AN ELEMENTARY PROOF OF QUILLEN’S THEOREM FOR COMPLEX COBORDISM

A thesis submitted by
Christian Carrick,

advised by
Michael Hopkins,

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1. Introduction

In ordinary cohomology of topological spaces, one has the Steenrod squares and powers. These are cohomology operations

\[ Sq^i : H^k(X; \mathbb{Z}/2) \to H^{k+i}(X; \mathbb{Z}/2) \]

\[ p_j^i : H^k(X; \mathbb{Z}/p) \to H^{k+2j(p-1)}(X; \mathbb{Z}/p) \]

that can be characterized axiomatically. They are extremely useful and one only needs to know their axioms in practice, but their construction is not very straightforward. The \( p \)-th power operation is constructed by noticing that, for a space \( X \), the \( p \)-fold cartesian product \( X^p \) is acted on cyclically by \( \mathbb{Z}/p \) and fits into a fibration

\[ X^p \to \mathbb{E}/p \times_{\mathbb{Z}/p} X^p \to B\mathbb{Z}/p \]

Then if \( u \in H^*(X; \mathbb{Z}/p) \), one constructs the \( k \)-fold cross-product \( u \times \cdots \times u \in H^*(X^p, \mathbb{Z}/p) \) in the obvious way on the chain level. This class is then extended to a class \( \tilde{u} \in H^*(\mathbb{E}/p \times_{\mathbb{Z}/p} X^p; \mathbb{Z}/p) \) by a similar construction on the chain level, and one pulls \( \tilde{u} \) back along the map

\[ \mathbb{E}/p \times_{\mathbb{Z}/p} \Delta : \mathbb{E}/p \times_{\mathbb{Z}/p} X \to \mathbb{E}/p \times_{\mathbb{Z}/p} X^p \]

to give an element of \( H^*(B\mathbb{Z}/p \times X; \mathbb{Z}/p) \). Applying the Kunneth formula and taking the classes in \( H^*(X; \mathbb{Z}/p) \), one defines the classes \( p^l p(u) \).

Since this construction uses heavily the chain-level description of cohomology, it is not clear how to extend such a construction to other generalized cohomology theories. If \( E \) is a highly commutative ring spectrum in the sense that it admits a factorization

\[ E^\wedge p \to (E^\wedge p)_{h\mathbb{Z}/p} \to E \]

where subscript \( h\mathbb{Z}/p \) denotes the homotopy quotient, and \( X \) is a space with \( u : \Sigma^\infty_+ X \to E \in E^*(X) \), we take the map

\[ \Sigma^\infty_+ X \xrightarrow{\Delta} (\Sigma^\infty_+ X)^\wedge p \to E^\wedge p \]

taking the homotopy quotient of this map with respect to \( \mathbb{Z}/p \) gives a map

\[ \Sigma^\infty_+ (X \times B\mathbb{Z}/p) \to (E^\wedge p)_{h\mathbb{Z}/p} \to E \]

in \( E^*(X \times B\mathbb{Z}/p) \). But this requires some relatively advanced concepts, like the language of highly commutative ring spectra. This approach would work and could prove all the results in this paper [8], but we take the more elementary approach of Quillen. Namely, when \( E = MU \) and \( X \) is a smooth manifold, \( MU^*(X) \) has an elementary description as a particular set of smooth maps into \( X \). If \( f : Z \to X \) is such a map, one can simply take the map \( \mathbb{E}/p \times_{\mathbb{Z}/p} (f^\wedge p) \) and pull it back along \( \mathbb{E}/p \times_{\mathbb{Z}/p} \Delta \), after approximating the infinite complex \( \mathbb{E}/p \) by manifolds. Not much generality is lost here since every finite CW complex is homotopy equivalent to a smooth manifold via a neighborhood of an embedding into Euclidean space.

These operations in \( MU^* \) turn out to have useful properties like the Steenrod powers, and when one compares them to the Landweber-Novikov operations - \( MU \) operations which are defined via \( MU \) Chern classes - one finds they are closely related. In his 1971 paper Elementary Proofs of Some Results of Cobordism Theory, Quillen discovers this relationship and uses it to give a remarkable description of \( MU^*(X) \) for \( X \) a finite CW complex [7]. He then uses it to re-prove his theorem that the map \( L \to \pi_*(MU) \) from the Lazard ring classifying the formal group law on \( \pi_*(MU) \) is an isomorphism. His paper is notoriously difficult to follow, and this paper is an exposition of Quillen’s results that fills in many of the details left out in his paper, motivates some of his results, and attempts to give a sense of why his paper is important. I also include an appendix a detailed proof that \( MU^*(X) \) has the claimed geometric description for manifolds.

2. Prerequisites

Since the spirit of Quillen’s paper is that his proof is elementary - in the sense that it does not make any use of the Adams’ Spectral Sequence or the structure of \( H^*(MU) \) as a module over the Steenrod Algebra, and only relies on one result from homotopy theory (5.2.1) - I cover some basic concepts in stable homotopy theory that I make use of in the paper. Most of this material is not crucial to the main arguments, but serves well to put things in a broader context. I assume the reader has familiarity with smooth manifolds, singular cohomology of spaces, basic category theory, and vector bundles and principal bundles. The material covered
in this section can be found in greater detail in [1] and [5].

**Definition 2.1.** A spectrum $E$ is a sequence of pointed topological spaces $\{E_i\}_{i \in \mathbb{Z}}$ together with pointed structure maps

$$\Sigma E_i = S^1 \wedge X \rightarrow E_{i+1}$$

for all $i$. If $X$ is a pointed space, we let $\Sigma^\infty X$ - its *suspension spectrum* - be the spectrum with $(\Sigma^\infty X)_i = \Sigma^i X = S^i \wedge X$.

One of the key motivations for defining spectra is Brown’s representability theorem. This says if $\{h^i\}_{i \in \mathbb{Z}}$ is a sequence of contravariant functors from the category of CW pairs to the category of abelian groups satisfying certain axioms satisfied by ordinary cohomology like homotopy invariance and excision - what one calls a *generalized cohomology theory* - then for each $i$ there is a space $E_i$ such that $h^i(X, x_0) \cong [X, E_i]$, where $X$ is a CW complex with basepoint $x_0$, and $[-, -]$ denotes pointed homotopy classes. For ordinary cohomology, these $E_i$ are the Eilenberg-MacLane spaces $K(\mathbb{Z}, i)$. Because every generalized cohomology theory has suspension isomorphisms, the sequence $\{E_i\}$ then forms a spectrum $E$. We would thus like to define the category of spectra so that the spectrum $E$ represents $\{h^i\}$ in the sense that a cohomology class in $h^i(X)$ corresponds to a morphism $\Sigma^\infty X \rightarrow E$. That $h^i(X, x_0) \cong [X, E_i]$ tells us already that our notion of morphism should be defined modulo homotopy in a suitable sense. It is easy to define a function between spectra $E$ and $F$ - one takes a sequence of maps $f_i : E_i \rightarrow F_{i-k}$ such that the diagram

$$\begin{array}{ccc}
\Sigma E_i & \xrightarrow{\Sigma f_i} & \Sigma F_{i-k} \\
\downarrow & & \downarrow \\
E_i & \xrightarrow{f_i} & F_{i-k}
\end{array}$$

commutes for all $i$, and we say $\{f_i\}$ is a function of degree $k$. But it takes a bit more work to define a morphism that satisfies the above property. We take it as given that there is a notion of morphism between spectra such that

$$[\Sigma^\infty X, E]_k \cong \text{colim}_{i \rightarrow \infty} [\Sigma^i X, E_{i-k}]$$

where $[-, -]_k$ denotes morphisms of degree $k$, and when $E$ comes from a generalized cohomology theory $\{h^i\}$, we have $\text{colim}_{i \rightarrow \infty} [\Sigma^i X, E_{i-k}] \cong [X, E_{-k}] \cong h^{-k}(X, x_0)$. If $E$ is a spectrum, and we let $(\Sigma^k E)_i = E_{k+i}$, then one has isomorphisms

$$[E, F]_* \cong [\Sigma E, \Sigma F]_* \cong [\Sigma^2 E, \Sigma^2 F]_*$$

And since forming $\Sigma E$ is the same as forming the spectrum whose $i$-th space is $S^1 \wedge E_i$, $[E, F]_k$ is an abelian group for all $k$ by essentially the same reasoning that shows $\pi_k(X)$ is an abelian group for a space $X$ and $k > 1$. The isomorphism $h^{-k}(X, x_0) \cong [\Sigma^\infty X, E]_k$ given by Brown representability is then an isomorphism of abelian groups, and if the $h^i$ are a multiplicative cohomology theory in the sense that $h^*(X)$ forms a graded ring, we want to introduce a notion intrinsic to spectra that makes this isomorphism a ring isomorphism. We take the following result as given:

**Proposition 2.2.** There exists a bifunctor $- \wedge -$ on the category of spectra that is associative, commutative, and has the sphere spectrum $S$ (i.e. $\Sigma^\infty S^0$) as a unit, all up to coherent natural isomorphism. If $X$ is a pointed space then $\Sigma^\infty X \wedge E$ is isomorphic to the spectrum whose $i$-th space is $X \wedge E_i$. A spectrum $E$ with maps $\mu : E \wedge E \rightarrow E$ and $\eta : S \rightarrow E$ such that $\mu$ is associative and unital with respect to $\eta$ in a suitable sense is called a *ring spectrum*.

**Definition 2.3.** If $E$ is a spectrum and $X$ a pointed space, we define $\tilde{E}^n(X) := [\Sigma^\infty X, E]_{-n} \cong \text{colim}_{i \rightarrow \infty} [\Sigma^i X, E_{i+n}]$ and $\tilde{E}_n(X) := [S, \Sigma^\infty X \wedge E]_n \cong \text{colim}_{i \rightarrow \infty} \pi_i(X \wedge E_{i-n})$, the reduced $E$ cohomology and homology of $X$ respectively. When $X$ is an unpointed space we define $E^n(X) = \tilde{E}^n(X_\ast)$ where $X_\ast := X \sqcup \ast$ and similarly for homology. Homology and cohomology of a pair $(X, A)$ are defined by taking reduced cohomology of $X/A$. One also has a splitting

$$E^*(X) \cong \tilde{E}^*(X) \oplus E^*(\ast)$$
as in ordinary cohomology. \( E^*(\cdot) \) is often abbreviated \( E^* \).

When \( E \) is a ring spectrum, \( \tilde{E}^*(X) \) forms a graded ring and the isomorphism from Brown representability when \( E \) is built from \( \{ h^i \} \) is then an isomorphism of graded rings. The ring structure on \( \tilde{E}^*(X) \) is given by

\[
\tilde{E}^*(X) \otimes \tilde{E}^*(X) = [\Sigma^\infty X, E] \otimes [\Sigma^\infty X, E] \to [\Sigma^\infty X \wedge \Sigma^\infty X, E \wedge E] \xrightarrow{\Delta} [\Sigma^\infty (X \wedge X), E] \xrightarrow{\Delta^*} [\Sigma^\infty X, E] = \tilde{E}^*(X)
\]

where \( \Delta : X \to X \wedge X \) is the diagonal and we use the fact that \( \Sigma^\infty X \wedge \Sigma^\infty X \cong \Sigma^\infty (X \wedge X) \).

**Example 2.4.** Let \( EO(n) \to BO(n) \) be the universal vector bundle of rank \( n \) and let \( MO(n) \) be its Thom space. Then since the Thom space construction is functorial, we have a map \( \text{Thom}(EO(n) \oplus 1) \to \text{Thom}(EO(n+1)) = MO(n+1) \) where \( 1 \) denotes the trivial bundle of rank 1. However, \( \text{Thom}(EO(n) \oplus 1) = \text{Thom}(EO(n)) \wedge S^1 = \Sigma MO(n) \), hence the sequence \( \{ MO(n) \} \) forms a spectrum we call \( MO \). Applying the same construction to the universal bundles \( EU(n) \to BU(n) \) we obtain a spectrum \( MU \), where in this case we have maps \( \Sigma^2 MU(n) \to MU(n+1) \), so we let the even spaces be \( MU(n) \), and \( (MU)_{2n+1} = \Sigma MU(n) \).

In fact if \( \{ G_n \} \) is a sequence of topological groups with maps \( G_n \to BO(k_n) \), satisfying certain compatibility conditions (e.g. \( \{ BU(n) \} \) with \( k_n = 2n \), or \( \{ BSO(n) \} \) with \( k_n = n \)) we may form a spectrum \( MG \) in the same way, and there is a canonical morphism \( MG \to MO \). We call these Thom spectra.

\( MO \) and \( MU \) are ring spectra because the direct sum maps \( BO(n) \times BO(m) \to BO(n+m) \) under the Thom construction give maps \( MO(n) \wedge MO(m) \to MO(n+m) \) and similarly for \( MU \). These maps patch together in the limit to give a ring map \( \mu \).

**Definition 2.5.** A complex oriented cohomology theory is a ring spectrum \( E \) with a chosen class \( x \in \tilde{E}^2(\mathbb{CP}^\infty) \) such that under the map

\[
\tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(\mathbb{CP}^1) = \tilde{E}^2(S^2) \cong E^0(\cdot)
\]

induced by inclusion \( \mathbb{CP}^1 \to \mathbb{CP}^\infty \), \( x \) is sent to 1 in the ring \( E^0(\cdot) \).

Arguing by universality building up from the universal complex line bundle, one can show that this is the same as saying that there is a Thom class in \( E^{2n}(V, V - X) \) for every complex vector bundle \( V \to X \) of rank \( n \), and these Thom classes are multiplicative and natural under pullbacks. Then since every complex vector bundle is orientable in the sense of ordinary cohomology and thus has a Thom class, we see that the Eilenberg-Maclane spectrum \( (HZ)_n = K(\mathbb{Z}, n) \) (i.e. the one that represents ordinary cohomology) is complex oriented. \( MU \) is a complex oriented cohomology theory because the zero section \( \mathbb{CP}^\infty \to BU(1) \to EU(1) \) induces a homotopy equivalence \( \mathbb{CP}^\infty \to MU(1) \), which determines a class \( x \in [\Sigma^\infty \mathbb{CP}^\infty, MU]_{-2} \), and it pulls back to the 1 in \( MU^0(\cdot) \) because the map \( \mathbb{CP}^1 \to \mathbb{CP}^\infty \to MU(1) \) induces a map \( S \to MU \), and this is by definition the unit for the ring spectrum \( MU \). \( MU \) is in fact the universal complex oriented cohomology theory:

**Proposition 2.7.** Every complex oriented cohomology theory \( E \) receives a unique ring map from \( MU \) that respects the classes \( x \).

**Proof:** We sketch a proof of this. Since the universal bundle over \( BU(n) \) is a complex vector bundle there exists a Thom class \( u \in E^{2n}(EU(n), EU(n) - BU(n)) \cong \tilde{E}^{2n}(MU(n)) \) and hence we have an element of \( [\Sigma^{-2n} \Sigma^\infty MU(n), E]_0 \cong \tilde{E}^{2n}(MU(n)) \) for all \( n \). Then since \( MU = \colim_n \Sigma^{-2n} \Sigma^\infty MU(n) \), we may ask if these maps define a map from the colimit \( MU \). This can be deduced from the Mihor sequence [1], noting that the \( \lim^1 \) terms vanish since the maps \( E^*(MU(n+1)) \to E^*(MU(n)) \) are surjections since the maps \( E^*(BU(n+1)) \to E^*(BU(n)) \) are. We omit the proof that this map respects the ring structures, except to say that it follows by building \( E^* \) Chern classes in the method of 3.3 and applying the Cartan formula. \( \square \)

We finish off with an important property about complex oriented cohomology theories, which follows by a computation using the Atiyah-Hirzebruch spectral sequence.

**Proposition 2.8.** If \( E \) is a complex oriented cohomology theory, then \( E^*(\mathbb{CP}^\infty) \cong E^*|x|/x^{n+1} \). \( \square \)
3. **Geometric Cobordism**

3.1. **The Geometric Model of $MU^*(X)$**.

