every module has a resolution by free modules).
4. In $E_\infty$-algebras over $R$, pushouts are $\otimes$’s.

For every commutative ring $R$, we can define the $E_\infty$-algebra over $R$

$$C^*(X; R) = \lim_X HR.$$ 

We have a functor $\text{Top} \to \text{EAlg}(R)^{op}$ which has an adjoint. From now on, “algebra” means $E_\infty$-algebra.
Let $A$ be an $R$-algebra. Then
\[
\text{Map}_{E\infty}(A, C^*(X; R)) = \text{Map}_{E\infty}(A, \lim X H R) = \lim_X \text{Map}_{E\infty}(A, H R)
\]
so
\[
\text{Map}_{E\infty}(A, C^*(X; R)) = \text{Map}(X, \text{Map}_{E\infty}(A, H R)) = \lim_X \text{Map}_{E\infty}(A, H R)
\]
and $\text{Map}_{E\infty}(\cdot, H R)$ is the adjoint. In particular, it takes pushouts to pullbacks.

For rational homotopy theory, consider the case $R = \mathbb{Q}$. This gives a fully faithful embedding of rational spaces into $E\infty$-algebras over $\mathbb{Q}$. We will try to do the same thing for $p$-local spaces. You might try to take $R = \mathbb{F}_p$, but that doesn’t work – you don’t get a fully faithful embedding. Instead, you want to take $R = H\mathbb{F}_p$. The idea is that the Dyer-Lashof action should be thought of as a Frobenius action.

To sum up, we have a functor $\text{Top} \to \text{EAlg}(\mathbb{F}_p)^{op}$ sending $X \mapsto C^*(X; \mathbb{F}_p)$. We want to know for which spaces $X$ this is fully faithful.

There is an abstract nonsense thing that helps here.

**Definition 4.1.** $X$ is $p$-resolvable if $X \to \text{Map}_{E\infty}(C^*(X, \mathbb{F}_p), H\mathbb{F}_p)$ (the unit of the adjunction above) is an equivalence.

Then for formal reasons, the above functor is fully faithful on $p$-resolvable spaces. But we don’t know what they are yet.

**Proposition 4.2.** Suppose there is an equivalence $X \to \lim_n X_n$ where $X_n$ is $p$-resolvable and $H^*(X; \mathbb{F}_p) \leftarrow \text{colim} H^*(X_n, \mathbb{F}_p)$. Then $X$ is $p$-resolvable.

The idea is to apply this to Postnikov towers. (Actually, you don’t need the map $X \to \lim_n X_n$ to be an equivalence.)

**Proof.** Let $UA = \text{Map}_{E\infty}(A, H R)$. We have a map $X \to UC^*(X)$, and we can upgrade that to a tower
\[
\begin{array}{cccccc}
X & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \ldots \\
\downarrow & & \sim & & \sim & & \\
UC^*(X) & \longrightarrow & UC^*(X_n) & \longrightarrow & UC^*(X_{n-1}) & & \\
\end{array}
\]
Now use the fact that $C^*(X) = \text{colim} C^*(X_n)$.

**Theorem 4.3** (Eilenberg-Moore spectral sequence). Suppose
\[
\begin{array}{cccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \ \\
Z & \longrightarrow & W
\end{array}
\]
is a homotopy pullback such that for all \( v \in W \), \( \pi_1(W, w) \) acts nilpotently on \( H^*(Y_W; K) \) (here \( Y_W \) is the fiber). Then you get an equivalence
\[
C^*(X; K) \simeq C^*(Y; K) \otimes_{C^*(X; K)} C^*(Z; K).
\]
From this you get a spectral sequence that converges

See \( X, Y, \) and \( Z \) as spaces with a fibration to \( W \). Construct a local system \( w \mapsto C^*(Y_W) \).

Then \( H^*Y \) is the global sections of this local system.

We have
\[
\begin{array}{ccc}
X & \rightarrow & UC^*(X) \\
\downarrow & & \downarrow \\
UC^*(Y) = Y & \rightarrow & UC^*(W) = W \\
\downarrow & & \downarrow \\
Z = UC^*(Z) & \rightarrow & UC^*(W) = W
\end{array}
\]
Both the inner square (by applying a functor) and outer squares (by assumption) are pullback squares, so \( X \rightarrow UC^*(X) \) is an equivalence.

Warning: we will use homological grading, not cohomological grading.

\( C^*(K(\mathbb{F}^p, n); \mathbb{F}_p) \) has a fundamental class \( \iota_{-n} \) of degree \( -n \). So there is a map \( \text{Free}(S^{-n}) \rightarrow C^*(K(\mathbb{F}^p, n); \mathbb{F}_p) \). We know that \( Q^0 \iota_{-n} = \iota_{-n} \). This is a highly nontrivial fact. This means that the composition \( S^{-n} \xrightarrow{Q^0-1} \text{Free}(S^{-n}) \rightarrow C^*(K(\mathbb{F}^p, n); \mathbb{F}_p) \) is nullhomotopic, so we can choose a specific nullhomotopy
\[
\begin{array}{ccc}
\text{Free}(S^{-n}) & \rightarrow & C^*(K(\mathbb{F}^p, n); \mathbb{F}_p) \\
\uparrow & & \uparrow \\
\text{Free}(S^{-n}) & \rightarrow & H^P
\end{array}
\]

**Main theorem 4.4.** This is a pushout square.

**Corollary 4.5.** \( K(\mathbb{F}_p, n) \) is \( p \)-resolvable.

**Proof.** \( U \text{Free}(S^{-n}) = \text{Map}_{E_{\infty}}(\text{Free}(S^{-n}), \mathbb{F}_p) = \text{Map}_{Sp}(S^{-n}, H^P) = K(\mathbb{F}_p, n) \). I claim there is a pushout (pullback?) square
\[
\begin{array}{ccc}
UC^*(K(\mathbb{F}_p, n)) & \rightarrow & K(\mathbb{F}_p, n) \\
\downarrow & & \downarrow \varphi^{-1} \\
* & \rightarrow & K(\mathbb{F}_p, n)
\end{array}
\]
Here \( \varphi \) is the Frobenius. (Why is this what this map is? You need to compute what it does on homotopy groups.) \( \square \)
We have SES’s
\[ 0 \to \mathbb{F}_p \to \mathbb{F}_p^{\frac{1}{p^{-1}}} \to \mathbb{F}_p \to 0 \]
\[ K(\mathbb{Z}/p^{i+1}, n) \to K(\mathbb{Z}/p^i, n) \to K(\mathbb{Z}/p^i, n+1). \]
Use induction to show that \( K(\mathbb{Z}/p^i, n) \) is \( p \)-resolvable.

First, we will show that \( \pi_* \text{Free}(S^{-n}) \to \pi_* \text{Free}(S^{-n}) \) is a free algebra map (so the latter is a free module, and the Tor in the Künneth spectral sequence collapses), and \( \pi_* \text{Free}(S^{-n}) \otimes \pi_* \text{Free}(S^{-n}) \to H^*(K(\mathbb{F}_p, n)) \) is an equivalence.

From now on, \( p = 2 \). (There are analogous results for \( p > 2 \), but the formulas are different.)

We have the extended Dyer-Lashof algebra (the “big Steenrod algebra”) with generators \( \{Q^i\}_{i \in \mathbb{Z}} \) and relations
\[ Q^r Q^s = \sum_{i+j=r+s} \binom{j-1-s}{2j-r} Q^i Q^j \text{ for } r > 2s. \]

We have admissible elements \( Q^I = Q^{i_1} \cdots Q^{i_r} \) where \( i_j \leq 2i_{j+1} \). Also
\[ \pi_* \text{Free}(S^{-n}) = \mathbb{F}_p[Q^I x : I \text{ is admissible}, e(I) > -n] \]
(where \( e(I) = i_1 - \sum_{j \geq 2} i_j \) means excess). We also know
\[ H^*(K(\mathbb{F}_p, n)) = \mathbb{F}_p[Q^I x : I \text{ admissible}, e(I) > -n, I = (i_1, \ldots, i_r), i_r < 0]. \]

\( Q^0 - 1 \) is the map \( Q^I x \mapsto Q^I Q^0 x - Q^I x. \) Define \( B_n \) to be the free \( \mathbb{F}_p \)-module on \( \{Q^I x : e(I) \geq -n\} \). This is endowed with the operation \((\cdot)^2 : B_n \to B_n \) sending \( Q^I x \mapsto Q^{|I|-n} Q^I x. \) A module with such an operation is called a restricted module. (Recall \( Q^{[m]} y = y^2. \))

The enveloping algebra \( UM \) over such a restricted algebra \( M \) is the free commutative algebra on \( M \) under the relation \( m^2 = m \cdot m \). I claim \( \pi_* \text{Free}(S^{-n}) = UB_n. \)

Let \( \mathcal{A}_n \) be the free \( \mathbb{F}_p \)-module spanned by \( \{Q^I x : e(I) \geq -n, I < 0\} \). We have to define the “squaring” operation:
\[ (Q^I x)^2 = \begin{cases} Q^{|I|-n} Q^I x & \text{if } n = 0, I = \emptyset \\ x & \text{if } n > 0 \end{cases} \]

We have \( U \mathcal{A}_n = H^*(K(\mathbb{F}_p, n), \mathbb{F}_p) \). We have a square of restricted algebras
\[ \begin{array}{ccc} B_n & \to & B_n \\ \downarrow & & \downarrow \\ 0 & \to & \mathcal{A}_n \end{array} \]

The map \( B_n \to \mathcal{A}_n \) is
\[ Q^I x \mapsto \begin{cases} Q^I x & I < 0 \\ Q^{i_r} x & I = (i_1, \ldots, i_r, 0, 0, \ldots, 0), i_r < 0 \\ 0 & i_r > 0. \end{cases} \]
When you apply the enveloping algebra functor, you get the square that’s supposed to be a pushout square.

\[
\begin{array}{ccc}
\pi_* \text{Free}(S^{-n}) & \longrightarrow & \pi_* \text{Free}(S^{-n}) \\
\downarrow & & \downarrow \\
\mathbb{F}_p & \longrightarrow & H^*(K(\mathbb{F}_p, n))
\end{array}
\]

But you also need a flatness condition. We will claim that the first square is split exact. Clearly there is an injection \( A_n \to B_n \).