In all that follows we assume all manifolds and vector bundles to be smooth. For a compact manifold $X$, we define the unoriented bordism groups of $X$ as the set

$$\Omega^O_n(X) := \{ f : M \to X : f \text{ is smooth, and } \dim M - \dim X = n \} / \sim$$

where $(f_1 : M_1 \to X) \sim (f_2 : M_2 \to X)$ if there is a smooth map $g : W \to X$ for a compact manifold $W$, with $\partial W = M_1 \cup M_2$ and $\partial g = f_1 \cup f_2$. $\Omega^O_n(X)$ becomes a group under disjoint union, and Thom’s theorem says that there is an isomorphism of groups $\Omega^O_n(X) \cong [S, MO \wedge \Sigma^n \mathbb{X}]_n =: MO_n(X)$, where $MO$ is the Thom spectrum of $BO$. Since $MO$ is the universal Thom spectrum, this theorem may be seen as a way to geometrically interpret a Thom spectrum $MG$ as a homology theory by asking that the geometric interpretation of $MO$ factor through the map $MG \to MO$ in some sense.

For the proof of Thom’s theorem in the case $X = \ast$, one takes an embedding $M \hookrightarrow \mathbb{R}^N$ and forms the normal bundle $v_N$, with $\dim M = n$. Then by the tubular neighborhood theorem [4] there is an open embedding $v_N \hookrightarrow \mathbb{R}^N$ so that the following diagram commutes

$$\begin{array}{ccc}
\nu_N & \hookrightarrow & \mathbb{R}^N \\
\downarrow \text{0-section} & & \\
M & \hookrightarrow & \mathbb{R}^N
\end{array}$$

Since one point compactification is a contravariant functor with respect to open inclusions of locally compact spaces, we get a map $S^N \cong (\mathbb{R}^N)^+ \to (v_N)^+$. Since $v_N$ is a vector bundle, it admits $v_N \to EO(N - n)$, and applying the Thom functor, we have a map $(v_N)^+ \to MO(N - n)$, recalling that $\text{Thom}(v_N) \cong v_N^+$ since $M$ is compact. Putting these together, we have a map

$$S^N \to \text{Thom}(v_N) \to MO(N - n)$$

which gives an element of $[S, MO]_n = MO_n(\ast)$ The choice of embedding of $M$ and the cobordism relation corresponds exactly to two maps obtained in the above way being stably homotopic, i.e. equivalent in $[S, MO]$, where $S$ is the sphere spectrum. Said another way, the element in $[S, MO]_n$ only depends on the class of $v_N$ in $[M, BO]$ and the cobordism class of $M$. For a different Thom spectrum $MG$, we may therefore ask that the classifying map $v : M \to BO$ factor through the map $BG \to BO$ that forgets the $G$ structure on a vector bundle, and thus a factorization of the maps $S \to MG$ and $S \to MO$ through $MG \to MO$.

This approach works to define geometric bordism (i.e. a geometric model for $MG_*$) for arbitrary $MG$. Defining geometric cobordism (i.e. a geometric model for $MG^*$) is a bit trickier, and one of the interesting parts of Quillen’s paper is that he constructs such a geometric model for $MU^*$. His model can be applied to other $MG$ as well [2]. The model is as follows:

**Definition 3.1.1.** A **complex oriented map** is a pair $(f, v)$ where $f$ is a smooth proper map $f : Z \to X$ of even dimension (i.e. $\dim f := \dim Z - \dim X$ is even), and $v \in [Z, BU]$ is such that for some $n$, we have that $f$ factors as $Z \hookrightarrow X \times \mathbb{C}^n \to X$ for an embedding $i : Z \hookrightarrow X \times \mathbb{C}^n$, and $v$ has a complex structure so that the class of

$$Z \xrightarrow{v} BU((2n - \dim f)/2) \to BU$$

in $[Z, BU]$ is equal to $v$. We say that a complex orientation on an even dimensional map $f$ is a pair as above, and one on an odd dimensional map is a pair as above for the map $(f, 0) : Z \to X \times \mathbb{R}$.

**Example 3.1.2.** If $E \to Z$ is a complex vector bundle, then the zero section $i : Z \to E$ is a complex oriented map (where $E$ is playing the role of $X$ in 3.1.1). Since $i$ is an embedding, we may factor $i$ as $Z \to E \times \mathbb{C}^0 \to E$ and $v_i \cong E$. More generally, if $i : Z \to X$ is a closed embedding of manifolds such that $v_i$ has a complex structure, then it is a proper map, and we may factor $i$ through $X \times \mathbb{C}^0$, and thus $i$ is complex oriented.

**Pullbacks:**
If \( f : Z \to X \) has a complex orientation and \( g : Y \to X \) is transverse to \( f \), we have the maps

\[
h : Y \times_X Z \to Y
\]
\[
v' : Y \times_X Z \to Z \xrightarrow{\nu} BU
\]
Since \( f \) factors through an embedding \( i \) into \( X \times C^n \), we can pullback this bundle over \( X \) to \( Y \times C^n \) over \( Y \), and \( Y \times_X Z \) factors through an embedding \( j \) into \( Y \times C^n \). Then \( v_j \) is the pullback of \( v_i \) along \( \tilde{g} : Y \times_X Z \to Z \), hence \( v_j \) is the same class in \( [Y \times_X Z, BU] \) as \( v' \) since \( v_i \) is in the same class as \( v \). Furthermore, there is no dependence on the choice of representative for the homotopy class \( v \in [Z, BU] \), since all such representatives will yield the same class for \( v' \). It is not hard to see that since all the spaces are Hausdorff, the pullback of a proper map is proper. \( h \) is thus complex oriented.

**Definition 3.1.3.** Two complex oriented maps \((f_0 : Z_0 \to X, v_0)\) and \((f_1 : Z_1 \to X, v_1)\) are said to be **cobordant** if there is a complex oriented map \((g : W \to X \times [0, 1], \tau)\) with \( \epsilon_i : X \to X \times [0, 1] \) sending \( x \mapsto (x,i) \) transversal to \( g \) for \( i = 0, 1 \), and \( \epsilon^*_i(g, \tau) = (f_i, v_i) \).

Cobordism forms an equivalence relation because if \( f_1 : Z_1 \to X, f_2 : Z_2 \to X \), and \( f_3 : Z_3 \to X \) have complex orientations such that \( f_1 \) is cobordant to \( f_2 \), and \( f_2 \) is cobordant to \( f_3 \), there exist manifolds \( W \) and \( W' \) with the conditions as above. In particular, \( Z_2 \) is a submanifold of \( W \) and \( W' \), so we can form the connected sum of \( W \) and \( W' \) along \( Z_2 \), and the resulting manifold exhibits a cobordism between \( f_1 \) and \( f_3 \). Symmetry and reflexivity are clear. There is a bit more to be said here regarding orientations and gluing, but we refer the reader to [2] since these details are non-essential here. We thus define \( U^g(X) \), the complex cobordism groups of \( X \) to be the set of complex oriented maps into \( X \) of dimension \( -q \) modulo cobordism. The negative dimension here is in place so that the gradings in \( U^g \) and \( MU^g \) coincide.

**Group Structure:**

If \((f, v), (f', v') \in U^g(X)\), we define their sum to be \((f \sqcup f', v \sqcup v')\). This is a complex oriented map because the factorizations of \( f \) and \( f' \) through embeddings \( i, i' \) into trivial bundles \( n \) and \( n' \) respectively give a factorization of \( f \sqcup f' \) through an embedding into their direct sum, and the corresponding normal bundle is \((n \oplus n') \sqcup (n' \oplus n) \to Z \sqcup Z'\), whose class in \([Z \sqcup Z', BU]\) is equal to \( v \sqcup v' \). The group law respects cobordism because a cobordism on either side of a map \( Z_1 \cup Z_2 \to X \) gives one on the sum of the maps by holding the other side constant. Regarding the empty map \( \emptyset \to X \) as a map of dimension \(-q\) gives an identity element. We note that any \((f, v)\) has an inverse given by the same map \( f \), with the same factorization via an embedding \( i \) into \( n \), but \( C^n \) is given the complex structure and \( v_i \) is thus given the structure induced on the quotient. This gives an inverse for \((f, v)\) because the orientation on the manifold \( X \times [0, 1] \) is the one with the orientation for \( X \) on \( X \times \{0\} \) and the negative orientation \((\text{i.e. } -X)\) on \( X \times \{1\} \). When the manifold \( X \times [0, 1] \) is oriented via a complex structure on the normal bundle of an embedding into \( C^n \), the negative orientation is induced on \( X \times \{1\} \) by taking the negative of the \( n \)-th normal vector, as above. Thus the horseshoe map \( X \times [0, 1] \to X \times [0, 1] \) has the corresponding complex orientation, and gives a cobordism between \((f, v) + (-f, v)\) and the empty map. For more details, see [2].

**Ring Structure:**

\( U^g \) is a contravariant functor - at least to \textbf{Sets} for now - because for any \( g : Y \to X \) and any \((f, v) \in U^g(X)\), we may find \( \hat{g} \) homotopic to \( g \) that is transverse to \( f \) [4], and then define \( g^*((f, v)) \) to be the pullback of \((f, v)\) along \( \hat{g} \). As above, this of course does not depend on the choice of \( \hat{g} \) since all such choices will be homotopic and they will thus pull back \((f, v)\) to the same cobordism class, and for the same reason we see that any two homotopic maps \( \hat{g}, g' \) have the same pullback.

For \((f, v), (f', v') \in U^g(X)\), we can define \((f \times f', v \times v') \in U^g(X \times X)\), where \( v \times v' \) is the homotopy class

\[
Z \times Z' \xrightarrow{v \times v'} BU \times BU \to BU
\]
using the \( H \)-space structure on \( BU \) given by direct sums of vector bundles. This gives a map \( U^g(X) \times U^g(X) \to U^g(X \times X) \), and we postcompose this with the map \( \Delta^g : U^g(X \times X) \to U^g(X) \) to define \( U^g(X) \)
as a ring, where $\Delta : X \to X \times X$ is the diagonal map. The identity map of $X$ with the obvious orientation serves as a multiplicative identity.

$U^*(X)$ is moreover an algebra over $U^*(\ast)$ because we may use the same procedure to define a map $U^*(Y) \times U^*(X) \to U^*(Y \times X)$ for any $Y$, and when $Y = \ast$, we identify $X \cong \ast \times X$ to obtain an action.

**Lemma 3.1.4.** $U^*$ is a contravariant functor of $U^*(\ast)$-algebras.

**Proof:** We have already shown $U^*$ to be functorial via pullbacks, it suffices to show that $g^*$ is a map of $U^*(\ast)$-algebras for $g : Y \to X$ smooth. $g^*$ is additive because $g^*(f \cup f', v \cup v')$ is just $g^* f \cup g^* f'$ with the normal bundles pulled back, as any map $\overline{g}$ with which we replace $g$ that is transverse to $f \cup f'$ must be transverse to both $f$ and $f'$. Hence when we pull back, we get the sum of the maps obtained from pulling back $f$ and $f'$ along $\overline{g}$. $g^*$ is multiplicative since $g^*(f_1 \cdot f_2) = (\Delta_Y \circ g)^*(f_1 \times f_2)$, and

$$g^*(f_1) \cdot g^*(f_2) = \Delta_X (g \times g)^*(f_1 \times f_2) = (g \times g \Delta_X)^*(f_1 \times f_2) = (\Delta_Y \circ g)^*(f_1 \times f_2)$$

$g^*$ is unital since the identity map pulls back to the identity. Finally, $g^*$ is $U^*(\ast)$-linear since the choice of a replacement for $g$ that is transverse to a map $r \times f : M \times Z \to \ast \times X$ is the same as one that is transverse just to $f$, and when we pull back we just get $r \times g^*(f)$. All of the normal bundle data carries through the above arguments naturally because $\overline{g}$ - the map at the top of the pullback square involving $g$, pulls back the normal bundles to their vector bundle pullbacks, and such pullbacks are natural with respect to sums and products. $\square$

**Pushforwards:**

If $(f, v) \in U^*(X)$ and $(g : X \to Y, \tau) \in U^*(Y)$, we may define $g_*(f)$ to be the map $g \circ f$. To see that this map has a complex orientation, recall that we can factor $f$ as $Z \to X \times \mathbb{C}^n \to X$ for an embedding $i$ and $g$ as $X \to Y \times \mathbb{C}^m \to Y$ for an embedding $j$, and thus we may factor $g \circ f$ as

$$Z \xrightarrow{i} X \times \mathbb{C}^n \xrightarrow{i \times 1} (Y \times \mathbb{C}^m) \times \mathbb{C}^n \to Y$$

and the normal bundle of the embedding $(j \times 1) \circ i$ is $v_i \oplus f^* v_j$ and thus comes equipped with a complex structure coming from those of $v_i$ and $v_j$. We then have an element $(g \circ f, v_{(i \times 1)\circ i}) \in U^*(Y)$ which depends on the choices of $i$ and $j$ only up to their class in $[Z, BU]$ and $[X, BU]$ since we take the class $v_{(i \times 1)\circ i} \in [Z, BU]$. It is easy to check that these pushforwards are group homomorphisms and are $U^*(\ast)$-linear, but they are not in general multiplicative. We record some basic facts about how the pushforwards and pullbacks interact.

**Lemma 3.1.5.** If $(f : Z \to X, v) \in U^*(X)$ and $g : Y \to X$ is transverse, and $g', f'$ are the maps in the following diagram

$$
\begin{array}{ccc}
Y \times X Z & \xrightarrow{g'} & Z \\
\downarrow f' & & \downarrow f \\
Y & \xrightarrow{g} & X
\end{array}
$$

and $f'$ is given the pullback complex orientation, then $g^* f_* = f'_* g'^*$.

**Proof:** Let $(h : A \to Z) \in U^*(Z)$, and let $f$ and $h$ factor through embeddings $j$ and $i$, respectively. Then $g'^*(h)$ is given by the pullback map $(Y \times X Z) \times_Z A \to Y \times X Z$ with the pullback complex orientation, and thus $f'_* g'^*(h)$ is the map

$$(Y \times X Z) \times_Z A \to Y \times X Z \xrightarrow{f'} Y$$

with the pullforward complex orientation. The other way around, $f_* (h)$ is the map $A \xrightarrow{h} Z \xrightarrow{f} X$, so $g^* f_* (h)$ is the pullback map $Y \times X A \to Y$. But $Y \times X A = \{(y, a) : g(y) = f(h(a))\}$ and $(Y \times X Z) \times_Z A = \{(y, z, a) : g(y) = f(z)\}$, and these are isomorphic and the maps are the same. Let $g''$ be the map $Y \times X A \to A$. The normal bundle given by going around counterclockwise is $g''^* v_i \oplus g'^* f^* v_j$, and going clockwise we have $g''^* (v_i \oplus f^* v_j)$ hence the result follows from the fact that pullbacks of vector bundles commute with direct sums. $\square$
Lemma 3.1.6. Let \( f : Z \to X \) be a complex oriented map, and let \( x \in U^*(X) \), then \( f_*f^*(x) = f_*(1) \cdot x \).

Proof: It is sometimes useful to refer to the class \( f \) also as \( f_*(1) \) to make clear we are referring to the cobordism class and not the map itself. Represent \( x \) by a map \( x : M \to X \), then \( f^*(x) \) is the map from the fiber product of \( f' \) and \( x \) for \( f' \) homotopic to \( f \) with \( f' \) transverse to \( x \). To say \( f' \) is transverse to \( x \) is the same as saying the map \( f' \times x : Z \times M \to X \times X \) is transverse to the diagonal \( \Delta : X \to X \times X \). Then it is easy to check that the latter fiber product is just the pullback \( f_*f^*(x) \), thus by the definition of products in \( U^*(X) \), we have the above equation. An argument similar to 3.1.5 shows that the normal bundles match. □

With these definitions in place, we may now state the analogue of Thom’s theorem for \( MU^* \). We give the proof in Appendix B, since the details of it are not crucial to understand moving forward.

Theorem 3.1.7. Regarding \( MU^* \) and \( U^* \) as functors from the category of smooth manifolds to the category of graded rings, there is a natural isomorphism \( U^* \to MU^* \). For any manifold \( X \), \( U^*(X) \) has the structure of a graded algebra over \( U^*(*) \), and similarly for \( MU^*(*) \). Under the isomorphism \( U^*(*) \to MU^*(*) \), the ring isomorphism \( U^*(X) \to MU^*(X) \) is an isomorphism of graded \( U^*(*) \)-algebras. If \( A \) is a strong deformation retract of an open neighborhood \( U \) in \( X \), we may similarly identify

\[
U^*(X, X - A) := \{ \text{Complex-oriented maps } f : Z \to X : f(Z) \subset A \}
\]

with \( MU^*(X, X - A) \).