**Proposition 4.6.** \( \ker(B^* \to A^*) = B^*(Q^0 - 1) \).

**Proof.** Step 1: if \( r > 0 \), \( Q^r(Q^0)^r = 0 \). By induction on \( r \) using the Adem relations:

\[
Q^1 Q^0 = \sum_{i+j=1} \binom{j-1}{2j-1} Q^i Q^j = 0
\]

We have \( Q_j Q^0(Q^0)^{r-1} = \sum_{i+j=r}(j-1)Q^i Q^j(Q^0)^{r-1} \) for \( 0 < j < r \). By induction, \( Q^j(Q^0)^{r-1} = 0 \).

Step 2: \( \ker(B^* \to A^*) = \mathbb{F}_2(Q^I x, Q^I Q^0 x - Q^I x) \) where \( I = \{i_1, \ldots, i_r\}, \ i_r > 0 \). First suppose \( i_r > 0 \). We know \( Q^I x = Q^I (1 - (Q^0)^{i_r}) x = -Q^I (1 + \cdots + (Q^0)^{i_r-1})(Q^0 - 1)x \). \( \square \)

We want to construct a section of \( B_n \to B_n \to A_n \), specifically a map \( f : B_n \to B_n \) such that \( f(Q^I Q^0 x - Q^I x) = Q^I x \). Recall \( B_n \) is free.

First suppose \( I = (i_1, \ldots, i_r) \) where \( i_r < 0 \). Then those are not in the image anyway, so define \( f(Q^I x) = 0 \).

Now suppose we’re in the case where \( I = (i_1, \ldots, i_r) \) where \( i_r > 0 \). Then define \( f(Q^I x) = Q^I (1 + Q^0 + (Q^0)^2 + \cdots) \). This makes sense because the last term is positive, so the sum is finite.

Suppose \( I = (i_1, \ldots, i_r, 0) = (I', 0) \). Then send \( f(Q^I x) = Q^{I'} x + f(Q^{I'} x) \).

We only need to check for \( I = (i_1, \ldots, i_r) \) with \( i_r > 0 \). Need to check using the Adem relations that \( Q^I Q^0 x = \sum Q^I \). We have \( f(Q^I Q^0 x) = Q^I IQ^0(1 + (Q^0) + (Q^0)^2 + \cdots)x \) and \( f(Q^I x) = Q^I (1 + Q^0 + \cdots)x \). So \( f(Q^I Q^0 x - Q^I x) = Q^I (1 - Q^0)(1 + Q^0 + \cdots)x = Q^I x \), and hence \( f \) is a section.

You should also check it’s a map of restricted modules, but that’s easy.

5. **October 26, Danny Shi: The Hopkins-Miller Theorem**

Outline:

(1) Introduction: theorem statement
(2) Obstruction theory using the Bousfield-Kan spectral sequence

(3) Computations

The idea is that you can put an $A_\infty$ structure on $E_n$. But I have to be more specific on the exact statement.

5.1. Introduction. Let $\text{FGL}$ be the category of formal group laws; the objects are pairs $(K, \Gamma)$ where $K$ is a perfect field of characteristic $p$ and $\Gamma$ is a height $n$ formal group law. A morphism $(K_1, \Gamma_1) \to (K_2, \Gamma_2)$ is a map $i : K_1 \to K_2$ along with an isomorphism of formal group laws $\Gamma_1 \cong i^* \Gamma_2$.

Consider the universal deformation of $(K, \Gamma)$; this is a formal group law over $E(K, \Gamma) = W(K)[[u_1, u_2, \ldots, u_{n-1}][u^\pm]]$, where $|u| = -2$ (we want to get something 2-periodic). There is a map $F : MU_* \to E(K, \Gamma)$ classifying this, and we hope to make a homology theory out of it.

**Fact 5.1.** $F$ is Landweber exact, so you get a homology theory $E_{(K, \Gamma)}$.

$E(K, \Gamma)$ is a commutative ring. By the Landweber exact functor theorem, $E_{(K, \Gamma)}$ is a multiplicative homology theory. So the spectrum representing this homology theory is homotopy commutative. Can we put more structure on these spectra? The Hopkins-Miller theorem (what we'll talk about) gives an $A_\infty$ structure; Goerss-Hopkins gives an $E_\infty$ structure.

**Theorem 5.2** (Hopkins-Miller). There exists a lift

\[
\begin{array}{ccc}
A_\infty\text{-ring} & \xrightarrow{\pi} & \text{homology theories} \\
\text{FGL}^{\text{op}} & \xrightarrow{} & \text{FGL}^{\text{op}}
\end{array}
\]

where $\pi$ is the forgetful functor, and the bottom map sends $(K, \Gamma) \mapsto E_{(K, \Gamma)}$. Furthermore, suppose we start with $(K_1, \Gamma_1)$ and $(K_2, \Gamma_2)$ in $\text{FGL}^{\text{op}}$. Then

\[
\text{Map}_{A_\infty\text{-ring}}(E_{K_1, \Gamma_1}, E_{K_2, \Gamma_2}) \to \text{FGL}((K_2, \Gamma_2), (K_1, \Gamma_1))
\]

is an equivalence. Here $\text{Map}$ is the mapping space – what this means is that the mapping space is actually homotopy equivalent to a set with the discrete topology (the RHS).

What is this good for? We can construct homotopy fixed point spectra $E^h_G$ (where $G$ is the Morava stabilizer). You can lift the action to an action by $A_\infty$-maps, so you can lift the action to spectra (not just the homotopy category). The second application is that you can construct power operations.

Reference: Charles Rezk’s notes on the Hopkins Miller theorem. It’s not always clear what the order is you prove things. He took two Lubin-Tate theories $F, E$ and assumed you can put $A_\infty$-structures on them, and then computed $\text{Map}_{A_\infty}(F, E)$. This turned out to be homotopy discrete. There are obstructions for this, and those turn out to be zero. Then he turns to the original problem, namely how to put an $A_\infty$-ring structure on $E$. The obstructions that you need to put an $A_\infty$-structure on $E$ have actually already been computed in the first step.
computing the mapping space). This ends up being equivalent to computing \( \text{Map}_{A_\infty}(E, E) \) (which is why he did things in this order).

5.2. Spectral sequence. We want to use a spectral sequence, and for that we need a filtration. Let \( C \) be the \( A_\infty \)-operad. Start by resolving \( F \) by the free \( C \)-algebra on \( F \). What I mean is

\[
F \xleftarrow{} CF \xleftarrow{} CCF \xleftarrow{} \ldots
\]

where \( CF \simeq S^0 \vee F \vee (F \wedge F) \vee (F \wedge F \wedge F) \vee \ldots \) is the free \( E_\infty \)-algebra on \( F \). Let \( F_* = CF \xleftarrow{} CCF \ldots \); this is a pointed simplicial object.

Let \( Y^* = \text{C-alg}(C^{n+1}F, E) \); this is a cosimplicial space. A cosimplicial space has a totalization (dual to the geometric realization of a simplicial space). There is a tower of fibrations called the “Tot tower”

\[
\text{Tot}^0 Y^* \xleftarrow{} \text{Tot}^1 Y^* \xleftarrow{} \text{Tot}^2 Y^* \xleftarrow{} \ldots
\]

whose colimit is the totalization \( \text{Tot} Y^* \).

Fact 5.3. In our case, it turns out that \( \text{Tot} Y^* \simeq \text{Map}_{A_\infty}(F, E) \).

So the Tot tower gives a filtration on the thing we want to compute, so we get a spectral sequence to compute it, called the Bousfield-Kan spectral sequence. This has

\[
E_2 = \pi^s(\pi_t Y^*) \implies \pi_{t-s}(\text{Tot} Y^*, f)
\]

where \( f \) is the basepoint (an \( A_\infty \)-map \( F \to E \) that we assume we’ve already chosen at this point) and \( \pi^s \) is cohomotopy: if we fixed \( t \), then there are maps

\[
\pi_t Y^0 \xlongleftarrow{} \pi_t Y^1 \xlongleftarrow{} \pi_t Y^2 \xlongleftarrow{} \ldots
\]

Then cohomotopy is the homotopy of this simplicial object. This has differential

\[
d_r : E^s_{r,t} \to E^s_{r+1,t+r-1}.
\]

Here is what this looks like with the Adams grading \((t - s, s)\):

<table>
<thead>
<tr>
<th>( \pi_1 Y^4 )</th>
<th>( \pi_0 Y^3 )</th>
<th>( \pi_1 Y^3 )</th>
<th>( \pi_0 Y^2 )</th>
<th>( \pi_1 Y^2 )</th>
<th>( \pi_0 Y^1 )</th>
<th>( \pi_1 Y^1 )</th>
<th>( \pi_2 Y^1 )</th>
<th>( \pi_0 Y^0 )</th>
<th>( \pi_1 Y^0 )</th>
<th>( \pi_2 Y^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -3 )</td>
<td>( -2 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 3 )</td>
<td>( 4 )</td>
<td></td>
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</tr>
</tbody>
</table>

We’re computing the homotopy groups of a space, so at the \( E_\infty \)-page, we only have things with \( t - s \geq 0 \) left.

This spectral sequence is really confusing. First, it is a fringed spectral sequence: at the \( t = 1 \) diagonal, we get groups (but not necessarily abelian), and at the \( t = 0 \) diagonal, we only get pointed sets. Of all the pointed sets, \( \pi_0 Y^0 \) is important: \( E_2^{0,0} = \text{Eq}(\pi_0, Y^0 \xrightarrow{} \pi_0 Y^1) \).
We started by choosing a base point \( f \), an \( E_\infty \)-map \( F \to E \). But how do we know there is one? This is actually a really hard question: an \( A_\infty \)-structure is a map \( A_\infty \to \text{End}_E \). Knowing that this mapping space is nonempty is all we want.

There is an obstruction to choosing a base point, and that lies in the \(-1\) stem of the spectral sequence. Why does this make sense? It turns out you don't need a base point to define the \( E_2 \) page.

So we want to find a point in \( \text{Tot} Y^\bullet \). Start by choosing a point in \( Y^0 = \text{Tot}^0 Y^\bullet \); this corresponds to an element of \( \pi_0 Y^0 \). Then you have to argue that this lifts to \( \text{Tot}^1 \), and that that then lifts to \( \text{Tot}^2 \), .... This is equivalent to asking whether a point \( x \in \pi_0 Y^0 \) lifts to \( \text{Tot}^1 \), so the point is in the equalizer of \( \pi_0 Y^0 \cong \pi_0 Y^1 \). Lifting this to \( \text{Tot}^2 \) is the same as showing \( d_2(x) = 0 \), etc. Note that this obstruction in \( E_2^{1,1} \) depends on the obstruction in \( E_1^{1,0} \). So you can imagine defining the spectral sequence in stages: once you do the first stage of defining the base point, you can define the second stage of the spectral sequence, and so on.

But this isn’t a problem for us, because it turns out that the spectral sequence collapses.

5.3. Computation. We have two tasks: identify the terms in the spectral sequence, and show that they are zero.

**Theorem 5.4** (Hopkins-Miller theorem).

\[
E_2^{0,0} \cong \text{Hom}_{E_\ast \text{-alg}}(E_\ast F, E_\ast) \cong FGL((K_1, \Gamma_1), (K_2, \Gamma_2))
\]

\[
E_2^{s,t} \cong \text{Der}_{E_\ast \text{-alg}}^s(E_\ast F, E_{s+t}) \cong 0
\]

So there are four parts, one for each \( \cong \). I’ll focus on the first \( \cong \).

We need some facts about Landweber exact theories and Lubin-Tate spectra. The slogan is “Landweber flat theories are ‘flat’, so they have a lot of nice properties, and Lubin-Tate theories are even nicer.” Let \( E, F \) be Landweber flat.

(1) \( E_\ast F \) is a flat \( E_\ast \)-module.

(2) There are Künneth spectral sequences \( \text{Tor}_{E_\ast}^*(E_\ast X, E_{s+t} Y) \implies E_{s+t}(X \wedge Y) \) and universal coefficient spectral sequences \( \text{Ext}_{E_\ast}^*(E \wedge X, E_{s+t}) \implies E^{t-s} X \).

(3) If \( E \) and \( F \) are Lubin-Tate, then \( E^* F \to \text{Hom}_{E_\ast}(E_\ast F, E_\ast) \) is an isomorphism. (Since they’re Lubin-Tate, the higher Ext terms vanish.)

We want \( \text{Eq}(Y^0 \cong Y^1) = \text{Eq}(\pi_0 C\text{-alg}(CF, E) \cong \pi_0 C\text{-alg}(C^2 F, E)) \). We want to convert this to something more algebraic. Suppose we’re interested in \( \pi_0 C\text{-alg}(Y, E) \) where \( Y \) is any \( C \)-algebra (e.g. \( CF \) or \( C^2 F \)). If I have a map \( Y \to E \) and I want to make it more algebraic, I can apply \( E_\ast(-) \) to get \( E_\ast Y \to E_\ast E \to E_\ast \). That is, we have a map \( \pi_0 C\text{-alg}(Y, E) \to \text{Hom}_{E_\ast \text{-alg}}(E_\ast Y, E_\ast) \). We want to show that this is an isomorphism.
In our case, \( Y = CF \simeq S \vee F \vee (F \wedge F) \vee \ldots \), and there’s something algebraic that’s analogous: if \( M = E_+ F \), there is the tensor algebra \( TM := E_+ \oplus M \oplus (M \otimes E_+ M) \oplus \ldots \).

**Fact 5.5.** Given a map \( X \to CX \) I get a map \( E_* X \to E_*(CX) \), and that also gives rise to a map \( T(E_* X) \to E_*(CX) \). Using the fact that \( E \) and \( F \) are Landweber flat, this is an isomorphism (in the case \( X = F \), or \( F^2 \), etc.).

We have a map \( \pi_0 C\text{-}alg(CX, E) \to \text{Hom}_{E_* \text{-}alg}(E_*(CX), E_*) \) (this is the map from before with \( Y = CX \), where \( X = F \) or \( F^2 \), etc.). \( CX \) is the free associative algebra on \( X \), so there are isomorphisms

\[
\pi_0 C\text{-}alg(CX, E) \xrightarrow{\cong} \text{Hom}_{E_* \text{-}alg}(E_*(CX), E_*)
\]

\[
[X, E] \xrightarrow[\cong]{\text{by flatness}} \text{Hom}_{E_*}(E_* X, E_*)
\]

The right vertical map comes from identifying \( E_*(CX) \cong T(E_* X) \).

So now we have reduced the problem to

\[
\text{Hom}_{E_* \text{-}alg}(T(E_* F), E_*) \Rightarrow \text{Hom}_{E_* \text{-}alg}(T^2(E_* F), E_*)
\]

I can turn \( E_* \text{-}alg \) into \( E_* \) by getting rid of one \( T \) on each side. Suppose I start with \( f : E_* F \to E_* \). The first map sends this to \( T(E_* F) \xrightarrow{\text{mult}} E_* F \xrightarrow{f} E_* \); the other map sends \( f \) to \( T(E_* F) \xrightarrow{T(f)} T(E_*) \xrightarrow{\text{mult}} E_* \). Originally, \( f \in \text{Hom}_{E_*}(E_* F, E_*) \). The only way for these to be equal is for \( f \) to be in \( \text{Hom}_{E_* \text{-}alg}(E_* F, E_*) \). (Think of this as the equalizer of \( f(ab) \) vs. \( f(a)f(b) \).)

---

6. November 2: Eva Belmont, HKR character theory

No notes here. Read Nat Stapleton’s expository paper “An introduction to HKR character theory”. His \( C_0 \) is what HKR call \( L(E^*) \).

7. November 9: Hood Chatham, \( E \)-theory of the symmetric group

We have a total power operation \( P^m : E^*(X) \to E^*(X \times B\Sigma_m) \cong E^*(X) \hat{\otimes}_{E_*} E^*(B\Sigma_m) \) (where the \( \cong \) is a Künneth isomorphism because \( B\Sigma_m \) is good). This has a formula

\[
P^m(x + y) = \sum_{i + j = m} \text{Tr}_{\Sigma_i \times \Sigma_j}^m (P^i(x) \times P^j(y))
\]

where \( \text{Tr} \) means transfer. If we wanted this to be additive, we might want to kill the ideal \( I_{tr} \) of transfers in \( E^*(X) \otimes E^*(B\Sigma_m) \). You don’t even need to take the ideal generated by them, because they already form an ideal: \( x \cdot \text{Tr}(y) = \text{Tr}((\text{Res}(x) \cdot y)) \). Let \( \overline{R}_k = E^*(B\Sigma_m)/I_{tr} \). So the total power operation modulo transfers is a map \( E^*(X) \to \overline{R}_k \).

The goal is to prove:
\textbf{Theorem 7.1} (Strickland).
\[ \text{Spf } \mathcal{R}_k = \text{Sub}_{p^k}(G) \]
(where Sub means subgroups).

This is too ambitious, so we’ll construct a map \( \text{Spf } \mathcal{R}_k \to \text{Sub}_{p^k}(G) \).

Note that there is a closed injection \( \text{Sub}_{p^k}(G) \hookrightarrow \text{Div}_{p^k}(G) \) sending \( K \) to the formal sum of its elements.

If \( K \) is a subgroup, I want a factorization
\[
\begin{array}{ccc}
K \times K & \longrightarrow & G \times G \\
\downarrow & & \downarrow m \\
K & \longrightarrow & G
\end{array}
\]

This corresponds to a map
\[
\begin{array}{ccc}
E^*[[x, y]]/(f_K(x), f_K(y)) & \longleftarrow & E^*[[x, y]] \\
& \uparrow & \uparrow \\
E^*[[x]]/(f_K(x)) & \longleftarrow & E^*[[x]]
\end{array}
\]

We have \( f_K(x + F y) = 0 \mod (f_K(x), f_K(y)) \). Let \( K \) be the universal divisor over \( \text{Div}_{p^k}(G) \), and get \( \text{Sub}_{p^k}(G) \).

First, we’ll construct a factorization
\[
\begin{array}{ccc}
\text{Spf } \mathcal{R}_k & \longrightarrow & \text{Sub}_{p^k}(G) \\
& \downarrow & \downarrow \\
& \text{Div}_{p^k}(G)
\end{array}
\]

We already have \( \text{Spf } E^*B\Sigma_{p^k} \to \text{Div}_{p^k}(G) = \text{Spf } E^*(BU(p^k)) \). This is given by \( E^*(\mathbb{P}(\xi_m)) \cong E^*(BU(p^k))[t]/(\sum c_n - t^m) \) where \( \xi_m \) is the canonical \( m \)-bundle and the \( c_i \) are Chern classes (maybe this means \( c_p \) ?).

The map \( \Sigma_{p^k} V_{\text{perm}} \to U(p^k) \) gives rise to a map \( \mathcal{R}_k = \text{Spf } E^*B\Sigma_{p^k} V_{\text{perm}} \to \text{Spf } E^*BU(p^k) = \text{Div}_{p^k}(G) \).

I claim \( c_{p^k}(V_{\text{perm}}) \), and \( \text{Res}_{\Sigma_i \times \Sigma_j}(c_{p^k-i}(V_{\text{perm}})) = 0 \). So \( c_k \cdot \text{tr}(x) = \text{tr}((\text{Res } c_k)x) = 0 \).

\textbf{Fact 7.2.} \( I = \text{ann}(c_k) \)

This uses character theory.

The moral of character theory is that if you want to know about \( \text{Spf } E^*BG \), probing it with maps \( \text{Spf } E^*BA \to \text{Spf } E^*BG \) (where \( A \) is abelian) is enough to figure it out rationally.
We have a diagram *that does not necessarily commute*

\[
\begin{array}{ccc}
\text{Spf } E^* BA & \longrightarrow & \text{Spf } E^*(B \Sigma^p) \\
\downarrow & = & \downarrow \\
\text{Hom}(A^* \mathbb{G}) & \longrightarrow & \text{Div}_{p^k}(\mathbb{G})
\end{array}
\]

(Here \(A^* = \text{Hom}(A, S^1)\).) The bottom map takes \(f \mapsto [f(A)]\). The left = comes from the fact that \(G = \text{Spf } E^* BU\). I would like the bottom map to factor through subgroups, but it doesn’t. If it did, then maybe I’d be able to get a factorization.

Divisors are kind of like subsets, but they like to count multiplicity. An element of \(\text{Hom}(A^*, G)\) that is not injective might give rise to a “subgroup with multiplicity”. You want to pick out the injections, but this isn’t functorial. Instead, form the pullback

\[
\begin{array}{ccc}
\text{Hom}(A^*, G) & \longrightarrow & \text{Div}_{p^k}(\mathbb{G}) \\
\uparrow & & \uparrow \\
\text{Level}(A^*, \mathbb{G}) & \longrightarrow & \text{Sub}_{p^k}(\mathbb{G})
\end{array}
\]

**Theorem 7.3** (Greenlees-Strickland). There is a surjection \(\bigsqcup_{A \subset G} \text{Level}(A^*, \mathbb{G}) \twoheadrightarrow \text{Spf } E^*(BG)\).