3.2. The Thom Isomorphism.

Since \( MU \) is a complex-oriented cohomology theory, if \( E \to X \) is a complex vector bundle, we have a Thom isomorphism \( MU^*(X) \to MU^*(E, E - X) \) which pulls back a class in \( MU^*(X) \) along \( E \to X \) and takes its product with the Thom class \( \pi \in MU^*(E, E - X) \). If \( i : X \to E \) is the zero section of this bundle, it is a complex oriented map as in 3.1.2, and we thus have a pushforward \( i_* : U^*(X) \to U^*(E, E - X) \), which is obviously injective. It is also surjective because, for \((f, v) \in U^*(E, E - X)\), since \(f\) lands in \(X\), we can factor \(f\) as

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \times \mathbb{C}^n \\
\downarrow{j} & & \downarrow{j}
\end{array}
\]

where \(i\) and \(j\) are embeddings. \(v_{j\circ i}\) has a complex structure and splits as \(v_i \oplus i^*v_j\). \(v_i\) is just the normal bundle of \(X \to E\) (i.e. \(E\)), and hence we may give \(v_i\) the complex structure of \(v_{j\circ i}/i^*v_j\). This complex oriented map now pushes forward to recover \((f, v)\). This argument goes through unchanged for an embedding \(i : A \to X\) with a complex structure on \(v_i\). It will be very useful to know that \(i_*\) corresponds to the usual Thom isomorphism in the following way.

Proposition 3.2.1. If \( \pi : E \to X \) is a complex vector bundle with zero section \( i : X \to E \), the following diagram commutes

\[
\begin{array}{ccc}
U^*(X) & \xrightarrow{i_*} & U^*(E, E - X)
\end{array}
\]

\[
\begin{array}{ccc}
MU^*(X) & \xrightarrow{\text{Thom}} & MU^*(E, E - X)
\end{array}
\]

Proof: We first show that \(i_* = \pi^*(f) \cdot i_*(1) \) for \( f \in U^*(X) \), where \( 1 \) is the identity map of \( X \) with the obvious complex orientation. \(f\) is transverse to \(\pi\) since \(\pi\) is a submersion, so we may take the fiber product and we see that \(\pi^*(f)\) is the map \(f^*E \to E\), with the complex structure pulled back from \(\pi\). Then \(\pi^*(f) \cdot i_*(1)\) is the map in the pullback square

\[
\begin{array}{ccc}
P & \to & X \times f^*E \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
E & \xrightarrow{\Delta} & E \times E
\end{array}
\]

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Proposition 3.3.1. Thom classes in $\mathbf{U}^*$ are multiplicative and natural under pullbacks, then since $i^*$ is a ring homomorphism, and zero sections always pull back to zero sections, Euler classes in $\mathbf{MU}^*$ are multiplicative and natural under pullbacks. But $U^* \to \mathbf{MU}^*$ is natural and multiplicative, and it sends the classes $i^*i_s(1)$ to Euler classes in $\mathbf{MU}^*$.

3.3. Characteristic Classes in $\mathbf{U}^*$.

It is easy to define Chern classes in $U^*$ (and therefore $\mathbf{MU}^*$) using the Grothendieck construction, which we recall here. There is a clean way to organize these classes into polynomials - not the usual Chern polynomials - so that on a line bundle, the polynomial is given by the sum of the powers of the Chern class. Via the Thom isomorphism, these polynomials thus define elements of $\mathbf{MU}^*(MU) := [\mathbf{MU}, MU]_*$ which are called the Landweber-Novikov operations. These operations are the key tools in Quillen’s proof.

**Proposition 3.3.1.** Let $E \to X$ be a complex vector bundle of rank $n$ with zero section $i : X \to E$. There exist unique classes $c_i \in U^{2i}(X)$ for $i = 0, 1, \ldots, n$ such that the following hold:

\[(1) \quad c_0(E) = 1\]
(2) For \( f : Y \to X, f^*(c_i(E)) = c_i(f^*E) \)

(3) \( c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cdot c_j(E_2) \).

(4) If \( L_i^N \to \mathbb{CP}^N \) is the tautological line bundle, \( c_1(L_i^N) = i^*i_*(1) \).

**Proof:** Taking the projective bundle of \( E \), one has a fibration

\[ \mathbb{CP}^{n-1} \to \mathbb{P}(E) \to X \]

Then since \( MU^*(\mathbb{CP}^{n-1}) \) is a free \( MU^*(*) \) module by 2.8, the Leray-Hirsch theorem says that there is a class \( y \in MU^2(\mathbb{P}(E)) \) that restricts to the element \( x_{n-1} \in MU^2(\mathbb{CP}^{n-1}) = \overline{MU}^2(\mathbb{CP}^{n-1}) \) that is pulled back from the complex orientation \( x \in MU^2(\mathbb{CP}^\infty) \). Here we are using the fact that \( MU^2(\mathbb{CP}^{n-1}) = \overline{MU}^2(\mathbb{CP}^{n-1}) \oplus MU^2(*) \), and \( MU^2(*) \cong U^2(*) = 0 \) since there are no manifolds of negative dimension. One also has that the set \( \{1, y, \ldots, y^{n-1}\} \) forms a basis for \( MU^*(\mathbb{P}(E)) \) as a module over \( MU^*(B) \). Thus there exist unique classes \( c_i(E) \in MU^2(B) \) such that

\[ y^n - c_1(E)y^{n-1} + \cdots + (-1)^nc_n(E) = 0 \]

and these are the Chern classes of \( E \), defining \( c_0(E) = 1 \). Properties (2) and (3) are proved in the usual way as in cohomology. To show (4), we note that when we take the line bundle \( L_i^\infty \to \mathbb{CP}^\infty \), we get the fibration

\[ \ast \to \mathbb{P}(L_i^\infty) \to \mathbb{CP}^\infty \]

and so the latter map is a homotopy equivalence, and the Leray-Hirsch theorem just tells us that the class \( y \) as above is just the complex orientation \( x \) pulled back along the homotopy equivalence. The above equation then becomes \( x - c_1(L_i^\infty) \cdot 1 = 0 \). Pulling back along the isomorphism \( U^* \to MU^* \), we get classes in \( U^* \) for smooth \( E \to X \). Now the restriction of the complex orientation \( x \in MU^2(\mathbb{CP}^\infty) \) to \( MU^2(\mathbb{CP}^N) \) pulls back to \( i^*i_*(1) \in U^*(\mathbb{CP}^N) \) by the arguments of 3.2 as Thom classes in \( MU \) are defined so that \( x \) is the pullback of the Thom class \( u \) along the zero section \( i \).

### A Basis of \( MU^*BU \)

Just as the Steenrod operations are given by elements of \([HZ/2, HZ/2] = HZ/2^*HZ/2\), one is led to look at elements of \( MU^*MU \) for cobordism operations. But by the Thom isomorphism, it suffices to look at elements of \( MU^*BU \). In ordinary cohomology, one has \( H^*(BU(1)) \cong \mathbb{Z}[x] \) with \( x \in H^2(BU(1)) \), and

\[ H^*(BU(n)) \subset H^*(BU(1))^\otimes n \cong \mathbb{Z}[x_1, \ldots, x_n] \]

is the subalgebra of symmetric functions. Moreover, the Chern classes of the tautological bundle over \( BU(n) \) correspond to the elementary symmetric polynomials in the \( x_i \)'s. Thus in the limit we have

\[ H^*(BU) \cong \lim_{\leftarrow n} \mathbb{Z}[\sigma_1, \ldots, \sigma_n] \]

where \( \sigma_i(x_1, \ldots, x_n) \) is the \( i \)-th elementary symmetric polynomial. This gives a \( \mathbb{Z} \)-basis for \( H^*(BU) \) consisting of the monomials in the elementary symmetric functions in the \( x_i \), or equivalently the monomials in the Chern classes. Since \( BU(1) \cong \mathbb{CP}^\infty \), and \( MU \) has the complex orientation \( x \in MU^2(BU(1)) \), the same results hold by using the Chern classes we have just built above and 2.8. We thus get an \( MU^* \) basis for \( MU^*BU \). In fact, for this reason we refer primarily to \( H^*(BU) \) in this section for simplicity, but all of the same methods work for \( MU^* \).

This is perhaps the first basis - and the simplest - that one would try, but it is not the one that is used most commonly because there is a different basis that behaves much more naturally with respect to direct sums of vector bundles. In particular, \( H^*BU \) is a ring under the cup product furnished by the diagonal map \( \Delta : BU \to BU \times BU \). \( H^*MU \), however does not have a cup product because \( MU \) is not a space and thus does not have a diagonal map. This reflects the fact that the Thom isomorphism is an isomorphism of \( \mathbb{Z} \)-modules, not \( \mathbb{Z} \)-algebras. But the basis we have chosen above is built from the cup product structure on spaces - we took polynomials in the classes \( x_i \), so this is not a particularly natural basis to choose.
$H^*MU$ does have the structure of a $\mathbb{Z}$-coalgebra, however, as we have a map

$$[MU, HZ] \to [MU \wedge MU, HZ] \cong [MU, HZ] \otimes_{HZ} [MU, HZ]$$

induced by the ring map $\mu: MU \wedge MU \to MU$, which comes from direct sums on vector bundles. We thus look for a basis that fits more naturally into this structure. Starting again with the basis $\{1, x, x^2, \ldots\}$ of $H^*(BU(1))$, we take the dual basis $\{b_0, b_1, \ldots\}$. Via the injection $BU(1) \to BU$, we send the $b_i$’s along the map $H_*(BU(1)) \to H_*(BU)$. This map is an injection because $U(1)$ is a retract of $U(n)$ for all $n$ via continuous group maps, namely we have the sequence

$$U(1) \hookrightarrow U(n) \xrightarrow{\text{det}} U(1)$$

so we know the $b_i$’s are linearly independent in $H_*(BU)$. Now since $BU$ is an $H$-space via direct sum of vector bundles, the $b_i$’s can be multiplied in the ring $H_*(BU)$. So we take the set of all monomials in the $b_i$: each one is given by $b^a = b_1^{a_1} \cdots b_n^{a_n}$ for some $n$ and some sequence of nonnegative integers $a = (a_1, a_2, \ldots)$ with all but finitely many $a_i = 0$. Using the fact that $H_*(BU(1)) \to H_*(BU)$ is a monomorphism, the fact that the $b_i$’s form a basis of $H_*(BU(1))$, and iteratively running the Serre spectral sequence on the fibration $S^{2n-1} \to BU(n-1) \to BU(n)$, one may prove that the $b_i$’s form a basis of $H_*(BU)$. Our basis of $H^*(BU)$ will be the one dual to this: we define $c_a$ to be the element dual to $\prod b_i^{a_i}$. We already see that direct sums are built into this basis since we defined the monomials $\prod b_i^{a_i}$ using multiplication in $H_*(BU)$ from direct sums.

**Bases of the Ring of Symmetric Functions**

We explain the relationship between the two bases of $H^*(BU)$ discussed. The ring of symmetric functions in $n$ variables $\Lambda_n$ is very special in that it is a polynomial ring, it is self-dual, and there are many standard bases of it. The most commonly used bases are the elementary symmetric polynomials, the complete homogeneous symmetric polynomials, the monomial symmetric functions, and the Schur polynomials. Since the standard Chern classes in $H^*(BU)$ correspond to the elementary symmetric polynomials in this ring, we may ask if the basis we have cooked up corresponds to one of these other standard bases. If we can prove such a thing, then we may use facts from algebra to go between our two bases. We briefly explain how one might prove this, and the full result can be pieced together from [1] and [6]. In the case of the elementary symmetric polynomials, we have the Chern classes $c_1, \ldots, c_n \in H^*(BU(n))$, and the direct sum map $BU(1)^{\times n} \to BU(n)$, and thus a sequence of maps

$$\mathbb{Z}[c_1, \ldots, c_n] \to H^*(BU(n)) \xrightarrow{\varphi^*} H^*(BU(1)^{\times n}) \cong H^*(BU(1))^\otimes n \cong \mathbb{Z}[x]^\otimes n \cong \mathbb{Z}[x_1, \ldots, x_n]$$

where $x$ is the generator as above. One finds that $c_i$ is sent to the elementary symmetric polynomial $\sigma_i$ in this sequence, and the first map in the sequence is an isomorphism. Taking duals, we have

$$H_*(BU(1)^{\times n}) \xrightarrow{\varphi_*} H_*(BU(n)) \to (\mathbb{Z}[c_1, \ldots, c_n])^*$$

and replacing $x_j$ with its dual $(b_j)_i$ (by which we mean the element $b_j \in H_*(BU(1))$ on the $i$-th factor), we have that the "polynomial" dual to $\sigma_i$ in the $(b_j)_i$’s is sent to the element dual to the $i$-th Chern class $c_i$. The quotes here are meant to reflect the fact that we are not taking powers of the $b_i$’s, the "polynomial" $\sum a_1 b_1^{a_1} \cdots t_n^{a_n}$ in indeterminates $t_1, \ldots, t_n$ refers to the sum $\sum a_1(b_{a_1})_i \cdots (b_{a_n})_n$. Applying $\oplus_*$ to this then sends $(b_j)_i$ to $b_i \in H_*(BU(n))$, and so may write the elements dual to the monomials in the $c_i$’s as a polynomial in the $b_i$’s, under direct sum. Taking duals and then taking limits, we can write the basis of $H^*(BU)$ consisting of monomials in the elementary symmetric polynomials in terms of our new basis. We find that for each integer sequence $a$, there is a unique polynomial $P_a$ such that $c_a$, as defined above, can be written as $P_a(c_1, c_2, \ldots)$. Moreover, if we set $n = \sum a_i$, then the $P_a$ are the *Schur polynomials* defined by the property that if $\sigma_i(x_1, \ldots, x_n)$ is the $i$-th elementary symmetric polynomial, then

$$P_a(\sigma_1, \ldots, \sigma_n) = \sum x_1^{m_1} \cdots x_n^{m_n}$$

where the sum ranges over $n$-tuples $(m_1, \ldots, m_n)$ such that $a_1$ of the $m_i$’s are 1, $a_2$ of the $m_i$’s are 2, and so on.

Perhaps a clearer way to describe the Schur polynomials is to say that for each nonnegative integer sequence $a = (a_1, a_2, \ldots)$ with all but finitely many $a_i = 0$ we may find $k$ such that $a_i = 0$ for all $i > k,$
then we can associate a nonincreasing integer sequence \( \beta = (\beta_1, \beta_2, \ldots) \) which begins with \( \alpha_k \) many \( k \)'s, then \( \alpha_{k-1} \) many \( (k-1) \)'s, and so on. This nonincreasing integer sequence has the property that \( \beta_i = 0 \) for all \( i > n \), again setting \( n = \sum \alpha_i \). Then to this nonincreasing integer sequence with \( \beta_i = 0 \) for all \( i > n \), we may associate a monomial symmetric function given by the formula

\[
\left( \prod_i \frac{1}{\alpha_i!} \right) \sum_{i_1, \ldots, i_n} x_{i_1}^{\beta_1} \cdots x_{i_n}^{\beta_n} = c_\alpha
\]

As an example, if we begin with the sequence \( \alpha = (1, 2, 0, 0, \ldots) \), we get the nonincreasing sequence \( \beta = (2, 2, 1, 0, \ldots) \) and the monomial symmetric function

\[
\frac{1}{2} \sum_{i,j,k} x_i^2 x_j x_k = x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + \cdots
\]

We can check for instance that when we take \( \alpha = (i, 0, \ldots) \), we recover the \( i \)-th elementary symmetric polynomial in the \( x_i \)'s, and therefore the \( i \)-th Chern class. We also see that the classes \( c_\alpha \) live in degree \( 2 \cdot \sum_j j \mu_j \).

### 3.4. Operations in \( U^* \).

Building the classes \( c_\alpha \)'s in \( MU^*(BU) \) in exactly the same way, they give characteristic classes for complex vector bundles by pulling back from universal bundles, and thus they give characteristic classes in \( U^* \). By analogy with the other basis of \( H^*(BU) \), we define a 'total' Chern class so that we can state a sort of Cartan formula satisfied by the \( c_\alpha \). This formula will be much easier to prove, however, because we have built direct sums into this basis. From there we will define operations in \( U^* \) and see that the fruit of our labor in defining this basis is that our operations will be ring homomorphisms.

**Definition 3.4.1.** The total Chern class of a complex vector bundle \( E \to X \) is the sum

\[
c_t(E) := \sum_{\alpha} c_\alpha(E) t^a \in U^*(X)[t_1, t_2, \ldots]
\]

where the \( c_\alpha(E) \) are defined by pulling back the classes \( c_\alpha \) defined above, and \( t^a := t_1^{a_1} \cdots t_n^{a_n} \) where the \( t_i \)'s are indeterminates of degree \(-2i\), and \( t_0 = 1 \).