If you quotient on the left by the conjugation of \(G\) and rationalize, then this is an isomorphism. The proof of this is almost the same as the proof of normal character theory.

Step 1: get a pullback

\[
\begin{array}{ccc}
W & \longrightarrow & \text{Spf } R_k \\
\downarrow & & \downarrow \\
\bigsqcup_{A \subset \Sigma_{p^k}} \text{Level}(A^*, \mathbb{G}) & \longrightarrow & \text{Spf } E^* B \Sigma_{p^k}
\end{array}
\]

and compute \(W\).

Step 2: show that there is a commutative diagram

\[
\begin{array}{ccc}
W & \longrightarrow & \text{Sub}_{p^k}(\mathbb{G}) \\
\downarrow & & \downarrow \\
\text{Spf } R_k & \longrightarrow & \text{Div}_{p^k}(\mathbb{G})
\end{array}
\]

We need another fact:

**Fact 7.4.** \(D(A) := \mathcal{O}_{\text{Level}(A^*, \mathbb{G})}\) is a domain. (Here \(D(A)\) is the Drinfeld ring. *I think this is the image of \(E^* BA \rightarrow S^{-1} E^* BA\) in my talk, i.e. \(L(E^*)\) before you invert \(p\).*)

I claim \(W = \bigsqcup_{A : i^* c_k \neq 0} \text{Level}(A^*, \mathbb{G})\).
We have a map \( A = \prod A_i \to \Sigma^k \), and if you pull back the permutation representation on the RHS, you get \( \bigoplus V^i \) (where \( V^{reg} \) is the regular representation).

If \( A \) is not transitive, \( i^*(c_k) = 0 \) (“because there are multiple trivial things in \( \bigoplus V^i \)”). If \( A \) is transitive, then \( i^*(c_k) \neq 0 \), and the pullback to the level structures is nonzero. So we get

\[
W = \bigsqcup_{A \text{ transitive}} \text{Level}(A^*, \mathbb{G}).
\]

The following diagram is not commutative for all \( A \)

\[
\begin{array}{ccc}
\text{Hom}(A^*, \mathbb{G}) & \longrightarrow & \text{Div}_{p^k}(\mathbb{G}) \\
\downarrow & & \downarrow \\
\text{Spf } E^* \text{BA} & \longrightarrow & \text{Spf } E^* \text{BU}(p^k) \\
\downarrow V^{reg} & & \downarrow \\
\text{Spf } E^* B\Sigma^k & \longrightarrow & \text{Spf } E^* \Sigma^k \text{BA}
\end{array}
\]

This commutes when the pullback of the permutation representation is the regular representation. This happens exactly when \( A \) is transitive.

If \( A = A_1 \times A_2 \), the top map is just taking the image (so that’s a subgroup), but the bottom map \( \text{Hom}(A^*, \mathbb{G}) \to \text{Spf } E^* B\Sigma^k \to \text{Spf } E^* \text{BU}(p^k) \) corresponds to the map \( \text{Hom}(A^*_1 \times A^*_2, \mathbb{G}) \to \text{Div}_{p^k}(\mathbb{G}) \) with image \([f(A^*_1 \times 1)] + [f(1 \times A^*_2)]\) (where \([f(A^*_1 \times 1)]\) comes from \( V^{reg}_1 \) and \([f(1 \times A^*_2)]\) corresponds to \( V^{reg}_2 \)) and that’s going to have multiplicity. \( c_k \) is going to be zero if there is more than one trivial representation. But it turns out that exactly when it’s not a subgroup, there is more than one trivial representation. Issue: \( V^{reg} \) might have two copies of the trivial representation. The image is inside of subgroups exactly when there is one trivial representation, and that happens exactly when \( c_k \neq 0 \).

So when \( A \) is transitive, this diagram commutes, and we get the diagram in (2).


Goal: Using Strickland’s computation \( E^0_n(B\Sigma_m)/(tr) \) (as explained by Hood), describe all of the structure that adheres to \( \pi_0 \) of a \( K(n) \)-local commutative (i.e. in the \( E_\infty \) sense) \( E_n \)-algebra, at least in the case of an algebra \( A \) where \( \pi_0 A \) is \( p \)-torsion-free. This tells you about the additive operations, which in some sense correspond to \( E^0_n(B\Sigma_m)/(tr) \). All operations, in
some sense, correspond to $E_n^0(B\Sigma_m)$. So we’re trying to describe all operations in terms of the additive operations.

The way we formalize this is by defining a monad $T : M \text{Mod}_{E_0} \to \text{Mod}_{E_0}$ (here $E$ means $E_n$, so $E_0$ means $\pi_0E_n$) such that there is a factorization

$$
\begin{array}{c}
\text{CAlg}_{E}^{K(n)\text{-loc}} \\
\pi_0 \\
\downarrow \\
\text{Alg}_T \\
\downarrow \\
\text{Mod}_{E_0}
\end{array}
$$

and $T$ gives “all” of the structure on $\pi_0$ in the following sense:

$$[T(\pi_0 M)]_m \cong \pi_0 L_{K(n)}^p M \text{ if } M \in \text{Mod}_E \text{ has } \pi_0 M \text{ flat over } E_0. \text{ (Here } m \text{ is the maximal ideal in } E_0.)$$

Up to issues of completion, $T$ is good enough for Goerss-Hopkins obstruction theory.

Further, the duals of $E^0(B\Sigma_m)/(\text{tr})$ assemble into a “graded twisted $E_0$-bialgebra”, which means it has the following structure:

$$
\Gamma = \bigoplus_{k \geq 0} \Gamma[k] \quad \Gamma[k]^\vee = E^0(B\Sigma_{p^k})/(\text{tr})
$$

(here $\Gamma$ is a commutative graded algebra with $k$-graded piece $\Gamma[k]$, with co-commutative, co-associative coproduct) and there is a factorization

$$
\begin{array}{c}
\text{CAlg}_{\Gamma} \\
\text{Alg}_T \\
\downarrow \\
\text{CAlg}_{\Gamma} \\
\downarrow \\
\text{Mod}_{E_0}
\end{array}
$$

$\text{CAlg}_{\Gamma}$ is where we just remember the additive operations.

$\text{CAlg}_{\Gamma}$, by Strickland’s theorem, has a description in terms of formal groups

$$
\text{Spf}(E^0(B\Sigma_{p^k})/(\text{tr})) \cong \text{Sub}_{\text{G}}^{p^k}
$$

and $\text{G}/E_0$ is the universal deformation.

**Theorem 8.1** (Congruence criterion). $\text{Alg}^{p\text{-tf}}_{E_0} \to \text{CAlg}^{p\text{-tf}}_{\Gamma}$ is fully faithful with essential image as follows:

there exists $\sigma \in \Gamma[1]$ such that $A \in \text{CAlg}^{p\text{-tf}}_{\Gamma}$ is in the essential image iff $x\sigma \equiv x^p \pmod p$ for all $x \in A$.

Here $p\text{-tf}$ means algebras whose underlying module is $p$-torsion free.
**Example 8.2.** At height 1: Alg$_T$ is the category of $\theta$-algebras over the Witt ring $W(k)$, where $\theta : A \to A$ “defined” by $\psi^p = (-)^p + p\theta$ is a ring homomorphism (note that the difference of two possible $\theta$’s is $p$-torsion). Furthermore, in this case, $\Gamma = \mathbb{Z}_p[\psi^p]$ (free $\mathbb{Z}_p$-module) and $\sigma = \psi^p$. All the congruence criterion is saying is that, if $A$ is $p$-torsion free, then $\theta$ is uniquely determined by $\psi^p$ and can be defined iff $\psi^p(x) \equiv x^p \pmod{p}$.

**Definition 8.3.** $T := \bigoplus_{m \geq 0} T_m$, where $T_m$ are defined as follows. Let Mod$^\text{ff}_E$ be the category of finite free $E$-modules. Then $T_m$ is the left Kan extension of the following diagram

$$
\begin{array}{ccc}
\text{Mod}^\text{ff}_E & \xrightarrow{\pi_0 L_{K(n)} P_m} & \text{Mod}_{E_0} \\
\pi_0 \downarrow & & \downarrow T_m \\
\text{Mod}_{E_0} & & \\
\end{array}
$$

(This exists because Mod$^\text{ff}_E$ is small.) Here $P_m$ is the $m$-th extended power.

Since we have an equivalence $\pi_0 : \text{Mod}^\text{ff}_E \cong \text{Mod}^\text{ff}_{E_0}$, $T(E_0) = \bigoplus_{m \geq 0} E_0^\wedge (B \Sigma_m)$.

$T$ inherits a monad structure from $L_{K(n)} P$, and we get a factorization

$$
\begin{array}{ccc}
\text{CAlg}^{K(n)}_{\text{loc}} & \xrightarrow{\pi_0} & \text{Alg}_T \\
\downarrow & & \downarrow \text{Alg}_T \\
\text{Mod}_{E_0} & & \\
\end{array}
$$

**Fact 8.4.** $T$ factors through $\text{CAlg}_{E_0}$. More precisely, $T$ is a functor $(\text{Mod}_{E_0}, \bigoplus) \to (\text{Mod}_{E_0}, \otimes)$. This implies that there is a factorization

$$
\begin{array}{ccc}
\text{Alg}_T & \xrightarrow{\text{Alg}_{E_0}} & \text{Alg}_{E_0} \\
\downarrow & & \downarrow \text{Alg}_{E_0} \\
\text{Mod}_{E_0} & & \\
\end{array}
$$

The proof isn’t too bad but takes a long time.

The forgetful functor $\text{Alg}_T \to \text{Set}$ is equal to $A \mapsto \text{Alg}_T(T(E_0), A)$. By Yoneda, the monoid of endomorphisms of Forget is equal to $\text{Alg}_T(T(E_0), T(E_0))$; as a set, this is just equal to the free $T$-algebra $T(E_0)$. 

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**Definition 8.5.** Define $\Gamma$ to be the sub-monoid of $T(E_0)$ consisting of additive endomorphisms.

$T(E_0)$ has a Hopf algebra structure with coproduct coming from the transfer along $\Sigma_i \times \Sigma_{m-i} \hookrightarrow \Sigma_m$.

By the formula $P_m(x+y) = P_m(x) + P_m(y) + \sum_{0<i<m} \text{tr}_{i,m-i}(P_i(x), P_{m-i}(y))$ (here tr means transfer), we have $\Gamma = \text{Prim}(T(E_0))$, and that says $\Gamma_k = \ker(\text{tr}) \subset \hat{E}_0(B\Sigma_k)$ is dual to $E_0(B\Sigma_k)/(\text{tr})$.

Furthermore, $\Gamma_k = 0$ unless $k = p^r$. Reindex and write $\Gamma[k] = \Gamma_p^k$, and $\Gamma = \bigoplus_{m \geq 0} \Gamma[k]$.

Facts about $\Gamma$:

1. $\Gamma$ is naturally a graded ring.

2. $B\Sigma_{p^k} \to \ast$ and the diagonal map $B\Sigma_{p^k} \xrightarrow{\Delta} B\Sigma_{p^k} \times B\Sigma_{p^k}$ give it the structure of of a “twisted graded $E_0$-bialgebra”.

By definition, we get a factorization

$$
\begin{array}{ccc}
\text{Alg}_T & \xrightarrow{\Delta} & \text{CAlg}_\Gamma \\
\downarrow & & \downarrow \\
\text{Mod}_{E_0}
\end{array}
$$

Now I’m going to give a sketch of the proof of the congruence criterion. Now that we’ve gotten rid of the grading on $E$ (just looking at $E_0$), it is not that bad. The steps are as follows:

1. Get a rational equivalence $\text{Alg}_{T,\mathbb{Q}} \to \text{CAlg}_{\Gamma,\mathbb{Q}}$.

   This is just the classification of graded Hopf algebras over $\mathbb{Q}$. There is a natural map

   $$\text{Sym}_{E_0}^*(\Gamma \otimes M) \otimes \mathbb{Q} \to T(M) \otimes \mathbb{Q}.$$  

   I claim that if this is an isomorphism (of monads), then this implies step 1. Using the way these behave w.r.t. colimits you can reduce to $M$ being free. So we’re looking at $\text{Sym}_{E_0}^*(\Gamma) \otimes \mathbb{Q} \to T(E_0) \otimes \mathbb{Q}$, and I claim (maybe by Kashiwabara’s results) $\Gamma \otimes \mathbb{Q} \cong \text{Prim}(T(E_0) \otimes \mathbb{Q})$. (You need to know that tensoring with $\mathbb{Q}$ commutes with primitives, and for that you need the fact that each graded piece is finite free. Actually maybe you need this Kashiwabara fact...)

2. If $A$ is a $p$-torsion free $\Gamma$-algebra, then $A \to A \otimes \mathbb{Q}$ induces a $T$-algebra structure (coming from $A \otimes \mathbb{Q}$) iff $\psi(T_p(A)) \subset A$. (Here $\psi$ is the map $T(A \otimes \mathbb{Q}) \to A \otimes \mathbb{Q}$.)

   For this part, I claim that there is a factorization

   $$
   \begin{array}{ccc}
   T(A) & \xrightarrow{\psi} & A \\
   \downarrow & & \downarrow \\
   T(A \otimes \mathbb{Q}) & \xrightarrow{\psi} & A \otimes \mathbb{Q}
   \end{array}
   $$
Also, this factors as

\[
\begin{array}{ccc}
T(A) & \longrightarrow & A \\
\downarrow & & \downarrow \\
T(A) \otimes \mathbb{Q} & \longrightarrow & A \otimes \mathbb{Q}
\end{array}
\]

This comes down to \( p \)-Sylow subgroups of symmetric groups.

(3) The image of \( T_p(A) \to T_p(A) \otimes \mathbb{Q} \) can be described as follows. There is a map \( u : A \otimes \Gamma[1] \to T_p(A) \). Since this is an algebra, there is a map \( v : \text{Sym}^p_{E_0}(A) \to T_p(A) \). Then the image is spanned by the image of \( u \), the image of \( v \), and elements of the form \( (x\sigma - x^p)/p \) for all \( x \) in \( A \) (where \( \sigma \) is still mysterious).

\( A \) is already a \( \Gamma \)-module, and an algebra. So this part of the image is inside \( A \). So the only condition left is \( (x\sigma - x^p)/p \). Asking this to be in \( A \), and not \( A \otimes \mathbb{Q} \) is precisely the congruence criterion. So the theorem follows from these three steps.

Step 3 is the key part, and this comes down to almost a trick. Since we defined \( T_p \) by left Kan extension from finite free things, we can look at \( E_0 \). There is a pushout diagram of \( E_0 \)-modules.

\[
\begin{array}{ccc}
E_0 & \longrightarrow & E_0 \\
\downarrow^{(\sigma,-1)} & & \downarrow^{\beta} \\
\Gamma[1] \oplus E_0 & \longrightarrow & T_p(E_0)
\end{array}
\]

This implies 3 immediately. But you still have to prove it. The desired pushout is equivalent to having a pullback of \( E_0 \)-modules

\[
\begin{array}{ccc}
E^0(B\Sigma_p) & \longrightarrow & E^0(B\Sigma_p)/(\text{tr}) \oplus E^0 \\
\downarrow^{\beta^*} & & \downarrow \\
E^0 & \longrightarrow & E^0
\end{array}
\]

Using the double coset formula, I claim that

\[
\begin{array}{ccc}
E^0(B\Sigma_p) & \longrightarrow & E^0(B\Sigma_p)/(\text{tr}) \\
\downarrow & & \downarrow \sigma^* \\
E^0 & \longrightarrow & E^0/\text{pE}^0
\end{array}
\]

is a pullback diagram, because the transfer ideal is just the transfer coming from the trivial subgroup inside \( \Sigma_p \). It is free on one generator, and it maps surjectively... Here \( \sigma^* \) is dual to a class \( \sigma \in \Gamma[1] \otimes E^0/\text{pE}^0 \).

This is equivalent to having a SES

\[
0 \to E^0(B\Sigma_p) \xrightarrow{(\pi^*,\text{res}^*)} E^0(B\Sigma_p)/(\text{tr}) \oplus E^0 \xrightarrow{(\sigma^*,-1)} E^0/\text{pE}^0 \to 0
\]
\[ \beta^* \text{ is the map that makes this diagram commute:} \]

\[ \begin{array}{ccc}
0 & \xrightarrow{} & E^0(B\Sigma p) \\
\downarrow^{\beta^*} & & (\pi^*, res^*) \\
0 & \xrightarrow{} & E^0(B\Sigma p)/(\text{tr}) \oplus E^0 \\
\downarrow^{(\pi^*, -1)} & & \downarrow^{(\sigma^*, -1)} \\
0 & \xrightarrow{} & E^0/pE^0
\end{array} \]

\[ \sigma \text{ is an arbitrary lift of the } \overline{\sigma} \text{ defined above.} \]

9. November 30: Jeremy Hahn, Rezk’s logarithm

There has been a major program to construct a map \( \text{MString} \to \text{tmf} \) (the Witten orientation) and (more classically) the Atiyah-Bott-Shapiro orientation \( \text{MSpin} \to \text{ko} \). Eric will show how to construct the latter in a way that generalizes to the first. This has to do with orientation theory.

For \( G \) an abelian group, you can form the commutative ring \( \mathbb{Z}[G] \). From a commutative ring \( R \), you can form the abelian group \( \text{GL}_1(R) \). There is an adjunction:

\[ \mathbb{Z}[-] : \text{Ab} \rightleftarrows \text{Comm. Rings} : \text{GL}_1(-). \]

This generalizes to homotopy theory; the analogue of an abelian group is a connective spectrum. If \( G \) is a connective spectrum, I can form the \( E_\infty \)-ring \( S[G] \simeq \Sigma^\infty_{+} \Omega^\infty G \) (later I’ll say why this has a natural \( E_\infty \)-ring structure). Conversely, for \( R \) an \( E_\infty \)-ring, you can define the space \( \text{GL}_1(R) \) to be the homotopy pullback of the diagram

\[ \begin{array}{ccc}
\text{GL}_1(R) & \xrightarrow{} & \Omega^\infty R \\
\downarrow & & \downarrow^{\pi_0} \\
\pi_0(\Omega^\infty R)^{\times} & \xrightarrow{} & \pi_0(\Omega^\infty R)
\end{array} \]

This gives a space \( \text{GL}_1(R) \).

**Fact 9.1.** \( \text{GL}_1(R) \) is naturally \( \Omega^\infty \) of some spectrum \( gl_1(R) \), and there is an adjunction

\[ S[-] : \text{Connective spectra} \rightleftarrows E_\infty \text{-rings} : gl_1(-). \]

**Remark 9.2.** \( \pi_i(gl_1 R) = \pi_i(\Omega^\infty gl_1 R) = \pi_i(\text{GL}_1 R) \). In particular, \( \pi_0(gl_1 R) \cong (\pi_0 R)^{\times} \) and \( \pi_i(gl_1 R) \cong \pi_i R \) for \( i > 0 \). (But these are very different spectra, even though their homotopy groups are almost the same.)

It’s “easy” to construct \( E_\infty \)-ring maps \( S[X] \to R \): these are just spectrum maps \( X \to gl_1(R) \).

**Definition 9.3.** \( \text{MString} \) is the Thom \( E_\infty \)-ring spectrum associated to the composite map of \( \infty \)-loop spaces

\[ \text{BString} \to BO \xrightarrow{J} B\text{GL}_1(S) \]

I mean that each of these maps are maps of spectra, so it’s perhaps better to view this as

\[ \Omega^\infty(bstring \simeq \tau_{\geq 8} \text{ko} \to bo \simeq \tau_{\geq 2} \text{ko} \xrightarrow{J} \Sigma gl_1(S)). \]
**Construction 9.4.** Let $X$ be a spectrum, and $f : X \to \Sigma gl_1(S)$ a map of spectra. You get $\Omega^{\infty} f : \Omega^{\infty} X \to BGL_1(S)$, and a functor $F : \Omega^{\infty} X \to \text{Spectra}$ (think of both $\Omega^{\infty}$ and $BGL_1(S)$ as groupoids, the latter whose one point is the sphere spectrum and morphisms are automorphisms $S \to S$; the functor sends the one point to $S$). Let $Mf = \text{colim} F$; this is the Thom spectrum of $F$.