**Proposition 3.4.2.** The total Chern \( c_t(E) \) class of a complex vector bundle \( E \to X \) with zero section \( i : X \to E \) satisfies the following properties

1. If \( E = L \) is a complex line bundle, \( c_t(L) = \sum_{j \geq 0} t^j c_j(L) \) where \( c_j(L) = i^* i_*(1) \) is the Euler class of \( L \).
2. For vector bundles \( E_1, E_2 \), \( c_t(E_1 \oplus E_2) = c_t(E_1) \cdot c_t(E_2) \)

**Proof:** For item 1, since the complex orientation \( x \in MU^2(\mathbb{C}P^\infty) \) pulls back to \( c(\mathbb{L}_1^N) \) under the map \( \mathbb{C}P^N \to BU(1) \) and \( U^* \to MU^* \) by 3.3.1, it suffices to prove

\[
c_t(\mathbb{L}_1^N) = \sum_{j \geq 0} t^j x^j \in MU^*(BU(1))[t_1, t_2, \ldots]
\]

defining total Chern classes in \( MU^* \) in exactly the same way. We again prove these in \( H^* \) for simplicity, noting the arguments may be carried out identically in \( MU^* \). Since the isomorphisms \( H^*(BU(n)) \cong \Lambda_n \), where \( \Lambda_n \) is the ring of symmetric functions, are compatible with the restrictions \( H^*(BU(n)) \to H^*(BU(k)) \) for \( k < n \), we get that the \( i \)-th elementary symmetric polynomial in \( n \) variables pulls back to the \( i \)-th elementary symmetric polynomial in \( k \) variables. In particular, since the \( i \)-th elementary symmetric polynomial in \( k \) variables vanishes for \( i > k \), the map \( r : H^*(BU(n)) \to H^*(BU(1)) \) sends \( c_i \) to 0 for \( i > 1 \) and \( c_1 \) to the generator \( x \). It is a general fact about Schur polynomials that for \( \alpha = (0, 0, \ldots, 1, \ldots) \) where 1 is in the \( i \)-th spot,

\[
P_\alpha(\sigma_1, \sigma_2, \ldots) = \sum_j \sigma_j^i
\]
Moreover, for any $a$ not of this form, for each monomial in the sum $P_a(e_1,e_2,\ldots)$, there exists an $i > 1$ such that $e_i$ is a factor of this monomial. Therefore for $a = (0,0,\ldots,1,\ldots)$, $r(c_a) = 1$, and for any other $a$, $r(c_a) = 0$, thus proving 1.

To show item 2, one may use the Cartan formula for integer Chern classes and prove combinatorially using the formulas defining Schur polynomials that we have the desired formula for $H^*(BU)$, and the fact that the identity map is a ring map means the following diagram commutes

$$H_s(BU) \otimes H_s(BU) \overset{1 \otimes 1}{\longrightarrow} H_s(BU) \otimes H_s(BU) \overset{\oplus_*}{\longrightarrow} H_s(BU) \otimes H_s(BU).$$

This seems trivial, but since the basis elements are monomials in the $b_i$'s, this says that if we express two elements of $H_s(BU)$ in terms of this basis, multiplying them corresponds to multiplying the corresponding monomials, and this multiplication comes from direct sums of vector bundles. In particular, under the isomorphism $\text{Hom}(H_s(BU),H_t(BU)) \cong \text{Hom}(Z,H_s(BU) \otimes H^*(BU))$ (this isomorphism holds for projective modules $M$ over a ring and it comes via the isomorphism $M \otimes M^* \rightarrow \text{End}(M)$), this diagram corresponds to the diagram

$$Z \longrightarrow H_s(BU) \otimes H^*(BU) \longrightarrow \bigoplus H_s(BU) \otimes (H^*(BU))^\otimes 2 \overset{\text{flip}}{\longrightarrow} (H_s(BU) \otimes H^*(BU))^\otimes 2 \otimes (H^*(BU))^\otimes 2.$$

The map $1 \otimes \mu$ does not come from the isomorphism described, we just include it for reference. The map in the top row sends $1 \mapsto \sum b_a \otimes c_a$ since the $b_a$ and $c_a$ are dual, and interpreting $\sum b_a \otimes c_a$ as $c_t = \sum_i t^a c_a$, the clockwise map followed by $1 \otimes \mu$ expresses $c_t(E)$ as $c_t(E_1 \oplus E_2)$ after collecting terms via $1 \otimes \mu$. The other way around followed by $1 \otimes \mu$ represents $c_t(E_1) c_t(E_2)$. We are relying heavily on the facts that

$$\oplus_* (b_i \otimes b_j) = b_i b_j$$

$$\otimes^* (c_a) = \sum_{\beta \gamma = \alpha} c_\beta \otimes c_\gamma$$

where by addition of integer sequences in the above sum we mean entry-wise addition. These come for free to us since we have defined these elements directly using direct sums, and the $c_a$ are dual to the $b_a$. \hfill \square

**Operations via the Chern Classes**

We have defined classes $c_a \in M^*(MU)$ and made sense of them as characteristic classes in $U^*(X)$ by pulling back from the universal bundles. But we want to interpret these operations $MU \rightarrow MU$ geometrically as operations $U^*(X) \rightarrow U^*(X)$. The natural thing to do would be to associate a complex vector bundle bundle over $X$ to $(f,v) \in U^*(X)$ that depends on both $f$ and $v$, and then take its Chern classes to get another element of $U^*(X)$. We of course have such a bundle since we can factor $f$ through an embedding $i : Z \hookrightarrow X \times \mathbb{C}^n$ and take the complex vector bundle $v_i$.

**Definition 3.4.3.** The Landweber-Novikov operation on $X$ is the map

$$s_t : U^*(X) \rightarrow U^*(X)[t_1, t_2, \ldots]$$

$$(f,v) \mapsto \sum t^a s_a (f,v) := \sum t^a f_s (c_a (v_i))$$

13
which we can think of as \( f_\ast(c_i(v)) \) if we adopt the convention that \( f_\ast \) moves past the indeterminates. It does not depend on the choice of \( v \); because total Chern classes are invariant under vector bundle isomorphism, and stably so because they are multiplicative with respect to direct sums and trivial bundles have vanishing Chern classes. We will relate these to our other key \( U^r \) operations, which are much easier to define.

**Remark 3.4.4.** The Landweber-Novikov operations are additive in the sense that \( s_\ast : U^r(X) \to U^r(X) \) are group homomorphisms because the \( c_q \)'s are additive which can be seen directly from the definition of addition in \( U^r(X) \). They are multiplicative in the sense that \( s_\ast \) is a ring homomorphism, which is the content of (2) in 3.4.2. This fact is the best justification for using the basis of \( MU^r(BU) \) we chose.

**Definition 3.4.5.** If \( Q \to B \) is a smooth principal \( \mathbb{Z}/p \)-bundle, then the power operation \( P \) associated to \( Q \) is the map

\[
P : U^{-2q}(X) \to U^{-2q}(Q \times_{\mathbb{Z}/p} X^p) \xrightarrow{(Q \times_{\mathbb{Z}/p} \Delta)^\ast} U^{-2q}(B \times X)
\]

where \( \mathbb{Z}/p \) acts cyclically on \( X^p \). We defined these as suggested in the introduction, except we have not used the bundle \( EZ/p \to BZ/p \) since these are not manifolds, but we will model this bundle with various smooth \( Q \to B \). We are giving \( X \) the trivial \( G \)-action, and identifying \( Q \times_{\mathbb{Z}/p} X \cong B \times X \), as is true for any trivial \( G \)-space \( X \). \( q \) can be any integer, but we take \( U^{-2q} \) in the above definition since it will make calculations clearer in our main theorem.

**Remark 3.4.6.** We are only considering maps of even dimension here because in order for \( Q \times_{\mathbb{Z}/p} X^p \) to be a complex oriented map, we need the factorization of \( f \) as \( Z \hookrightarrow X \times \mathbb{C}^n \to X \) to be equivariant, namely \( X \times \mathbb{C}^n \) needs to be a \( G \)-bundle in order for \( Q \times_{\mathbb{Z}/p} (X \times \mathbb{C}^n) \) to make sense, and the complex structure on \( v_i \) needs to be equivariant in order for \( vQ \times_{\mathbb{Z}/p} \) to have a complex structure. These conditions are not necessarily satisfied when \( f \) has odd dimension because if \( f \) is complex oriented, what we really mean is that the map \( Z \to X \times \mathbb{R} \) is complex oriented. Taking \( p = 2 \), for instance, this tells us the map \( Z \times Z \to X \times \mathbb{R} \times X \times \mathbb{R} \) is complex oriented, but in order to say the map of odd dimension \( f \times f \) is complex oriented, we need an orientation on \( Z \times Z \to X \times X \times \mathbb{R} \). Our only hope is to project \( \mathbb{R}^2 \to \mathbb{R} \) and give \( X \times X \times \mathbb{R} \) the swap action of \( \mathbb{Z}/2 \) on the first two factors, so that the bundle \( E \times E \to (X \times \mathbb{R})^2 \) may be regarded as a bundle over \( X \times X \times \mathbb{R} \). However, this projection map is not equivariant, since the action of \( \mathbb{Z}/2 \) on \( (X \times \mathbb{R})^2 \) is the swap map.

**Lemma 3.4.7.** If \( b \in B \) and \( i : \{b\} \to B \), then \( i^\ast(Px) = x^p \).

**Proof:** Let \( d \) be an approximation to \( \Delta \) transverse to \( x^{p} \). The map \( (Q \times_{\mathbb{Z}/p} \Delta) \circ i \) is then an approximation to \( Q \times_{\mathbb{Z}/p} \Delta \) transverse to \( Q \times_{\mathbb{Z}/p} x^{p} \). Thus, \( i^\ast(Px) \) is given by forming the fiber product of \( (Q \times_{\mathbb{Z}/p} d) \circ i \) with \( Q \times_{\mathbb{Z}/p} x^{p} \), which gives \( d^\ast(x^{p}) = \Delta^\ast(x^{p}) = x^p \).

4. **Localizing at the Fixedpoint Set**

The point of this section is to relate the two \( U^r \) operations we have just defined. The basic idea will be to modify the formula of 3.1.5 when \( g \) and \( f \) are not transverse and then specialize to the case of the diagram with \( g = Q \times_{\mathbb{Z}/p} \Delta \) and \( f = Q \times_{\mathbb{Z}/p} x^{p} \) defining the operations \( P \).

4.1. **Fixedpoint Formula.**

We now have our power operations \( P : U^{-2q}(X) \to U^{-2q}(B \times X) \) for \( Q \to B \) a principal \( \mathbb{Z}/p \)-bundle. These operations factor through \( U^{-2q}(Q \times_{\mathbb{Z}/p} X^p) \), by sending an even dimensional \( f : Z \to X \) to \( Q \times_{\mathbb{Z}/p} f^{p} \) and then pulling back along the diagonal. \( Z^{p} \) and \( X^p \) carry an action of \( \mathbb{Z}/p \) by cyclic permutation of the factors and \( f^{p} \) is equivariant with respect to this action. Moreover, the fixed points of the action are given by the diagonals, hence pulling back along the diagonal can be thought of as restricting
to the fixed point set. The power operations thus carry certain equivariance data which we exploit to derive our key formula relating the power operations and the Landweber-Novikov operations.

Since \( f^{x}p \) is \( \mathbb{Z}/p \)-equivariant, and the diagonal maps are the inclusions of the fixed points, we have the diagram

\[
\begin{array}{cccc}
B \times Z & \xrightarrow{Q \times \mathbb{Z}/p \Delta Z} & Q \times \mathbb{Z}/p Z^p \\
Q \times \mathbb{Z}/p f & \downarrow & \downarrow \quad Q \times \mathbb{Z}/p f^x p \\
B \times X & \xrightarrow{Q \times \mathbb{Z}/p \Delta X} & Q \times \mathbb{Z}/p X^p
\end{array}
\]

If we knew the maps \( Q \times \mathbb{Z}/p \Delta X \) and \( Q \times \mathbb{Z}/p f^x p \) were transverse, by 3.1.5, we would have

\[
f^* \Delta^* \mathbb{Z} (a) = \Delta^* X f^* (a)
\]

where \( a \in U^* (Q \times \mathbb{Z}/p Z^p) \), suppressing the \( Q \times \mathbb{Z}/p - \) notation. Letting \( a = 1 \in U^* (Q \times \mathbb{Z}/p Z^p) \), we would have the formula

\[
P(f) = \Delta^* X f^* (1) = f^* \Delta^* \mathbb{Z} (1)
\]

We of course don’t have that the maps are transverse in general, so we look for a way to measure how far the maps are from being transverse.

**Definition 4.1.1.** For a diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & Z \\
\downarrow{f'} & & \downarrow{i} \\
Y & \xrightarrow{j} & X
\end{array}
\]

we define the excess bundle \( F \) to be the cokernel of the map \( TZ \mid W \oplus TY \mid W \to TX \mid W \). Thus \( i \) and \( j \) are transverse iff \( F = 0 \).

**Remark 4.1.2.** We want to derive a formula involving \( F \) that specializes to 3.1.5 when \( F = 0 \) so as to unpack \( P(f) \) in the above manner. We ignore the equivariance data for a moment and look at the simple case when all the maps are embeddings of closed submanifolds, and \( W = Z \cap Y \).

**Lemma 4.1.3.** In the diagram 4.1.1, let all maps be closed embeddings with \( W = Z \cap Y \), and let \( T_w W = T_w Y \cap T_w Z \) for all \( w \in W \) - such an intersection of submanifolds is called a clean intersection. Then there is a short exact sequence of vector bundles over \( W \)

\[
0 \to v_l \to j'^* v_l \to F \to 0
\]

**Proof:** For subspaces \( U, U' \subset V \) of a vector space, one has an isomorphism

\[
\frac{V \mid U}{U' \mid U \cap U'} \cong \frac{V}{U \oplus U'}
\]

Letting \( V = T_w X, U = T_w Z, \) and \( U' = T_w Y \), the result follows by the fact that \( W \) is a clean intersection. \( \Box \)

**Proposition 4.1.4.** In the conditions of 4.1.3, let \( v_l \) and \( v_r \) have complex structures so that the map \( v_l \to j'^* v_l \) in 4.1.3 is a morphism of complex vector bundles, and give \( F \) the complex structure that is compatible with the short exact sequence. Then, if \( z \in U^* (Z) \),

\[
j^* i^*_s (z) = i'_s (e(F) \cdot j'^* (z))
\]

in \( U^* (Y, Y - W) \).

**Proof:** We first prove the claim for the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{f'} & v_l \\
\downarrow{k'} & & \downarrow{k} \\
v_l & \xrightarrow{l} & v_{l \circ f'} \cong v_l \oplus v_r \oplus F
\end{array}
\]
where all the maps are zero sections. To see why the isomorphism in the bottom right corner holds, note that from the sequence

\[ v_{i'} \leftrightarrow f'^*v_i \leftrightarrow v_{i'0j} \]

we have a short exact sequence

\[ 0 \rightarrow f'^*v_i \rightarrow v_{i'0j} \rightarrow v_{i'0j} \rightarrow 0 \]

Moreover, the embedding \( k : v_{i'} \rightarrow v_{i'} \oplus v_{i'} \oplus F \) has normal bundle \( v_{i'} \oplus F \), which has a given complex structure matching the one on \( f'^*v_i \) via the sequence 4.1.3, by assumption. \( v_{i'} \) also has by assumption a complex structure, and so \( v_{i'} \) and \( v_k \) have complex structures, and thus \( k \) and \( k' \) are complex oriented, by 2.2, so we may speak of their pushforwards. Define \( h_1 : v_{i'} \rightarrow v_{i'} \oplus v_{i'}, h_2 : v_{i'} \rightarrow v_{i'} \oplus v_{i'}, \) and \( h_3 : v_{i'} \oplus v_{i'} \rightarrow v_{i'} \oplus v_{i'} \oplus F \) to be the zero sections, and \( \pi : v_{i'} \oplus v_{i'} \oplus F \rightarrow v_{i'} \oplus v_{i'} \) to be projection. Using 3.2.1, 3.2.3, and functoriality, we have the following computation

\[ l^*k_s(z) = h_1^*h_3^*h_2^*h_2(z) \]
\[ = h_1^*h_3^*(\pi^*(h_2^*(z)) \cdot h_2^*(1)) \]
\[ = h_1^*((\pi \circ h_2)^*(h_2^*(z)) \cdot h_3^*h_2^*(1)) \]
\[ = h_1^*(h_3^*h_2^*(1) \cdot h_2^*(z)) \]
\[ = h_1^*(e(v_{i'} \oplus v_{i'} \oplus F \rightarrow v_{i'} \oplus v_{i'}) \cdot h_2^*(z)) \]
\[ = h_1^*(e(v_{i'} \oplus v_{i'} \oplus F \rightarrow v_{i'} \oplus v_{i'})) \cdot h_1^*h_2^*(z) \]

But then by the following diagrams

\[ \begin{array}{ccc}
\nu_{i'} \oplus F & \rightarrow & \nu_{i'} \oplus \nu_{i'} \oplus F \\
\downarrow & & \downarrow \\
\nu_{i'} & \rightarrow & \nu_{i'} \oplus \nu_{i'} \\
\downarrow & & \downarrow \\
W & \rightarrow & \nu_{i'} \oplus \nu_{i'} \\
\downarrow & & \downarrow \\
& \rightarrow & F
\end{array} \]

\[ \begin{array}{ccc}
\nu_{i'} \oplus \nu_{i'} \oplus F & \rightarrow & F \\
\downarrow & & \downarrow \\
\nu_{i'} \oplus \nu_{i'} & \rightarrow & F \\
\downarrow & & \downarrow \\
W & \rightarrow & \nu_{i'} \oplus \nu_{i'}
\end{array} \]

we have

\[ h_1^*(e(v_{i'} \oplus v_{i'} \oplus F \rightarrow v_{i'} \oplus v_{i'})) = h_1^*(p_2^*(e(F))) = p_1^*(e(F)) \]

Then since \( h_1 \) and \( h_2 \) are transverse, we apply 3.1.5, and we have

\[ l^*k_s(z) = p_1^*(e(F)) \cdot h_1^*h_2^*(z) \]
\[ = p_1^*(e(F)) \cdot k_1^*l^*(z) \]
\[ = k_1^*(e(F) \cdot l^*(z)) \]

Now we need to reduce the general case to the above diagram, which we do by way of replacing \( X \) with a tubular neighborhood of \( W \). By pulling back the tubular neighborhood diagram for \( i \circ j' : W \hookrightarrow X \) to \( Y \) and \( Z \), we can arrange the following commutative diagram

\[ \begin{array}{ccc}
W & \rightarrow & v_{i'} \\
\downarrow & & \downarrow \\
W & \rightarrow & Z \\
\downarrow & & \downarrow \\
Y & \rightarrow & v_{i'0j'} \\
\downarrow & & \downarrow \\
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
& \rightarrow &
\end{array} \]

Then since the equation of interest lands in \( U^*(Y, Y - W) \) and the image of \( v_{i'} \) is an open set containing \( W \), we may use excision to show that the induced map \( f_2^* : U^*(Y, Y - W) \rightarrow U^*(v_{i'}) \) is an monomorphism. It
thus suffices to show that $j^*i_*(z)$ and $i'_*(e(F) \cdot f^*(z))$ are equal after applying this map. We remark that the left facing square is a transverse pullback square since $f_2$ is an embedding and $v_\rho$ has the same dimension as a manifold as $X$, and $f_2(v_\rho) \cap W = W$, the map $W \hookrightarrow f_2(v)$ being the image of the zero section. 3.1.5 then gives $f_2i'_* = k'_*$, hence we have

$$f_2(i'_*(e(F) \cdot f^*(z))) = f_2(i'_*(e(F) \cdot l^*f_1^*(z))) = k'_*(e(F) \cdot l^*f_1^*(z)) = l^*k_*(f_1^*(z)) = l^*(f_3i_*(z)) = f_2(j^*i_*(z))$$

where in line 4 we used the fact that the right facing square is a transverse pullback square, which follows from an argument similar to the above one since we obtained $f_1$ by pulling back the tubular neighborhood diagram involving $f_3$ to $Z$. 

We now look to bring equivariance back into the picture. We consider the fixedpoint diagram of an equivariant closed embedding of $G$-manifolds $i : Z \to X$ for a compact Lie group $G$

$$\begin{array}{ccc}
Z^G & \xrightarrow{r^Z} & Z \\
\downarrow{G} & & \downarrow{i} \\
X^G & \xrightarrow{r^X} & X
\end{array}$$

where $v_i$ is a $G$-bundle with an equivariant complex structure. Our case of interest of a map $f^{\times p} : Z^p \to X^p$ is a special case of this when $f$ is an embedding and has even dimension.

**Lemma 4.1.5.** A fixedpoint diagram as above is a clean intersection.

*Proof:* Since tangent spaces are determined locally, we may choose Euclidean neighborhoods, whereby the action of $G$ becomes a representation, and looking at the tangent bundles over these neighborhoods, the problem is reduced to the fact that the intersection of the fixedpoints of a representation with a subrepresentation is the fixedpoints of the subrepresentation. 

In order to apply 4.1.4, we look for a description of the excess bundle in this case. We have that $v_i$ is a $G$-bundle, and thus $r^Z_*(v_i)$ is a $G$-bundle over a trivial $G$-space. As such, the action of $G$ on $r^Z_*(v_i)$ is just a representation on each fiber. Then since $v_{G}$ is the sub-bundle fixed by $G$, we have $r^G_*(v_i) \cong v_{G} \oplus \mu_{i}$, where $\mu_{i}$ is the subbundle of $v_i$ on which $G$ acts nontrivially, or the direct sum of all the nontrivial irreducible representations coming from the action of $G$. Since the complex structure of $v_i$ is equivariant, $v_{G}$ inherits a complex structure, and by 4.1.3, $\mu_{i}$ is the excess bundle, and the conditions of 4.1.4 are satisfied. We thus have the formula $r^G_*(z) = r^G_*(e(\mu_{i}) \cdot r^Z_*(z))$.

**Remark 4.1.6.** If instead of $U^*(\cdot)$, we used $U^*(Q \times_G \cdot)$ for a principal G-bundle $Q \to B$ on a diagram as in 4.1.5, the proof of 4.1.4 would go through unchanged since we have made all the necessary equivariance assumptions. In this setting, we suppress the $Q \times_G \cdot$ notation for pushforwards, so that we write $f_*$ for $(Q \times_G f)_*$ and $e(E)$ for $(Q \times_G i)^*(Q \times_G l)_*(1)$ where $i$ the zero section of a bundle $E \to X$, when it is clear from context what is meant.

We proceed in this way, and we remove the assumption that $i$ need be an embedding with that of an *equivariant complex oriented map*. Namely, let $f : Z \to X$ be an complex oriented G-map of even dimension such that it factors equivariantly through $i : Z \to X \times \mathbb{C}^n$ where $X \times \mathbb{C}^n$ is a complex G-bundle, and $v_i$ has an equivariant complex structure. Then define $\mu_{i}$ as above, and $\tau_{i}$ as above on the bundle $r^X_*(X \times \mathbb{C}^n)$.

**Proposition 4.1.7.** In a diagram as in 4.1.5 with $f$ an equivariant complex oriented map $f$ replacing the embedding $i$, we have

$$e(\tau_{i}) \cdot r^{G}_Xf_*(z) = f^{G}_*(e(\mu_{i}) \cdot r^{G}_Zz)$$
in $U^*(B \times Z^G)$.

**Proof:** Let $j : X \to n$ be the zero section. First working in $U^*(-)$, applying 4.1.4 to the diagrams

$$
\begin{align*}
X^G & \xrightarrow{r_X} X \\
\downarrow i^G & \quad \downarrow j \\
n^G & \xrightarrow{r_n} n
\end{align*}
\quad
\begin{align*}
Z^G & \xrightarrow{r_z} Z \\
\downarrow i^G & \quad \downarrow j \\
n^G & \xrightarrow{r_n} n
\end{align*}
$$

where we know $i$ is a closed embedding because $f$ is proper, we have the equations

$$
\begin{align*}
\rho_{ij}^* j^*_s(x) &= f^G_i(e(\tau_i) \cdot r_X^s(x)) \\
\rho_{ij}^* j^*_s(z) &= i^G_i(e(\mu_i) \cdot r_Z^s(z))
\end{align*}
$$

Then letting $x = f_s(z)$, since $j_s f_s = i_s$, $j^G i^G = i^G$, we have

$$
\begin{align*}
\rho_{ij}^* j^*_s(e(\mu_i) \cdot r_Z^s(z)) &= i^G_i(e(\mu_i) \cdot r_Z^s(z)) \\
&= \rho_{ij}^* j^*_s f_s(z) \\
&= f^G_i(e(\tau_i) \cdot r_X^s f_s(z))
\end{align*}
$$

If we choose a Riemannian metric on $n$ so that $i(Z) \subset Dn$ where $Dn$ is the disk bundle with respect to the metric, then we have that $i_s : U^*(Z) \to U^*(n, n - Dn)$. Then we have $j_s f_s = i_s$ as maps $U^*(Z) \to U^*(n, n - Dn)$, and when regarded this way, $j_s$ is an isomorphism as in 3.1. The same can be said of $f^G$ and $i^G$, and thus we have $j^G_s$ is an isomorphism, and so we may cancel it from both sides of the above. As before, we may carry out the arguments identically in $U^*(Q \times G -)$ since we have assumed everything to be equivariant. \hfill $\square$

**Proposition 4.1.8.** Fix a principal $\mathbb{Z}/p$-bundle $Q \to B$. Plugging in the data of the power operations to 4.1.6, let $f : Z \to X$ be a complex oriented map of dimension $-2q$ that factors through an embedding $i : Z \hookrightarrow X \times \mathbb{C}^n$. Let $\rho$ be the reduced regular representation of $\mathbb{Z}/p$, i.e., the quotient of the representation of $\mathbb{Z}/p$ on $\mathbb{C}^n$ given by cyclic permutation of the factors by the subrepresentation given by the diagonal in $\mathbb{C}^n$. Let $\rho$ also denote the corresponding trivial $\mathbb{Z}/p$-bundle over a trivial $\mathbb{Z}/p$-space: i.e., for a representation $\rho$ of $\mathbb{Z}/p$ on a complex vector space $V$, we can form the vector bundle $X \times V \to X$ that carries the action of $\mathbb{Z}/p$ coming from $\rho$. Then

$$
e(\rho)^n P(f) = f_s(e(\rho \otimes v_i))$$

in $U^{2n(p-1)-2q}(B \times X)$.

**Proof:** First working in $U^*$, we apply 4.1.7 to the diagram

$$
\begin{align*}
Z & \xrightarrow{\Delta_Z} Z^p \\
\downarrow f & \quad \downarrow f^p \\
X & \xrightarrow{\Delta_X} X^p
\end{align*}
$$

and we have

$$
e(\tau_{fr}) \cdot \Delta_X^* (f^p) = e(\tau_{fr}) \cdot \Delta_X^* f^p(1) = f_s(e(\mu_{fr}) \cdot \Delta_X^* (1)) = f_s(e(\mu_{fr}))
$$

Let $j : X \to X \times \mathbb{C}^n$ be the zero section. The normal bundle of $j^p$ is just $X^p \times (\mathbb{C}^n)^p$, hence its restriction to $X$ is just $X \times (\mathbb{C}^n)^p$, with the action of $\mathbb{Z}/p$ cyclically permuting the factors of $\mathbb{C}^n$. $\tau_i$ is the quotient of this bundle by the subbundle fixed by $\mathbb{Z}/p$, namely the diagonal in $(\mathbb{n})^p$. Thus we have $\tau_i \cong \rho \otimes \mathbb{n}$, and thus $e(\tau_i) = e(\rho)^m$. Similarly, the normal bundle of $i^p$ is just $v_i^\otimes i^p$, and its restriction to $Z$ is $v_i^\otimes i^p$ with the cyclic action of $\mathbb{Z}/p$, so we have $\mu_i \cong \rho \otimes v_i$. Applying the same reasoning to $U^*(Q \times \mathbb{Z}/p -)$, we have the stated formula. \hfill $\square$

**Remark 4.1.9.** The right side of this equation already looks very close to the definition of the Landweber-Novikov operations. If we took $v_i$ to be a line bundle, we could break $\rho \otimes v_i$ into a sum of line bundles, use the result 3.4.2 relating the total Chern class of a line bundle to a power series on its Euler class, then use
the splitting principle to derive a formula relating the right hand side to the total Chern class of \( v_i \), and thus to the Landweber-Novikov operation. There is, however, a better way to say all of this using the language of formal group laws. As we will show there is a formal group law - a certain bivariate power series - over the ring \( U^* (*) \) that comes from Euler classes of tensor products of line bundles, and this is the situation we have in 4.1.8 when \( v_i \) is a line bundle. This also has the advantage of giving us reason to believe that a formula like 4.1.8 may be used to say something about the structure of \( U^* (*) \), if we are already familiar with the fact that it is generated by the coefficients of the formal group law.

4.2. Formal Group Laws and the Key Formula.

We review briefly the notion of a formal group law and the one naturally occurring over the ring \( U^* (*) \), and use it to restate 4.1.8 in terms of the Landweber-Novikov operations.

**Definition 4.2.1.** A formal group law over a ring \( R \) is a power series \( F(x, y) \in R[[x, y]] \) satisfying axioms meant to mirror those of an abelian group, namely:

1. \( F(0, 0) = 0 \)
2. \( F(x, 0) = x \) and \( F(0, y) = y \)
3. \( F(x, F(y, z)) = F(F(x, y), z) \)
4. \( F(x, y) = F(y, x) \)

One can show that there always exists a power series \( i(x) \in R[[x]] \) called the inverse, satisfying \( F(x, i(x)) = 0 \), hence one may think of \( F \) as addition in an abelian group. We can interpret the action of \( \mathbb{Z} \) on an abelian group in this context and define \( [1]_F(x) = x \) and \( [n]_F(x) = F([n-1]_F(x), x) \) and extend to negative integers via \( i \). One checks for instance that the leading term of \( [n]_F(x) \) is \( nx \). We also define \( \langle n \rangle_F(x) = (1/x)[n]_F(x) \) and omit the subscript \( F \) when understood.

**Proposition 4.2.2.** There is a unique formal group law \( F(x, y) = \sum_{i,j \geq 0} c_{ij} x^i y^j \) over \( U^* (*) \) with \( c_{ij} \in U^{2-2i-2j}(* ) \) such that

\[
e(L_1 \otimes L_2) = F(e(L_1), e(L_2))
\]

for line bundles \( L_1, L_2 \) over the same base. This is usually stated in terms of the bundles’ first Chern classes, but of course these coincide with Euler classes, and the classes of interest in this paper are Euler classes. We let \( C \) denote the subring of \( U^* (*) \) generated by the \( c_{ij} \)’s.

**Proof:** We know by 2.8 and 3.3.1 that \( U^* (\mathbb{C}P^n) \cong U^*[x]/x^{n+1} \) and \( x = e(L^n_1) \), hence by the Kunneth formula, we have \( U^* (\mathbb{C}P^n \times \mathbb{C}P^n) \cong U^*[x_1, x_2]/(x_1^{n+1}, x_2^{n+1}) \) with \( x_i = p_{ri}^* e(L^n_i) \) where \( p_i : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n \) for \( i = 1, 2 \) are the projection maps. It follows that

\[
e(p_{11}^* L^n_1 \otimes p_{22}^* L^n_1) = \sum_{i,j \leq n} c_{ij} x_i^1 x_2^j
\]

for some coefficients \( c_{ij} \in U^* (*) \). The coefficients do not change as \( n \to \infty \) and hence one has a well-defined power series

\[
F(x, y) = \sum_{i,j} c_{ij} x^i y^j \in U^* [[x, y]]
\]

Then since every line bundle is pulled back from the tautological bundle \( L^n_1 \to \mathbb{C}P^n \) for some \( n \), every tensor product of line bundles can be pulled back from

\[
pr_1^* L^n_1 \otimes pr_2^* L^n_1 \to \mathbb{C}P^n \times \mathbb{C}P^n
\]

for some \( n \). Then since Euler classes are natural under pullbacks, the same expression \( F \) relates the Euler classes for an arbitrary tensor product of line bundles. By homogeneity the coefficients \( c_{ij} \) are forced to have
degree $2 - 2i - 2j$, and since tensor products of vector bundles are associative, commutative, and unital with respect to the trivial line bundle, and the trivial line bundle has a trivial Euler class, the axioms of 4.2.1 hold.$\square$

**Proposition 4.2.3.** The functor sending a commutative ring $R$ to the set of formal group laws over $R$ is corepresentable. That is, there exists a ring $L$, called the Lazard ring, equipped with a formal group law $F_{\text{Univ}}$ with coefficients $a_{ij}$ over $L$ such that for any ring $R$, there is a bijection

$$\text{Hom}_{\text{CRing}}(L, R) \rightarrow \{\text{Formal Group Laws over } R\}$$

that is natural in $R$, and is given by taking a map $f : L \rightarrow R$ and sending it to the formal group law over $R$ whose coefficients are $f(a_{ij})$. $\square$

The statement of Quillen’s theorem is that the map $L \rightarrow U^*(\ast)$ corresponding to the formal group law of 6.2 is a ring isomorphism. Before we use the formal group law to unpack 6.2, we prove some results about the representation $\rho$, in order to break it up as a sum of line bundles.