**Theorem 9.5.** $Mf$ is an $E_{\infty}$-ring. The space of $E_{\infty}$-ring maps $Mf \to R$ is the space of nullhomotopies of $X \to \Sigma gl_1(S) \xrightarrow{\Sigma gl_1(\text{unit})} \Sigma gl_1(R)$.

If one such $E_{\infty}$-map exists, the set of such maps up to homotopy is a torsor over $[X, gl_1(R)]$.

(If you have one nullhomotopy, i.e. way of filling in the square

\[
\begin{array}{ccc}
X & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \Sigma gl_1 R
\end{array}
\]

you can modify it to get another. By the universal property of colimits, this is the same as maps $\Sigma X \to \Sigma gl_1 R$, or maps $X \to gl_1 R$.)

Here's an exercise about 1-categories that might give you some intuition:

**Exercise 9.6.** Suppose $(C, \otimes_C)$ and $(D, \otimes_D)$ are monoidal categories. Given a lax monoidal functor $F : C \to D$ (e.g. equipped with natural transformation $FC_1 \otimes FC_2 \to F(C_1 \otimes C_2)$), colim $F$ is an algebra and algebra homomorphisms colim $F \to A$ are in bijection with lax monoidal lifts

\[
\begin{array}{ccc}
D/X & \xrightarrow{\pi} & D \\
\downarrow & & \downarrow \\
C & \xrightarrow{F} & D
\end{array}
\]

Think about why this implies the statement on Thom spectrum, when generalized to $\infty$-categories (or read Omar and Toby’s paper).

**Proposition 9.7.** A string orientation exists iff

\[
\Omega^{\infty} \tau_{\geq 2} ko \simeq \text{string} \to gl_1(S) \to gl_1(tmf)
\]

is null. If one exists, the space of such is a torsor over $[bstring, gl_1(tmf)]$.

A spin orientation exists iff

\[
\text{spin} \to gl_1(S) \to gl_1(ko)
\]

is null. The collection of such is a torsor over $[bspin, gl_1(ko)]$.

**Proposition 9.8.** Suppose $R$ is an $L_n$-local $E_{\infty}$-ring spectrum. Then $\tau_{\geq n+2} gl_1(R) \cong \tau_{\geq n+2} L_n gl_1(R)$.

The proof in Ando-Hopkins-Rezk is rather light on details...
(Here \( L_n \) is the chromatic localization.)

Note: by construction, string is 7-connected and spin is 3-connected. It suffices to think about \( L_2 gl_1(tmf) \) and \( L_1 gl_1(ko) \).

\( L_2 gl_1(tmf) \) is built out of 3 pieces: \( L_{K(2)} gl_1(tmf) \), \( L_{K(1)} gl_1(tmf) \), and \( L_{H\mathbb{Q}} gl_1(tmf) \). If we understand these three pieces, maybe we can assemble that to an understanding of \( gl_1 tmf \) itself.

**Theorem 9.9** (Bousfield-Kuhn). For each \( n > 0 \) there is a functor (the Bousfield-Kuhn functor) \( \Phi_n : \text{Spaces}_* \to \text{Spectra} \) such that

1. \( \Phi_n \) commutes with finite homotopy limits;
2. \( \Phi_n(\tau_{\geq k} X) \simeq \Phi_n(X) \);
3. \( \Phi_n(\Omega^\infty E) = L_{K(n)} E \).

Idea: to look at the \( K(n) \)-localization of a spectrum, it suffices to look at the underlying space.

Note: \( \Omega^\infty gl_1(tmf) \simeq GL_1 tmf \subset \Omega^\infty tmf \). On all homotopy groups but the 0th one, this induces an \( \cong \). If I take 1-connected covers, this becomes an isomorphism of spaces (just not of spectra). Using the Bousfield-Kuhn functor, we learn that there is a natural equivalence \( L_{K(n)} gl_1 R \to L_{K(n)} R \).

It looks like we’re almost done: if we assume we understand \( tmf \) pretty well, now we know \( L_{K(2)} gl_1(tmf) \), \( L_{K(1)} gl_1(tmf) \), and \( L_{H\mathbb{Q}} gl_1(tmf) \). What remains is putting the pieces together.

Suppose we want to calculate \( L_{K(1) \vee K(2)} gl_1(tmf) \). There is a fracture square

\[
\begin{array}{ccc}
L_{K(1) \vee K(2)} gl_1(tmf) & \to & L_{K(2)} gl_1(tmf) \simeq L_{K(2)} tmf \\
\downarrow & & \downarrow \\
L_{K(1)} gl_1(tmf) & \to & L_{K(1)} L_{K(2)} gl_1(tmf) \\
\cong & & \cong \\
L_{K(1)} tmf & \to & L_{K(1)} L_{K(2)} tmf
\end{array}
\]

But to understand this, we need to understand all the maps, and the bottom map is pretty mysterious – it’s not just the localization map. Hopkins and others call this the essential difficulty in working out the string orientation.

Rezk’s logarithm was invented to get at the mysterious map. For \( R \) an \( E_\infty \)-ring and \( n > 0 \), we get a map

\( \ell : gl_1(R) \to L_{K(n)} gl_1(R) \xrightarrow{\sim} L_{K(n)} R \)

and on \( \pi_0 \) we get a map \( \pi_0(R)^\times \to \pi_0(L_{K(n)} R) \).
**Theorem 9.10.** Suppose \( R \) is a \( K(1) \)-local \( \text{E}_{\infty} \)-ring with \( \pi_0(R) \) is \( p \)-torsion free (e.g. \( ko_{\hat{p}}, L_{K(1)}\text{tmf} \)). Then you get a logarithm (group homomorphism taking multiplication to addition)

\[
\ell : (\pi_0 R)^{\times} \rightarrow \pi_0 (L_{K(1)} R) \cong \pi_0 R
\]

which is given by

\[
\ell (x) = \frac{1}{p} \log \left( \frac{x^p}{\psi(x)} \right) = \sum_{k=1}^{\infty} (-1)^k \frac{p^k}{k} \left( \frac{\Theta (x)}{x^p} \right)^k
\]

where \( \psi (x) = x^p + p \Theta (x) \).

For any spectrum \( R \) you have a transfer homomorphism (additive map); the total transfer, denoted \( \text{tr} \), is a map \( R^0(\text{pt}) \rightarrow R^0 (\Omega^\infty S) \). This sends a map \( S \rightarrow R \) to the map \( \Sigma^\infty \Omega^\infty S \rightarrow S \rightarrow R \) (adjointing over the identity). Morally, \( \Sigma^\infty \Omega^\infty S \) should have some kind of Snaith splitting; it’s not connected, so all you get is a map \( \Sigma^+ B \Sigma_k \rightarrow \Sigma^\infty \Omega^\infty S \). This allows you to extend the transfer to \( R^0(B \Sigma_k) \) (which should be more familiar).

Suppose \( R \) is \( E_\infty \), so there is a \(gl_1(R) \). Given a map \( S^0 \rightarrow \Omega^\infty gl_1 (R) \), I can apply the additive transfer to get a map \( \Omega^\infty S \rightarrow \Omega^\infty gl_1 (R) \) which comes from the \( E_\infty \) structure. Finally, I can postcompose with the inclusion \( \Omega^\infty gl_1 (R) \hookrightarrow \Omega^\infty R \). This starts with an element in \( \pi_0(gl_1 (R)) \rightarrow R^0(\Omega^\infty S) \), i.e. a map \( (\pi_0 R)^{\times} \rightarrow R^0 (\Omega^\infty S) \). This is the map \( 1 + x \mapsto P(x) \) (the total power operation).

**Theorem 9.11.** Let \( R \) denote a \( K(n) \)-local \( E_{\infty} \)-ring. The logarithm \( \ell : \pi_0 (R)^{\times} \rightarrow \pi_0 (R) \) is given by capping the transformation \( 1 + x \mapsto P(x) \) with a specific class \( \lambda_R \in \tilde{R}_0 (\Omega^\infty S) \). Furthermore, the class \( \lambda_R \) is natural in \( R \); it is the image under the unit map map \( S \rightarrow R \) of a class \( \lambda_S \in \pi_0 (L_{K(n)} \Sigma^\infty \Omega^\infty S) \).

The class \( \lambda_S \) is obtained by applying \( \Phi_n \) to \( \Omega^\infty S \rightarrow \Omega^\infty \Sigma^\infty \Omega^\infty S \) (gotten by adjointing). So I get a map \( \lambda_S : L_{K(n)} S \rightarrow L_{K(n)} (\Sigma^\infty \Omega^\infty S) \).

**Proof.** This just uses formal properties of the Bousfield-Kuhn functor. \( \square \)

To understand the logarithm, you need to understand \( \lambda_S \) and \( L_{K(n)} \Sigma^\infty \Omega^\infty S \).

**Theorem 9.12.** Let \( R \) denote a \( K(n) \)-local \( E_{\infty} \)-ring. There are natural maps

\[
\tau : \tilde{R}_0 (\Omega^\infty S) \simeq \tilde{R}_0 (\Sigma^\infty \Omega^\infty S) \rightarrow \tilde{R}_0 (S)
\]

\[
\circ : \tilde{R}_0 (\Omega^\infty S) \otimes \tilde{R}_0 (\text{pt}) \tilde{R}_0 (\Omega^\infty S) \rightarrow \tilde{R}_0 (\Omega^\infty S).
\]

\( \tau \) is obvious. The element \( \lambda_R \) is uniquely characterized by:

1. \( \tau (\lambda_R) = 1 \)
2. For all \( x \in \tilde{R}_0 (\Omega^\infty S) \), \( x \circ \lambda_R = \tau (x) \lambda_R \).

When \( R \) is Morava \( E \)-theory, we understand \( \tilde{R}_0 (\Omega^\infty S) \). (This is what this seminar has been about.) Rezk wrote down \( \lambda_R \) in this case: he writes down an element plucked from the void and checks the properties. His selection of the element is complicated.
In the case \( n = 1 \) (corresponding to the Atiyah-Bott-Shapiro orientation), this gives a formula for \( \ell : gl_1(KU_p^\wedge) \to KU_p^\wedge \) (this is an older result by tom Dieck). This is an equivalence on 3-connected covers (a theorem of Adams and Priddy, but their proof is non-explicit, using Adams spectral sequence arguments). The Bousfield-Kuhn functor was originally invented by Bousfield to give a better proof of this fact.