**Lemma 4.2.4.** Let $\sigma$ be the representation of $\mathbb{Z}/p$ on the subspace of $\mathbb{C}^p$ given by

$$W := \{(z_1, \ldots, z_p) : \sum z_i = 0\}$$

where $\mathbb{Z}/p$ acts cyclically on the coordinates. Then $\sigma$ and $\rho$ are isomorphic representations. $\square$

**Proof:** Define a linear map $T : \mathbb{C}^p \rightarrow W$ by

$$(v_1, \ldots, v_p) \mapsto (v_1 - \frac{1}{p}\sum v_i, \ldots, v_p - \frac{1}{p}\sum v_i)$$

$T$ is equivariant since the sums $(1/p)\sum v_i$ are symmetric, and the kernel of $T$ is the diagonal, so we get a monomorphism of representations $\rho \rightarrow \sigma$ of dimension $p - 1$, and thus an isomorphism of representations.

**Lemma 4.2.5.** Let $\sigma$ be as above and let $\eta$ be the representation of $\mathbb{Z}/p$ on $\mathbb{C}$ given by multiplication by $e^{(2\pi i)/p}$. Then there is an isomorphism of representations

$$\bigoplus_{k=1}^{p-1} \eta^\otimes k \rightarrow \sigma$$

**Proof:** Let $\zeta := e^{(2\pi i)/p}$. We first remark that the representation $\eta^\otimes p$ is isomorphic to the one on $\mathbb{C}$ given by multiplication by $\zeta^k$. We define a linear map $T : \mathbb{C}^{p-1} \rightarrow W$ by $e_k \mapsto (1, \zeta^k, \ldots, \zeta^{(p-1)k})$, where $e_k$ is the $k$-th standard basis vector. $T$ lands in $W$ because since $p$ is prime and $1 \leq k \leq p - 1$, $\{1, \zeta^k, \ldots, \zeta^{(p-1)k}\}$ is the set of all $p$-th roots of unity, whose sum vanishes. Now, $T$ is injective because each $T(e_k)$ is an eigenvector for the action of $1 \in \mathbb{Z}/p$ on $W$ with eigenvalue $\zeta^k$, and eigenvectors with distinct eigenvalues are linearly independent. Since the dimensions match, it suffices to show $T$ is equivariant, and for that it suffices to check basis vectors. If $l \in \mathbb{Z}/p$, we have

$$T(l \cdot e_k) = T(\zeta^{lk} e_k) = \zeta^{lk} T(e_k) = (\zeta^{kl}, \zeta^{k(l+1)}, \ldots, \zeta^{k(l+p-1)})$$

which is what we would get if we cyclically permuted $T(e_k)$ $l$ times. $\square$

**Plugging into 4.1.8**
Let \( v = e(\eta) \) and \( w = e(\rho) \). With this information, we take a line bundle \( L \), and since an isomorphism of representations gives an isomorphism of their corresponding vector bundles as in 4.1.8, we have

\[
e(\rho \otimes L) = \prod_{k=1}^{p-1} e(\eta^k \otimes L)
= \prod_{k=1}^{p-1} F(e(\eta^k), e(L))
= \prod_{k=1}^{p-1} F([k]_F(v), e(L))
= \prod_{k=1}^{p-1} \left( [k]_F(v) + \sum_{j \geq 1} b_j(\rho) e(L)^j \right)
= \prod_{k=1}^{p-1} \left( [k]_F(v) \right) + \sum_{j \geq 1} a_j(v) e(L)^j
= w + \sum_{j \geq 1} a_j(v) e(L)^j
\]

for \( a_j(x), b_j(x) \in C[[x]] \), where \( C \) is the ring generated by the coefficients of the formal group law \( F \) as in 4.2.2, and we have

\[
w = \prod_{k=1}^{p-1} [k]_F(v) = (p-1)!v^{p-1} + \sum_{j \geq p} d_j v^j
\]

for \( d_j \in C \). Building up from line bundles, and working equivariantly with \( U^*(Q \times \mathbb{Z}/p) \), we obtain the following:

**Theorem 4.2.6.** If \( P \) is the \( q \)-th Power operation associated to a principal \( \mathbb{Z}/p \)-bundle \( Q \to B \), and \( v \) and \( w \) are the Euler classes as above, there exists \( n \) sufficiently large with respect to \( q \) and the dimension of \( X \) such that for \( x \in U^{-2q}(X) \), we have

\[
w^{n+q} px = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} \left( \prod_{j \geq 1} a_j(v)^{\alpha_j} \right) s_\alpha(x)
\]

in \( U^*(B \times X) \), where \( l(\alpha) = \sum \alpha_i \) and \( a_j(v) \in C[[v]] \).

**Proof:** We will derive this result from the above discussion and 4.1.8 and note that it suffices to prove this result for \( v \) a sum of line bundles by the splitting principle. The left side of the above equation is the same as the left side of 4.1.8, replacing \( n \) with \( n+q \) to make calculations easier in our main theorem, but 4.1.8 came from a factorization through \( X \times \mathbb{C}^n \). The statement about \( n \) being sufficiently large is just to say that we can bound \( n \) in this factorization for all \( x \) because if the map factors through a large trivial bundle over \( X \), we can push it into a smaller one \( n+q \) for some \( n \) without loss of generality since the dimension of \( X \) and \( q \) are finite. Then we may take all \( x \) to factor through \( n+q \) by adding the necessary number of trivial bundles to any given factorization.
Now let $v_i$ be a sum of $n$ line bundles $L_i$ (we know $v_i$ has dimension $n$ because $f$ has dimension $2q$ and $i$ is an embedding into $n+q$). With the convention that $a_0(v) = 1$, we compute

$$e(\rho \otimes v_i) = \prod_{i=1}^{n} e(\rho \otimes L_i)$$

$$= \prod_{i=1}^{n} (w + \sum_{j \geq 1} a_j(v) e(L_i)^j)$$

$$= \sum_{k=0}^{n} w^{n-k} \left( \sum_{j \geq 1} a_j(v) e(L_i)^j \right)$$

$$= \sum_{k=0}^{n} w^{n-k} \left( \prod_{i=1}^{k} a_j(v) e(L_i)^{j_i} \prod_{\ell=1}^{k} e(L_i)^{j_{\ell}} \right)$$

$$= \sum_{k=0}^{n} w^{n-k} \left( \sum_{j_{1,\ldots,j_k} \geq 0} \left( \prod_{l=1}^{k} a_{j_l}(v) \right) e(L_1)^{j_1} \cdots e(L_n)^{j_n} \right)$$

Now working backward, we know that for an integer sequence $\alpha$ with length $k$, if we let $\gamma(i)$ be the sequence with 0's in every entry except a 1 in the $i$-th entry and letting $\gamma(0)$ be the zero sequence, we have

$$c_{\alpha}(L_1 \oplus \cdots \oplus L_n) = \sum_{\beta(1) + \cdots + \beta(n) = \alpha} c_{\beta(1)}(L_1) \cdots c_{\beta(n)}(L_n)$$

for sequences $\beta(i)$ by 3.4.2. Then since sequences not of the form $\gamma(i)$ give vanishing Chern classes on line bundles and $c_{\gamma(i)}(L) = e(L)^i$ for a line bundle $L$, we have

$$c_{\alpha}(L_1 \oplus \cdots \oplus L_n) = \sum_{\gamma(j_1) + \cdots + \gamma(j_n) = \alpha} e(L_1)^{j_1} \cdots e(L_n)^{j_n} \sum_{j_i = 0 \text{ for exactly } n-k \text{ values of } i} e(L_1)^{j_1} \cdots e(L_n)^{j_n}$$

The second line follows because $l(\alpha) = l(\gamma(j_1)) + \cdots + l(\gamma(j_n))$ and $l(\gamma(j)) = 1$ or 0, and the third because to say $\gamma(j_1) + \cdots + \gamma(j_n) = \alpha$ is by definition of the $\gamma(j_i)$'s to say that $|\{j_k : j_k = j_i\}| = a_{j_i}$. We check that the terms appearing in the sum where we left off having the property that $|\{j_k : j_k = j_i\}| = a_{j_i}$ all have coefficient $\prod_{j \geq 1} a_j(v)^{a_j}$.

5. Quillen's Theorem

5.1. The Technical Lemma.

The purpose of this section is to show the existence of certain exact sequences that will be of use in applying 4.2.6 to prove our main theorem. It is difficult to motivate this section without having seen the proof of our main theorem that will follow, other than to say that 4.2.6 tells us that there is a formula involving any $x \in U^*(X)$ and the coefficients of the formal group law, i.e. the ring $C$, via the power operations. When we probe this formula for more information by, for instance, looking at the equation $p$-locally, we are led to some amazing consequences if we know there is an exact sequence involving multiplication by the class $x$ as
in 6.5. In fact, we could prove without much difficulty that there is an exact sequence
\[ \text{MU}^i(X) \xrightarrow{\langle p \rangle(v)} \text{MU}^i(B\mathbb{Z}/p \times X) \xrightarrow{i^*} \text{MU}^i+2(B\mathbb{Z}/p \times X) \]
where \( \langle p \rangle(v) \) is as in 6.1. But we wish to make sense of this in \( U^\ast \), so we prove the existence of a relation involving manifolds that approximate \( B\mathbb{Z}/p \), namely \( S^{2n+1}/\mathbb{Z}/p \) so that this relation approximates the above exact sequence as \( n \to \infty \). One can think of these as approximating \( B\mathbb{Z}/p \) because \( \mathbb{Z}/p \) acts freely on \( S^{2n+1} \subset \mathbb{C}^n \), the connectivity of which strictly increases as \( n \to \infty \), hence these approach \( E\mathbb{Z}/p \).

**Proposition 5.1.1.** Let \( B \) be compact, \( f: Q \to B \) a principal \( \mathbb{Z}/p \)-bundle, and \( L = Q \times_{\mathbb{Z}/p} \mathbb{C} \to B \) the line bundle associated to the representation \( \eta \) as in 4.2.5. Then \( f = \langle p \rangle(e(L)) \in U^0(B) \).

**Proof:** Let \( i: B \to L \) be the zero section and \( j: Q \hookrightarrow L \) be the embedding sending \( q \mapsto (q,1) \). \( f \) is proper and factors through \( L \), and its normal bundle has a complex structure since that of \( i \) does. Similar remarks apply to \( j \), so these maps are complex oriented, and we may speak of their pushforwards. Letting \( g: L \to B \) be the projection map, we form the line bundle \( g^*L = L \times_B L \to L \). Letting \( s \) be the diagonal section, note that \( s \) is nonvanishing away from the zero section of \( L \), hence it trivializes \( g^*L \) away from the zero section. We may therefore extend \( g^*L \) to a bundle \( M \) over \( L^+ \) (i.e. \( L \cup \{ \infty \} \)) since \( g^*L \) is trivial near infinity. \( s \) then extends to a section of \( M \) and is homotopic to the zero section since all sections are, and it is transverse to the zero section since the diagonal and the component on the left together span each tangent space. Thus we may compute the \( e(M) \) by forming the fiber product of \( s \) with the zero section, which is easily seen to be \( i_s(1) \).

We may similarly trivialize the bundle \( g^*(L^{\otimes p}) \) away from the zero section via \( s^{\otimes p} \) and form the bundle \( M^{\otimes p} \) over \( L^+ \). We now define a section \( t \) of \( M^{\otimes p} \) by \( t(q,z) = ((q,z),(q,z^p)) \). This is transverse to the zero section since the map on the right \( (q,z) \mapsto (q,z^p - 1) \) is a submersion fiberwise, and the fiber product of this with the zero section is \( \{(q,z) : z^p = 1\} = f(\{0\}) \) since pairs in \( L \) are subject to the equivalence relation \( (q,c^{2\pi i/p}z) \sim (q\sigma,z) \) where \( \sigma \) is the generator of \( Z/p \). Thus
\[
\begin{align*}
    j_s(1) &= e(M^{\otimes p}) \\
    &= [p]_f(i_s(1)) \\
    &= i_s1 \cdot \langle p \rangle(i_s(1)) \\
    &= i_s(\langle p \rangle(i^*_s(1))) \\
    &= i_s(\langle p \rangle(e(L)))
\end{align*}
\]
the last line following from 3.1.6. Now, we note that \( i_s: U^0(B) \to U^{0+2}(L^+ , \{ \infty \}) \) is an isomorphism, by 3.2.1 and the fact that the usual Thom isomorphism in \( MU \) can be seen as \( MU^i(B) \to \overline{MU}^{i+2}(\text{Thom}(L)) \), and since \( B \) is compact \( L^+ \cong \text{Thom}(L) \). If \( p: B \to L \) is the projection, then \( p_* \) is an inverse to \( i_s \), hence applying \( i_s^{-1} \) to both sides of the above we have \( f_s(1) = (p \circ j)_s(1) = \langle p \rangle(e(L)) \) since \( f = p \circ j \). Note that we could not have just used the fact that \( i_s: U^0(B) \to U^{0+2}(L,L-B) \) is an isomorphism - it was necessary to extend the bundles over a point at infinity since only then would all of the relevant classes live in \( U^0(L^+) \), allowing us to apply the inverse of \( i_s \) to \( j_s(1) \). In particular, if we ran the argument with the isomorphism into \( U^{0+2}(L,L-B) \), we could not have applied \( i_s^{-1} \) to \( j_s(1) \) since the image of \( j \) is not contained in the zero section. Note that \( L^+ \) is not necessarily a smooth manifold, so \( U^0(L^+) \) doesn’t make sense, but we can pass to \( MU^0(L^+) \), and then since \( U^* \to MU^* \) defines pushforwards for complex oriented maps in \( MU^* \), sends Euler classes to Euler classes, is natural, etc. the result holds for \( U^* \).

**Remark 5.1.2.** In the form of 5.1.1, this result doesn’t seem particularly useful with respect to studying the formula in 4.2.6, because the Euler class in 5.1.1 is of the bundle \( Q \times_{\mathbb{Z}/p} \mathbb{C} \to B \) associated to the representation \( \eta \), whereas the Euler class \( v \) in 4.2.6 is of the bundle \( Q \times_{\mathbb{Z}/p} (X \times \mathbb{C}) \to Q \times_{\mathbb{Z}/p} X \cong B \times X \). However, these are essentially the same thing because since \( X \) is a trivial \( \mathbb{Z}/p \)-space, the two bundles coming from \( \eta \)
\[
\begin{align*}
    Q \times_{\mathbb{Z}/p} (X \times \mathbb{C}) &\to B \times X \\
    (Q \times_{\mathbb{Z}/p} \mathbb{C}) \times X &\to B \times X
\end{align*}
\]

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are the same, and hence their Euler classes are the same. But if $\text{pr}_1 : B \times X \to B$ is projection, then the latter bundle is just $\text{pr}_1^*(L)$, where $L$ is the bundle of 5.1.1. Hence $v = \text{pr}_1^*(e(L))$, where $v$ is as in 4.2.6. One finds, for instance that the inclusion of a point $i : \{b\} \to B$ gives a ring homomorphism

$$U^*(B \times X) \xrightarrow{i^*} U^*(\{b\} \times X) \cong U^*(X)$$

that sets $v = 0$ because $i^* \text{pr}_1^*(e(L)) = (\text{pr}_1 \circ i)^*(e(L)) \in U^2(X)$, $\text{pr}_1 \circ i$ is null, and $U^2(\ast) = 0$.

The Gysin Sequence

If $p : E \to B$ is a complex vector bundle of rank $n$, and $\pi : SE \to B$ is the unit sphere bundle associated to $E$, one in general has the Gysin sequence

$$\cdots \to H^{q-2n}(B) \xrightarrow{e(E)} H^q(B) \xrightarrow{\pi^*} H^q(SE) \xrightarrow{\phi} H^{q-2n+1}(B) \to \cdots$$

where $\phi$ is the connecting map $H^q(SE) \cong H^q(E-0) \to H^{q+1}(E, E-0)$ followed by the inverse of the Thom isomorphism. The same sequence holds for any cohomology theory with Thom isomorphisms, i.e. complex oriented cohomology theories, so it holds for $MU^*$. In $U^*$, however, we have a more geometric interpretation of the Thom isomorphism as the pushforward $i_*$, and its inverse is $p_*$, and in fact it is not hard to show that $\phi = p_*|_{E-0} = \pi_*$. This pushforward makes sense because $\pi$ is proper as its fibers are spheres which are compact, and it factors through the embedding $SE \to E$. In $U^*$ the sequence thus becomes

$$\cdots \to U^{q-2n}(X) \xrightarrow{e(E)} U^q(B) \xrightarrow{\pi^*} U^q(SE) \xrightarrow{\pi_*} U^{q-2n+1}(B) \to \cdots$$

We apply this sequence to the following situation. Let $Z/p$ act on $C^n$ by having $1 \in \mathbb{Z}/p$ act as multiplication by $\zeta := \exp(\frac{2\pi i}{p})$. Then if $X$ is a trivial $Z/p$ space, we have the bundles $E_n := Q \times_{Z/p} (X \times C^n) \to B \times X$ and $(Q \times_{Z/p} C^n) \to B \times X$, which are the same by 5.1.2. There is an induced action of $Z/p$ on $S^{2n-1} \subset C^n$ since we may think of $S^{2n-1}$ as $\{z \in C^n : \sum |z_i|^2 = 1\}$, making it clear that $S^{2n-1}$ is invariant under $Z/p$ as $\zeta^{p_n} = 1$. Letting $Q \to B$ be the bundle $S^1 \to S^1/Z/p$, we thus conclude that $S(E_n) = (S^{2n-1} \times_{Z/p} S^1) \times X$.

Now let $p_n : (S^{2n-1} \times_{Z/p} S^1) \times X \to S^1/Z/p \times X$ be the projection of this spherical bundle. Then we have a Gysin sequence

$$\cdots \to U^{q-2n-2}(S^1/Z/p \times X) \xrightarrow{e(E_n)} U^q(S^1/Z/p \times X) \xrightarrow{p_*} U^q((S^{2n-1} \times_{Z/p} S^1) \times X) \xrightarrow{p_{n+1}} U^{q-2n-1}(S^1/Z/p \times X) \to \cdots$$

However, it is clear that the representation of $Z/p$ on $C^n$ described above is just $\eta^{p_n}$, and their corresponding bundles and unit sphere bundles are thus the same. In particular, $e(E_n) = v_1$, where $v_1$ is the Euler class in 4.2.6 with respect to the principal $Z/p$-bundle $S^{2n-1} \to S^{2n-1}/Z/p$. Now we want to see what happens when we stabilize these Gysin sequences against $n$, and we claim there is the following commutative diagram

\[
\begin{array}{ccc}
U^{q-2n-2}(S^1/Z/p \times X) & \xrightarrow{e(E_n)} & U^q(S^1/Z/p \times X) \\
\downarrow v_1 & & \downarrow \text{id} \\
U^{q-2n}(S^1/Z/p \times X) & \xrightarrow{\eta^n} & U^q((S^{2n-1} \times_{Z/p} S^1) \times X) \\
\downarrow j_n & & \downarrow \eta^n \\
& & U^{q-2n-1}(S^1/Z/p \times X)
\end{array}
\]

where $j_n : (S^{2n-1} \times_{Z/p} S^1) \times X \to (S^{2n-1} \times_{Z/p} S^1) \times X$ is induced by the inclusion $i : C^n \to C^{n+1}$. The left square obviously commutes, and the middle commutes as $p_n = p_{n+1} \circ j_n$. The right commutes by the following lemma.

**Lemma 5.1.3.** Let $E, F$ be complex vector bundles over $X$, with $f : S(E \oplus F) \to X$, $g : SE \to X$, and $j : SE \to S(E \oplus F)$. Then, for $z \in U^*(S(E \oplus F))$,

$$g_*j^*(z) = c(F) \cdot f_*(z)$$

**Proof:** We remark that $f$ and $g$ are proper as their fibers are spheres, and hence they are all complex oriented by 3.1.2 since they factor through obvious bundles, and we may take these bundles to be trivial by possibly embedding them in large trivial bundles. $j$ is proper as a closed embedding, and it has a complex structure.
on its normal bundle coming from the fact that the embedding $E \hookrightarrow E \oplus F$ does. Hence we may speak of the pushforwards of each of these maps. $f$ factors as

$$S(E \oplus F) \to E \oplus F \to X$$

so that $g = f \circ j$. Let $s : X \to F$ be the zero section, then the projection $p : S(E \oplus F) \to F$ is transverse to $f$ because locally $p$ is given by

$$X \times S^{2n+2m-1} \hookrightarrow X \times \mathbb{C}^n \times \mathbb{C}^m \to X \times \mathbb{C}^m$$

If $n = 0$, this map never intersects the zero section so there is nothing to check. When $n > 0$, the map $S^{2n+2m-1} \to \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ has 0 as a regular value, since a point sent to zero looks like $v = (z_1, \ldots, z_n, 0, \ldots, 0) \in \mathbb{C}^{n+m}$, i.e. a point in $S^{2n-1} \subset S^{2n+2m-1}$, and after identifying the tangent spaces with the vector spaces themselves, $T_p S^{2n+2m-1}$ is identified with the subspace of $\mathbb{C}^n \times \mathbb{C}^m$ orthogonal to $v$, which contains the factor of $\mathbb{C}^m$ on the right. We compute the fiber product $p^*(s) = j$ and therefore

$$j_* 1 = p^* s_* 1 = f^* s^* s_* 1 = f^* (e(F))$$

using the fact that the map $sf : S(E \oplus F) \to X \to F$ is homotopic to $p$ since $F$ deformation retracts onto the zero section. We thus make the following calculation

$$g^* f^* (z) = f^* s^* s_* (z) = f^* (f^* (e(F)) \cdot z) = e(F) \cdot f^* (z)$$

using 3.1.6 in lines 2 and 4, and using the above equation in line 3. Letting $g = p_n$, $j = j_n$, and $f = p_{n+1}$, since $E_n = E_{n-1} \oplus E_1$ in the above notation, this result proves that the above square commutes. \hfill \Box

The Key Lemma

With these in place, we may now prove the main result of this section, as referenced in the introduction. We remark that there is an action of $U^*(X)$ on $U^*(Y \times X)$ given by essentially the same action of $U^*(\ast)$ on $U^*(Y)$, when $Y$ has a basepoint. Namely, in our case of interest the inclusion of the $p$-th roots of unity $\mathbb{Z}/p \to S^1 \hookrightarrow S^{2n+1}$ induces a map $\ast = \mathbb{Z}/p/\mathbb{Z}/p \to S^{2n+1}/\mathbb{Z}/p$, i.e. we have a choice of basepoint in $S^{2n+1}/\mathbb{Z}/p$, and hence any complex oriented map $f : Z \to X$ may be considered as

$$\ast \times Z \to S^{2n+1}/\mathbb{Z}/p \times X$$

which thus may be multiplied by anything in $U^*(S^{2n+1}/\mathbb{Z}/p \times X)$. By the same reasoning as in the action of $U^*(\ast)$, one sees that pushforwards of the form $(f \times \text{id}_X)_\ast$, for $f \in U^*(S^{2n+1}/\mathbb{Z}/p)$ and pullbacks $(g \times \text{id}_X)^\ast$ are $U^*(X)$-linear.

Let $j_n$ be the map as above and define $j_n^* : S^{2n-1}/\mathbb{Z}/p \times X \to S^{2n+1}/\mathbb{Z}/p \times X$ also induced from the map $i : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$.

**Proposition 5.1.4.** If $x \in U^q(S^{2n+1}/\mathbb{Z}/p \times X)$ such that $x \cdot v_{n+1} = 0$, then there exists $y \in U^q(X)$ such that $y \cdot \langle p \rangle (v_n) = j_n^*(x)$. Since the $S^{2n+1}/\mathbb{Z}/p$ approximate $B\mathbb{Z}/p$, this may be thought of as approximating the exact sequence in the beginning of this section.

**Proof:** Let $\pi_{n+1} : S^{2n+1} / \mathbb{Z}/p S^1 \times X \to S^{2n+1}/\mathbb{Z}/p \times X$ be the unit sphere bundle of the line bundle induced from $\eta$, i.e. this is the sphere bundle of the bundle $F_{n+1}$ such that $e(F_{n+1}) = v_{n+1}$. Applying the Gysin sequence to this sphere bundle, we have an exact sequence

$$U^q + ((S^{2n+1} / \mathbb{Z}/p S^1) \times X) \xrightarrow{\pi_{n+1} \ast} U^q(S^{2n+1}/\mathbb{Z}/p \times X) \xrightarrow{\partial_{n+1}} U^{q+2}(S^{2n+1}/\mathbb{Z}/p \times X)$$
Hence if \( v_{n+1} \cdot x = 0 \), there exists \( z \in U^{i+1}((S^{2n+1} \times \mathbb{Z}/p \times X) \) such that \( x = \pi_{n+1} \cdot (z) \). Since \( j_n \) and \( j_n' \) are both induced from \( i \), the following square commutes

\[
\begin{array}{ccc}
(S^{2n-1} \times \mathbb{Z}/p) \times X & \xrightarrow{j_n} & (S^{2n+1} \times \mathbb{Z}/p) \times X \\
\pi_n & \downarrow & \pi_{n+1} \\
S^{2n-1} / \mathbb{Z}/p \times X & \xrightarrow{j_n'} & S^{2n+1} / \mathbb{Z}/p \times X 
\end{array}
\]

But \( \pi_{n+1} \) is a submersion and the square is a pullback, hence we have

\[
j_n' \circ (x) = j_n \pi_{n+1} \cdot (z) = \pi_n \cdot j_n' \cdot (z)
\]

by 3.1.5.

Recall that by 5.1.2, \( v_1 \in U^2(S^1 / \mathbb{Z}/p \times X) \) is equal to \( pr_1^* (e(L)) \), where \( pr_1 : S^1 / \mathbb{Z}/p \times X \to S^1 / \mathbb{Z}/p \) is projection, and \( L \) is the bundle \( S^1 \times \mathbb{Z}/p \mathbb{C} \rightarrow S^1 / \mathbb{Z}/p \) induced from \( \eta \), as in 5.1.1. Therefore \( e(L) \in U^2(S^1 / \mathbb{Z}/p) \cong U^2(S^1) = 0 \) since there are no manifolds of negative dimension, and hence \( v_1 = 0 \).

Since \( v_1 = 0 \), in the diagram preceding 5.1.3, we have \( p_n \circ j_n' \cdot (z) = v_1 \cdot p_{n+1} \cdot (z) = 0 \), so \( j_n' \cdot z \in \ker p_n \), and thus there exists \( z' \in U^{i+1}(S^1 / \mathbb{Z}/p \times X) \) such that \( p_n \circ j_n' \cdot (z') = j_n' \circ (z) \).

Hence

\[
j_n' \circ (x) = \pi_n \cdot j_n' \circ (z')
\]

Now by identifying \( S^1 / \mathbb{Z}/p \cong S^1 \), we may regard \( \phi : S^1 / \mathbb{Z}/p \times X \to X \) as the unit sphere bundle of \( \mathbb{C} \times X \), whose Euler class is trivial. We thus have that the corresponding Gysin sequence splits into short exact sequences

\[
0 \to U^{i+1}(X) \xrightarrow{\phi^*} U^{i+1}(S^1 / \mathbb{Z}/p \times X) \xrightarrow{\phi_*} U^i(X) \to 0
\]

These sequences split on the right via the map \( y \mapsto y \cdot i_* \) where \( i : X \cong \mathbb{C} \times X \to S^1 / \mathbb{Z}/p \times X \) is given by the inclusion of the basepoint in \( S^1 / \mathbb{Z}/p \), and \( y \) is shorthand for the action of \( U^* (X) \) described above. Computing \( \phi^* (y') = y' \cdot 1 \) where \( 1 \) is the cobordism class of the identity in \( U^* (S^1 / \mathbb{Z}/p \times X) \), we thus may write \( z' = y' \cdot 1 + y \cdot i_*(1) \). Hence

\[
j_n' \circ (x) = \pi_n \cdot p_n^* (y' \cdot 1) + \pi_n \cdot p_n^* (y \cdot i_*(1))
\]

We have by \( U^* (X) \)-linearity that

\[
\pi_n \cdot p_n^* (y' \cdot 1) = \pi_n (y \cdot p_n^* (1)) = \pi_n (y \cdot 1) = y \cdot \pi_n (1) = y \cdot \pi_n (p_n^* (1)) = 0
\]

using the Gysin sequence involving \( \pi_n \). One easily checks that \( p_n^* i_* (1) \) is the cobordism class of \( S^{2n-1} \times \mathbb{Z}/p \times X \cong (S^{2n-1} \times \mathbb{Z}/p) \times X \mapsto (S^{2n-1} \times \mathbb{Z}/p) S^1 \times X \), and hence \( \pi_n \cdot p_n^* i_* (1) \) is the cobordism class of \( S^{2n-1} \times X \to S^{2n-1} / \mathbb{Z}/p \times X \), which is \( (p) (e(L)) \) where \( L \) is the bundle \( (S^{2n-1} \times \mathbb{Z}/p \mathbb{C}) \times X \to S^{2n-1} / \mathbb{Z}/p \times X \), i.e. \( F_n \), noting that the proof of 5.1.1 goes through unchanged taking cartesian products with \( X \) everywhere. Hence we have

\[
j_n' \circ (x) = \pi_n \cdot p_n^* (y' \cdot 1) + \pi_n \cdot p_n^* (y \cdot i_*(1)) = 0 + y \cdot (p) (v_n)
\]

recalling that \( v_n \) is by definition the Euler class of \( F_n \). \( \square \)
5.2. The Main Theorem.

We now have all we need to prove the main result of Quillen’s paper, from which we will derive Quillen’s theorem as a corollary in the following section. The main maneuver will be to look at the equation 4.2.6 $p$-locally for each prime $p$ and to use the key lemma 5.1.4. We will need the following result from homotopy theory, which follows from the Serre finiteness theorem, and the fact that $MU$ has only finitely many cells in each degree. We also record an easy lemma to be used in our proof.

**Proposition 5.2.1.** If $X$ is a finite CW complex with basepoint $x_0$, $MU^q(X)$ is a finitely generated abelian group for all $q$. □

**Definition 5.2.2.** Let $X$ be a path-connected space with basepoint $x_0$, and let $\widetilde{MU}^*(X) \subset MU^*(X)$ be the ideal consisting of elements that vanish when restricted to the basepoint $x_0$. This agrees with the usual definition of $\tilde{MU}^*(X)$ up to isomorphism since one always has the splitting

$$MU^*(X) \cong \tilde{MU}^*(X) \oplus MU^*(\{x_0\})$$

but we are using this isomorphism to identify $\widetilde{MU}^*(X)$ and $MU^*(\{x_0\})$ as subsets of $MU^*(X)$ so that the splitting is an equality. Via the isomorphism $U^* \rightarrow MU^*$, we identify $\tilde{MU}^*(X)$ with $\tilde{U}^*(X) := U^*(X,x_0)$ when $X$ is a manifold.

**Lemma 5.2.3.** If $X$ is a space with basepoint $x_0$, one has $MU^*(\ast)$-linear suspension isomorphisms

1. $MU^{2q-1}(X) \cong \widetilde{MU}^{2q}(S^1 \wedge X_+)$
2. $MU^{2q}(X) \cong \widetilde{MU}^{2q+2}(S^2 \wedge X_+)$
3. $\widetilde{MU}^{2q-1}(X) \cong \widetilde{MU}^{2q}(S^1 \wedge X)$

**Proof:** (3) is the usual suspension isomorphism carried by any cohomology theory, which is almost tautological by 2.3. Recall that $\tilde{MU}(\ast)$ is the functor sending a pointed space $X$ to $[\Sigma^\infty X,\mu]$ and $MU(\ast)$ sends a space $X$ to $X_+$, i.e. $X$ with a disjoint basepoint, followed by $\tilde{MU}(\ast)$. Hence (1) and (2) follow immediately from (3) since $MU^*(X) = \tilde{MU}^*(X_+)$, and it is easy to see that (3) is $MU^*(\ast)$-linear, hence so are (1) and (2). □

**Lemma 5.2.4.** If $X$ is a pointed space with basepoint $x_0$ that is homotopy equivalent to a finite CW complex, the ideal $\widetilde{MU}^0(X) \subset MU^0(X)$ is nilpotent.