To understand the logarithm in full generality, at other heights, you need to understand the fiber sequence

\[
L_{K(1)} \Sigma^\infty \Omega^\infty S \to L_{K(1)}(KU \wedge \Sigma^\infty \Omega^\infty S) \xrightarrow{\psi_{l-1}} L_{K(1)}(KU \wedge \Sigma^\infty \Omega^\infty S).
\]

This is special to \( K(1) \).

10. December 14: Eric Peterson, The spin orientation of \( KO \)

The goal of today is to construct the Atiyah-Bott-Shapiro orientation \( \Phi_{ABS} \in E_\infty(\text{MSpin},KO) \).

We’re going to use chromatic fracture. There is a pullback

\[
\begin{array}{ccc}
\text{MSpin} & \longrightarrow & KO_{(p)} \\
\downarrow & & \downarrow \\
Q \otimes KO & \longrightarrow & Q \otimes KO_p
\end{array}
\]

(Instead of mapping into \( KO \), I’m going to map into \( KO_{(p)} \). I could safely drop this and use products instead.)

So I just have to understand how to map into the corners. The bottom two things are rational spectra, and the theory of rational orientations is pretty self-contained. I’m going to take a slightly simpler spectrum than MSpin, and a less concrete spectrum than \( KO \); I’m going to study maps \( MU \to Q \otimes R \) for some \( E_\infty \)-ring \( R \). \( S \to Q \otimes R \) factors through \( HQ \), and there’s also a factorization of \( S \to HQ \) through \( MU \) (“truncation to zero”). So we have

\[
\begin{array}{ccc}
MU & \longrightarrow & Q \otimes R \\
\downarrow & & \downarrow \\
S & \longrightarrow & HQ
\end{array}
\]

Jeremy told us that when \( E_\infty(MU,Q \otimes R) \neq \emptyset \), its \( \pi_0 \) is a torsor for \([bu,gl_1Q \otimes R]\). The diagram says that this mapping space is never empty. If I have a favorite reference point (e.g. the map marked “dumb”), then this will give me an isomorphism \([bu,gl_1Q \otimes R] \to \pi_*E_\infty(MU,Q \otimes R)\) that takes \( \alpha \) to dumb \( \cdot \alpha \). I also have \([bu,gl_1Q \otimes R[2,\infty]] = [bu,Q \otimes gl_1R] = [Q \otimes bu,Q \otimes gl_1R]\) and that is totally determined by what it does on homotopy groups. (You could also put the truncation in the other terms – these differ in degree 0 but \( bu \) doesn’t care about that). Call this \((t_{2k}) \in \prod_{k \geq 1} \Sigma^{2k}Q\).

Idea: the sequence \((t_{2k})\) “twists” the standard Thom class associated to the “dumb map”, which we’ll now call \( D \).
Theorem 10.1. Given the Thom class $x$ of $L$ over $\mathbb{C}P^\infty$ associated to $D$, any other Thom class differs from $D$ by a fraction $\frac{x}{\exp_F(x)}$ where $\exp_F$ is the exponential series for the form group law associated to the other orientation. For
\[\frac{x}{\exp_F(x)} = \exp\left(\sum_k \frac{t_k}{k!} \cdot x^k\right)\]
these coefficients $t_k$ track the behavior of the map $\pi_{2k}BU \to \pi_{2k}\mathbb{Q} \otimes R$.

This is called the characteristic series.

Proof. Start with the Bott class $S^{2k} \to BU$. This factors through $(\mathbb{C}P^\infty)^{\wedge k}$ (inclusion of the bottom cell of $CP^\infty$). If you want to calculate the behavior of the map above, you need to calculate the Thom class of the restricted bundle $L^{\times k}/S^{2k}$. The Thom class is related to the top Chern class of this. The total Chern class is $\frac{x}{\exp_F(x)}$, and if you want to grab the $k^{th}$ piece, the claim is that you get what is advertised, i.e. $(-1)^k t_{2k}$. $\square$

This was supposed to be the baby case of maps $MSpin \to \mathbb{Q} \otimes KO$. There are some issues.

Question 10.2. What about connective bordism theories?

In Eric’s course last spring, you learned that orientations $MU \langle 2k \rangle \to R$ were in bijection with symmetric $k$-variate 2-cocycles. There is a theorem (due to Ando-Hopkins-Strickland) that does this for complex oriented ring spectra. There is a special case for rational ring spectra $\mathbb{Q} \otimes R$, and these have more properties. Recall a symmetric $k$-variate 2-cocycle is a $k$-variate symmetric power series satisfying some ugly conditions that look like $f(x,y) f(t+x,y) f(t \times ty) = 1$ (but worse if $k$ is bigger). However, you can show that every such power series arises from a 1-variable series after applying $(\delta^1 f)(x,y) = \frac{f(x+y)}{f(x)f(y)}$ $k$ many times. Such a 1-variate series corresponds to a map $MU \to \mathbb{Q} \otimes R$.

There is an induced map $MU[2k, \infty] \to MU$, and that is applying $\delta^1$ $k$-many times.

\[
\begin{tikzcd}
MU[2k, \infty] \arrow{r}{(\delta^1)^k} \arrow{d}{MU} & \mathbb{Q} \otimes R \\
MU
\end{tikzcd}
\]

So there is nothing special about connective bordism theories.

Question 10.3. How do I deal with the difference between complex and real bordism?

There is a cofiber sequence
\[\Sigma KO \xrightarrow{\eta} KO \to KU \to \Sigma^2 KO\]
called Wood’s cofiber sequence. If I work rationally, $\eta = 0$, and there are splittings $KU \to KO$ (complexification) and $\Sigma KO$. I’m going to build two maps $KU \to KU$ that come from looking for the fixed points of complex conjugation: $\frac{1+\chi}{2}$. If 2 is inverted, you can do this. Think of $KO$ as the negative eigenspace, and $\Sigma^2 KO$ as the positive eigenspace. You get that
$KO$-cohomology is the $\chi$-fixed points of $KU$-cohomology, rationally. The idea is that we’re working with stuff that lives in the fixed-point set, so it lifts uniquely to $KO$.

Taking $\Omega^\infty$ of the above sequence of spectra begets fibrations like $BU \to BSO$ (“projections onto the positive eigenspace, rationally”). Complexifying allows you to extend to $BU \to BSO \to BU$, and the composite should be multiplication by 2 on homotopy.

Start with a map $MU \to \mathbb{Q} \otimes KU$. There is a usual one that selects the fgl $x + y - xy$. It has characteristic series $\frac{x}{e^x - 1}$ which has Taylor expansion $\sum_{k=0}^\infty B_k \frac{x^k}{k!}$ where $B_k$ are the Bernoulli numbers (this is the definition of the Bernoulli numbers).

This is not fixed by taking $x \mapsto -x$. There’s a standard way to fix this: replace this characteristic series with $\frac{x}{e^{x/2} - e^{-x/2}} = \exp(-\sum \frac{B_k x^k}{k!})$ (this is the $L$-genus).

$$
\begin{array}{ccc}
MU & \longrightarrow & \mathbb{Q} \otimes KU \\
\uparrow & & \uparrow \\
MSUe & \longrightarrow & \mathbb{Q} \otimes KO
\end{array}
$$

We saw that $MSU$-orientations are just taking $MU$-orientations and precomposing. Trade $MSU \to KO \otimes \mathbb{Q}$ for $MSpin \to KO \otimes \mathbb{Q}$ with characteristic series $\exp(-\sum \frac{B_k x^k}{2k})$. You’re supposed to get this stuff from the multiplication-by-2 on homotopy, but maybe that triangle is wrong.

In $\exp(-\sum \frac{B_k x^k}{2k})$, $\frac{B_k}{2k}$ is $t_{4k}$ for $MSpin \to KO \otimes \mathbb{Q}$.

The postcomposition $MSpin \to KO \otimes \mathbb{Q} \to \mathbb{Q} \otimes KO_p$ does nothing to the characteristic series.

Now let’s study $\mathcal{E}_\infty^\infty(MSpin, KO_p)$, and the map to $\mathcal{E}_\infty^\infty(MSpin, \mathbb{Q} \otimes KO_p)$. The problem, as outlined to us by Jeremy, is to find fillers for the following diagram

$$
\begin{array}{ccc}
\text{spin} & \downarrow j & \longrightarrow & \text{gl}_1 S \\
& & \downarrow \text{gl}_1 \eta_{KO_p} & \\
& & \text{gl}_1 KO_p & \downarrow ? \\
\Sigma^{-1} \mathbb{Q} / \mathbb{Z} \otimes \text{gl}_1 KO_p & \longrightarrow & \text{gl}_1 KO_p & \longrightarrow & \mathbb{Q} \otimes \text{gl}_1 KO_p & \longrightarrow & \mathbb{Q} / \mathbb{Z} \otimes \text{gl}_1 KO_p
\end{array}
$$

where $C_j$ is the cone on $j$. Call such a filler $A$ (the entire triangle on the right).
where $\Sigma^{-1}Q/Z\ldots$ is the fiber. The composite $gl_1 S \to Q/Z\ldots$ is null. There is always a filler of the lower triangle, and you can also fill

\[
\begin{array}{ccc}
\text{spin} & \rightarrow & gl_1 S \\
\downarrow & & \downarrow \text{filler} \\
\Sigma^{-1}Q/Z \otimes gl_1 KO_p & & \\
\end{array}
\]

If I compose two filled triangles, I get a filled square.

\[
\begin{array}{cccccc}
\text{spin} & \rightarrow & j & \rightarrow & gl_1 S & \rightarrow & C_j \\
\downarrow & & \downarrow A & & \downarrow B & & \downarrow C \\
\Sigma^{-1}Q/Z \otimes gl_1 KO_p & \rightarrow & gl_1 KO_p & \rightarrow & Q \otimes gl_1 KO_p & \rightarrow & Q/Z \otimes gl_1 KO_p \\
\end{array}
\]

Things start repeating themselves so you get a filler in the last top triangle. Call $A$ the possible filler for the first triangle considered. The space of $A$’s is the set of nullhomotopies of $\text{spin} \to gl_1 KO_p$. Checking that it’s nonempty is the same as checking that the composite is actually null.

By connectedness of various things (and Jeremy’s talk), $[\text{spin}, gl_01 KO_p] \cong [\text{spin}, L_1 gl_1 KO_p]$. The map $L_1 gl_1 KO_p \to L_{K(1)} gl_1 KO_p$ is $p$-adification: there is a pullback

\[
\begin{array}{ccc}
L_1 gl_1 KO_p & \rightarrow & L_{K(1)} gl_1 KO_p \\
\downarrow & & \downarrow \\
Q \otimes gl_1 KO_p & \rightarrow & Q \otimes L_{K(1)} gl_1 KO_p \\
\downarrow \text{inclusion} & \downarrow & \downarrow \\
\bigvee_{k \geq 1} \Sigma^{4k} H_{\mathbb{Q}_p} & \rightarrow & \bigvee_k \Sigma^{4k} H_{\mathbb{Q}} \\
\end{array}
\]

So by connectivity, I have $[\text{spin}, L_1 gl_1 KO_p] = [\text{spin}, L_{K(1)} gl_1 KO_p]$. Jeremy also said that the Rezk logarithm allows us to throw away the $gl_1$, so this is $[\text{spin}, KO_p] \cong [\Sigma^{-1} KO_p, KO_p]$. The claim is that this is 0.

Suppose fillers $A_1$ and $A_2$ gave rise to the same $C$. They differ by an element of $[\text{bspin}, gl_1 KO_p]$ which goes to zero in $[\text{bspin}, gl_1 KO_p] \otimes \mathbb{Q}$ (??), hence the difference is torsion. I would be able to show that the assignment from $A$’s to $C$’s is injective if I knew $[\text{bspin}, gl_1 KO_p]$ is torsion-free. But $[\text{bspin}, gl_1 KO_p] = [KO_p, KO_p]_{\text{tors}} = 0$. 

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I want to understand what $C$ does on homotopy (since the target is rational, this determines everything).

After a diagram chase you get

$$t_{4k}(C) = (1 - c^k)^{-1} b_{4k}(B')(1 - p^{k-1})^{-1}$$

I’m looking for $C$’s that have to do with the rational orientation I had a while ago, and also if it comes from one of these $B'$s. Here are conditions for “allowable $t_{4k}$’s”:

1. $t_{4k}(C) \equiv -\frac{B_k}{2k}$ (mod $\mathbb{Z}$)

2. $b_{4k}(B')$ (the effect $\pi_{4k}(KO \xrightarrow{f} KO)$) is the effect on homotopy of some map $KO_p \xrightarrow{f} KO_p$

The source material for this is kind of confused. There’s an alternate proof due to Haynes Miller, in *Universal Bernoulli numbers*. (This paper is pretty hard.)

(2) has to do with wanting $[KO_p, KO_p]_{\text{tors}} = 0$ and (earlier) $[\Sigma^{-1}KO_p, KO_p] = 0$.

10.1. Stable *K*-theory operations. Recall $\pi_0 L_{K(1)}(K_p \wedge K_p) = K_p^{\wedge} K_p$ is the ring of functions $O_{\text{Aut}(\hat{G}_m)}$. These are all pro-étale. But in the case of $\mathbb{G}_m$, this is actually constant:

$\text{Aut} \hat{\mathbb{G}}_m = \mathbb{Z}_p^{\times} = S_1 = \text{continuous functions } \mathbb{Z}_p^{\times} \to \mathbb{Z}_p$. Also, $\pi_1 = 0$.

$$K^0 K = \text{Hom}_{\text{cts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)}(\text{cts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p), \mathbb{Z}_p).$$

If I have a co-operation $S^0 \to K \wedge K$, I can make the Kronecker pairing

$$S^0 \xrightarrow{\varepsilon}K \wedge K \xrightarrow{1 \wedge \psi^\lambda}K \wedge K \xrightarrow{i_k}K$$

I’ve paired a stable operation and a stable co-operation to get an element of $\mathbb{Z}_p$. I claim that this pairing agrees with evaluation of $\psi^\lambda \in \text{Hom}(\text{cts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p), \mathbb{Z}_p)$ on $\tilde{c} \in \text{cts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)$ to get something in $\mathbb{Z}_p$.

$\psi^\lambda$ is associated with a $\lambda$-series $[\lambda] \in \text{Aut} \hat{\mathbb{G}}_m \cong \mathbb{Z}_p^{\times}$. I claim this has the effect of evaluation $\tilde{c} \in \text{cts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)$ at $\lambda$. 
I also claim \( c = \beta^{-k} \wedge \beta^k \) corresponds to the polynomial function \( x \mapsto x^k \). You can check this by feeding it into the Kronecker pairing and seeing what comes out: \( \mu(1 \wedge \psi^\lambda)(\beta^{-k} \wedge \beta^k) = \lambda^k \beta^k \cdot \beta^{-k} = \lambda^k \). Now use the previous claim.

**Theorem 10.4.** The assignment \( \text{Hom}(\text{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) \to \prod_{k \geq N} \mathbb{Z}_p \) sending \( f \mapsto (f(x \mapsto x^k))_{k \geq N} \) is injective. The image of this map is called the set of K" ummer sequences.

This is supposed to be giving a description of \( b_{4k} \) above. \( (f(x \mapsto x^k))_{k \geq N} =: (z_k) \) has the effect on homotopy of \( \psi^f \).

There’s also this auxiliary number \( N \), but it doesn’t matter.

\( p \)-adically: \( k + (p - 1)p^r \to k \) as \( r \to \infty \). If I have a continuous homomorphism, what that means is that there is a \( p \)-adic limit \( z_{k+(p-1)p^r} \to z_k \).

The other explanation is that \( K[2N, \infty) \wedge K \cong K \wedge K \).

Note \( K \wedge KO \cong K \wedge (K^{hC_2}) \). The issue is that smashing doesn’t commute with homotopy fixed points. But in this case, there is this property called ambidexterity that says \( K \wedge (K^{hC_2}) \cong K \wedge (K \wedge K)_hC_2 \cong (K \wedge K)^{hC_2} \).\( \pi_0((K \wedge K)^{hC_2}) = \text{cts}(\mathbb{Z}_p^\times/C_2, \mathbb{Z}_p) \).

(Everything is \( K(1) \)-localized.)

**Theorem 10.5** (Mazur). For each \( c \in \mathbb{Z}_p^\times \), there is a homomorphism \( E_{1,c} : \text{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \to \mathbb{Z}_p \) such that
\[
E_{1,c}(x \mapsto x^k) = (1 - c^k) \cdot (1 - p^{k-1}) \frac{B_k}{k}.
\]

Ando-Hopkins-Rezk show that you can put a 2 in the denominator.

Why?? Recall these facts about \( \zeta(s) \):

1. \( \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{dx}{e^x - 1} \). This integral is called a Mellin transform, the moments of the measure. \( \frac{1}{\Gamma(s)} \) is called the real Euler factor.

2. If \( k \in \mathbb{Z}_{>0} \) then \( \zeta(1 - k) = -\frac{B_k}{k} \).

Mazur wanted to construct a \( p \)-adic analogue of the zeta function, and take its Mellin transform. The idea is to build a \( p \)-adic approximation to this construction by building approximations to the integrand.

**Definition 10.6.** Write \( t e^{tx} = \sum_{k=0}^\infty \frac{B_k(x)}{k!} \cdot t^k \). Then \( B_k(x) \) is a Bernoulli polynomial.

I will define a Bernoulli distribution, a function out of \( \mathbb{Z}_p \). Start by building a function \( \mathbb{Z}/p^n \xrightarrow{F} \mathbb{Q} \subset \mathbb{Q}_p \) sending \( x \in [0, p^n) \mapsto \frac{x^{p^n - 1}}{p^n} B_k \left( \frac{x}{p^n} \right) \).
\[ \int_{\mathbb{Z}_p} 1 \cdot dE_k = \frac{1}{k} \sum_{a=0}^{p-1} B_k \left( \frac{a}{p} \right) = \frac{B_k(0)}{k} \]

(This is not obvious.) I want to construct an integral that I can evaluate on any continuous function, not just locally constant ones. These do not extend to functions \( \text{cts}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{Z}_p \) (too many \( p \)'s in the denominators). There is a standard trick for fixing this, called “regularization”.

Pick \( c \in \mathbb{Z}_p^\times \), and form \( E_{k,c}(x) = E_k(x) - c^k E_k(c^{-1} \cdot x) \). These distributions do extend to all continuous functions. You get

\[ \int 1 \cdot dE_{k,c} = (1 - c^k) \frac{B_k}{k}. \]

Claim (also non-obvious):

\[ \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} = \int_{\mathbb{Z}} dE_{k,c} = (1 - c^k) \frac{B_k}{k}. \]

Restrict \( \mathbb{Z}_p \) to \( \mathbb{Z}_p^\times \) in the integral limit, so that I can put any \( k \in \mathbb{Z}_p \) in the exponent. I just have to understand what that restriction does.

\[ \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} = \int_{\mathbb{Z}_p} - \int_{p\mathbb{Z}_p} \]

\[ = (1 - c^k) \frac{B_k}{k} (1 - p^{k-1}) \]

Mazur showed that \( f \mapsto \int_{\mathbb{Z}_p^\times} f \cdot dE_{1,c} \) is a \( p \)-adic replacement for \( \zeta \).

Note that \( 1 - c^k \) came from regularization, \( \frac{B_k}{k} \) came from special values of \( \zeta \), and \( 1 - p^{k-1} \) came from \( p \)-adic interpolation. In homotopy theory, these pieces also came from distinct places: \( (1 - c^k) \) came from the finite Adams resolution, \( \frac{B_k}{k} \) came from the characteristic series for the Atiyah-Bott-Shapiro orientation, and \( 1 - p^{k-1} \) came from the Rezk logarithm. This analogy has been explored to some extent, but no one knows how to compare these operations.

The second part of the paper, about string instead of spin, is analogous, but you need a chromatic fracture cube instead of a square. It’s harder, but not \emph{that} much harder... Bernoulli numbers are replaced by Eisenstein series.

The situation is Haynes’ paper is that we’re studying \( S^1 \wedge \mathbb{C}P^\infty \to \Sigma^{-1}Q/\mathbb{Z} \otimes gl_1 S^0 \). The way he does this is by applying the ANSS to each side, and looking at the map of \( E_2 \)-terms. This has to do with the \( S^1 \)-transfer. Try this for \((S^1)^{\times 2}\) instead – you’ll probably get some interesting things about number theory (but it’s also probably not doable).

Back in the \text{tmf} story, you have to understand \( S^1 \wedge \mathbb{C}P^\infty \to \Sigma^{-1}Q/\mathbb{Z} \otimes gl_1 S^0 \to \Sigma^{-1}Q/\mathbb{Z} \otimes gl_1 \text{tmf} \). That’s hard going.