**Proof:** Identifying $\widetilde{MU}^0(X) \cong MU^*(X,x_0)$, we recall that the relative cup product is the map

$$\widetilde{MU}^0(X) \otimes \widetilde{MU}^0(X) \cong MU^0(X,x_0) \otimes MU^0(X,x_0) \rightarrow MU^*(X \times X,X \vee X) \cong \widetilde{MU}^0(X \wedge X) \xrightarrow{\Delta^*_2} \widetilde{MU}^0(X)$$

where $\Delta_2$ is the map $X \rightarrow X \times X \rightarrow X \wedge X$. Then $\Delta_2$ is nullhomotopic for $n$ sufficiently large since we can take $X$ to be a CW complex, and take the map to be null by cellular approximation since forming $X \wedge X$ raises the degrees of the nonzero cells in $X$. □

**Theorem 5.2.5.** If $X$ is a connected space homotopy equivalent to a finite CW complex, then

$$MU^*(X) = C \cdot \bigoplus_{q \geq 0} MU^q(X)$$
$$\tilde{MU}^*(X) = C \cdot \bigoplus_{q > 0} MU^q(X)$$

where $C$ is the ring as in 4.2.2 that is generated by the coefficients of the formal group law.
Proof:

(i) Reducing to the Even Case: Suppose we have proven that
\[ \widetilde{MU}^{ev}(X) = C \cdot \bigoplus_{q>0} MU^{2q}(X) \]

where \( \widetilde{MU}^{ev}(X) \) is the subring of \( MU^*(X) \) consisting of elements of even degree. Then since \( C \) is concentrated in even degrees, we have
\[ C \cdot \bigoplus_{q>0} MU^{2q}(X) = \left( C \cdot \bigoplus_{q>0} MU^{2q}(X) \right) \oplus \left( C \cdot \bigoplus_{q>0} MU^{2q-1}(X) \right) = \widetilde{MU}^{ev}(X) \oplus \left( C \cdot \bigoplus_{q>0} MU^{2q-1}(X) \right) \]

By the splitting in 5.2.2, \( MU^{2q-1}(X) = \overline{MU}^{2q-1}(X) \oplus MU^{2q-1}(\{x_0\}) \), and since \( q > 0 \), \( MU^{2q-1}(\{x_0\}) \) vanishes since \( MU^{2q-1}(\{x_0\}) \cong U^{2q-1}(\ast) = 0 \) since \( q > 0 \). Hence we have
\[ C \cdot \bigoplus_{q>0} MU^{2q-1}(X) = C \cdot \bigoplus_{q>0} \overline{MU}^{2q-1}(X) \]

Applying the isomorphism (3) of 5.2.3 to \( \overline{MU}^*(X) \), we have that the images of \( C \cdot \bigoplus_{q>0} \overline{MU}^{2q-1}(X) \) and \( \overline{MU}^{odd}(X) \) are
\[ C \cdot \bigoplus_{q>0} \overline{MU}^{2q}(S^1 \wedge X) \quad \text{and} \quad \overline{MU}^{ev}(S^1 \wedge X) \]

respectively, noting that
\[ C \cdot \bigoplus_{q>0} \overline{MU}^{2q-1}(X) \subset \overline{MU}^*(X) \]

because the restriction map \( MU^*(X) \rightarrow MU^*(\{x_0\}) \) is \( MU^*(\ast) \)-linear. By the above assumption, these are equal, again using the fact that \( \overline{MU}^{2q}(S^1 \wedge X) = MU^{2q}(S^1 \wedge X) \) when \( q > 0 \). Thus the two preimages are equal, and we have
\[ C \cdot \bigoplus_{q>0} MU^{2q}(X) = \overline{MU}^{ev}(X) \oplus \left( C \cdot \bigoplus_{q>0} MU^{2q-1}(X) \right) = \overline{MU}^{ev}(X) \oplus \overline{MU}^{odd}(X) = \overline{MU}^*(X) \]

hence we recover the second claim of the theorem. To recover the first claim, we note that
\[ MU^{ev}(X) = \overline{MU}^{ev}(X) \oplus MU^{ev}(\{x_0\}) = \left( C \cdot \bigoplus_{q>0} MU^{2q}(X) \right) \oplus MU^{ev}(\{x_0\}) \]

If we apply the isomorphism (2) of 5.2.3 to \( C \cdot MU^0(\{x_0\}) \) and \( MU^0(\{x_0\}) \), we get
\[ C \cdot \overline{MU}^{2}(S^2) \quad \text{and} \quad \overline{MU}^{ev}(S^2) \]

respectively, and by our key assumption, the latter is
\[ C \cdot \bigoplus_{q>0} MU^{2q}(S^2) = C \cdot \overline{MU}^{2}(S^2) \]

using the splitting 5.2.2 and the fact that \( \pi_k(MU) \) vanishes for \( k < 0 \), as we showed since there are no manifolds of negative dimension. Putting these facts together we have
\[ MU^{ev}(X) = \left( C \cdot \bigoplus_{q>0} MU^{2q}(X) \right) \oplus \left( C \cdot MU^0(\{x_0\}) \right) \]
\[ = C \cdot \left( \bigoplus_{q>0} MU^{2q}(X) \right) \oplus MU^0(\{x_0\}) \]
\[ = C \cdot \bigoplus_{q>0} MU^{2q}(X) \]

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the last equality following from the fact that
\[ MU^0(X) = \tilde{MU}^0(X) \oplus MU^0(\{x_0\}) = \left( C \cdot \bigoplus_{q \geq 0} MU^{2q}(X) \right) \oplus MU^0(\{x_0\}) \]
where \([-]_0\) denotes all terms of degree 0. Using an identical argument with the isomorphism (1) in 5.2.3, we recover the first statement of the theorem.

(ii) Setting up Induction on the Key Formula: We now take a smooth manifold homotopy equivalent to \(X\), which we also call \(X\), and thereby reduce our statements to \(U^s\). By (i), letting
\[ R := C \cdot \bigoplus_{q \geq 0} U^{2q}(X) \]
it suffices to show \(\tilde{U}^{ev}(X) = R\), and it suffices to show this \(p\)-locally for every prime \(p\) by a localization-globalization argument since \(\tilde{U}^{ev}(X)\) and \(R\) are sub \(\mathbb{Z}\)-modules of \(U^s(X)\), and the prime ideals \((p) \subset \mathbb{Z}\) are all the maximal ideals in \(\mathbb{Z}\). Noting that both of these are concentrated in even degrees only, we assume as an inductive hypothesis that \(w_i v\). For the base case, if \(q = 0\) and \(j < q\), then \(-2j\) is positive, hence \(U^{-2j}(X) = U^{-2j}(X)\), and \(R^{-2j} \subset U^{-2j}(X)\) since \(U^s(\ast) \cdot U^s(X) \subset U^{s+1}(X)\) and \(U^{-2j}(X) \subset R^{-2j}\) since \(1 \in C\) and \(U^{-2j}(X)\) appears in the sum defining \(R\) as \(-j > 0\).

Applying 4.2.6 to the principal \(\mathbb{Z}/p\)-bundle \(S^{2m+1} \to S^{2m+1}/\mathbb{Z}/p\), there exists \(n \gg 0\) such that if \(x \in \tilde{U}^{-2j}(X) \subset U^{-2j}(X)\)
\[ w^{n+j} P_X = \sum_{l(a) \leq n} w^{n-l(a)} \left( \prod_{j \geq 1} a_j(v)^{a_j} \right) s_a(x) \]
with \(a_j(v) \in C[[v]]\), where in this case \(\nu = \nu_{m+1}\) as in 5.1.4. We also have that
\[ w = (p-1)! v^{p-1} + \sum_{j \geq p} d_j v^j = v^{p-1} = v^{p-1}((p-1)! + \sum_{j \geq p} d_j v^{j-p+1}) =: v^{p-1} \theta(v) \]
where \(d_j \in C\) and hence \(\theta(v) \in C[[v]]\) and after localizing at \(p\), \(\theta(v)\) becomes multiplicatively invertible because its leading term \((p-1)!\) is a unit after localizing at \(p\). When \(a = (0,0,\ldots)\), \(s_a(x) = x\), so that
\[ w^{n+j} P_X = w^p x + \sum_{0 < l(a) \leq n} w^{n-l(a)} \left( \prod_{j \geq 1} a_j(v)^{a_j} \right) s_a(x) \]
Remarkably that for \(a \neq 0\), \(s_a\) raises degree by an even number as in this case the Chern class \(c_a\) has positive even degree, localizing the above equation at \(p\), we have \(s_a(x) \in R_{(p)}^{-2j}\) for \(j < q\). Hence the above summation on the right lies in \(R_{(p)}[[v]]\). Furthermore since \(\theta(v)\) is invertible, we have
\[ w^n(w^p P_X - x) = \sum_{0 < l(a) \leq n} w^{n-l(a)} \left( \prod_{j \geq 1} a_j(v)^{a_j} \right) s_a(x) =: (v^{p-1})^n (w^p P_X - x) = \psi(v) \]
where \(\psi(v) \in R_{(p)}[[v]]\). Now we set \(m := n(p-1) > 0\), and we have
\[ v^m(w^p P_X - x) = \psi(v) \in U^s(S^{2m+1} / \mathbb{Z}/p \times X)_{(p)} \]
Hence the set of integers \(m \geq 0\) such that an equation as above - with the terms on the left and a power series in \(v\) with coefficients in \(R_{(p)}\) on the right, all living in \(U^s(S^{2m+1} / \mathbb{Z}/p \times X)_{(p)}\) is nonempty, so we choose \(m\) to be minimal. There is no cause for concern here in defining \(m\) as above because \(n\) depends only on the dimension of \(X\), and not on the dimension of the sphere defining the principal bundle in use.

(iii) \(m = 1\): If \(i = 0\) is as in 5.1.2, then \(i^s\) is a ring homomorphism that sets \(s = 0\), hence the above equation shows \(\psi(0) = 0\), so we may define the power series \(\psi_1(t) = (1/t)\psi(t)\), and thus we have
\[ v(v^{m-1}(w^p P_X - x) - \psi_1(v)) = 0 \]
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Then since \(|v| = 2\), \(|w| = 2(p - 1)\), and \(|P^x| = -2qp\), \(|v^{m-1}(w^qP^x - x) - \psi_1(v)| = 2(m - 1) - 2q\), hence 5.1.4 tells us there exists \(y \in U^{2(m-1)-2q}(X)\) with \(y \cdot (p)\)\((v_m) = j_m^*(v^{m-1}(w^qP^x - x) - \psi_1(v))\). But since \(j_m^*(v_{m+1}) = v_m\) and by \(v\) we have meant \(v_{m+1}\), we have that since \(j_m^*\) is a ring homomorphism

\[
v^{m-1}(w^qP^x - x) = \psi_1(v) + y \cdot (p)\langle v \rangle
\]
in \(U^*(\mathbb{S}^{2m-1}/\mathbb{Z} / \mathbb{Z} \times X)\). Now restricting this equation from \(X\) to its basepoint \(x_0\) is a ring homomorphism that sets \(x = 0\) since \(x \in \tilde{U}^{-2q}(X)\), the reduced cohomology of a point always vanishing. Similarly \(\psi_1(v)\) is set to zero because its coefficients lie in \(\mathcal{R}(p)\), which vanishes when restricted to a point since \(U^{2k}(\{x_0\}) = 0\) for \(k > 0\). Hence if \(y^*\) is the restriction of \(y\) to \(U^*(\{x_0\})\), we have \(y^*(p)\langle v \rangle = 0\). Thus

\[
v^{m-1}(w^qP^x - x) = \psi_1(v) + (y - y^*)\langle v \rangle
\]
and so we may assume \(y \in \tilde{U}^*(X)\) since \(y - y^*\) is. Now if \(m > 1\), then \(y \in \mathcal{R}(p)\) because \(|y| = 2(m - 1 - q)\) and \(m - 1 - q > -q\), so \(y\) is in the inductive range. Then the above equation would be of the type we are considering, and \(m > 1 \implies m - 1 > 1\), which would then contradict the fact that \(m\) is minimal. We thus conclude \(m = 1\).

(iv) **Finishing the Inductive Step:** Since \(m = 1\), we have \(w^qP^x - x = \psi_1(v) + y \cdot (v)\), hence applying \(i^*\) again we have \(i^*(y \cdot (v)) = y \cdot i^*(v) = p \cdot y\) since the leading term of \(\langle v \rangle\) is \(p\), and we have \(i^*P^x = x^p\) by 3.4.7. Therefore

\[
\psi_1(0) + py = \begin{cases} -x & \text{if } q > 0 \\ x^p - x & \text{if } q = 0 \end{cases}
\]
In the \(q > 0\) case, we thus have

\[
\tilde{U}^{-2q}(X) \subset \mathcal{R}^{-2q}(p) + p\tilde{U}^{-2q}(X) \subset \mathcal{R}^{-2q}(p) + p^2\tilde{U}^{-2q}(X) \subset \cdots
\]
then since \(\tilde{U}^{-2q}(X)\) is a finitely generated abelian group by 5.2.1, \(p^q\) eventually kills \(\tilde{U}^{-2q}(X)\), and we have \(\tilde{U}^{-2q}(X) \subset \mathcal{R}^{-2q}(p)\), thus completing induction for \(q > 0\), since the inclusion in the other direction is obvious.

In the \(q = 0\) case, using the same argument, we can reduce to showing that \(\tilde{U}^0(X) \subset \mathcal{R}^0(p) + p\tilde{U}^0(X)\).

The function \(x \mapsto x^p - x\) on \(\tilde{U}^0(X)\) lands in \(\mathcal{R}^0(p) + p\tilde{U}^0(X)\), and it descends to a function on the quotient

\[
\frac{\tilde{U}^0(X)}{\mathcal{R}^0(p) + p\tilde{U}^0(X)} \rightarrow \frac{\mathcal{R}^0(p) + p\tilde{U}^0(X)}{\mathcal{R}^0(p) + p\tilde{U}^0(X)}
\]
Then since \(\tilde{U}^0(X)\) is a nilpotent ideal by 5.2.4, \(x \mapsto x^p\) is a nilpotent endomorphism of \(\tilde{U}^0(X) / p\tilde{U}^0(X)\), hence there exists \(N >> 0\) such that \(x^{p^N} = 0\) for all \(x\). Thus applying \(x \mapsto x^p - x\) \(N\) times, we have \(x^{p^N} = x^{p^{N-1}} + \cdots\), the sum lying in \(S := (\mathcal{R}^0(p) + p\tilde{U}^0(X)) / p\tilde{U}^0(X)\), but since \(x^{p^N} = 0\) and the terms after \(-x^{p^{N-1}}\) all lie in \(S\) as well since they are in the image of \(x \mapsto x^p - x\), we thus have \(x^{p^{N-1}} \in S\). Arguing similarly in the next term in the sum, we see that \(x^{p^{N-2}} \in S\) and so on, that \(x \in S\). Thus \(x \in \mathcal{R}^0(p) + p\tilde{U}^0(X)\) up to a term in \(p\tilde{U}^0(X)\), so \(x \in \mathcal{R}^0(p) + p\tilde{U}^0(X)\). This completes the induction and the proof of the theorem.

**Corollary 5.2.6.** The map \(L \rightarrow MU^*(\ast) = \pi_*(MU)\) corresponding to the formal group law of 4.2.2 is a surjection.

**Proof:** The image of the above map is the ring \(\mathcal{C}\), and 5.2.5 says \(MU^*(\ast) = \mathcal{C} \cdot MU^0(\ast)\) since \(MU^0(\ast)\) vanishes for \(q > 0\). \(\mathcal{C} \subset \mathcal{C} \cdot MU^0(\ast)\) since \(1 \in MU^0(\ast)\), and \(U^0(\ast) \cong \mathbb{Z}\) since the manifolds of dimension 0 are the finite discrete sets. Therefore \(MU^0(\ast)\) is generated by 1, so we have \(\mathcal{C} \cdot MU^0(\ast) \subset \mathcal{C}\), and thus \(MU^*(\ast) = \mathcal{C}\). Since \(\mathcal{C}\) is concentrated in even degrees, we also see that \(MU^{ev}(\ast) = \mathcal{C}\) and \(MU^{odd}(\ast) = 0\). □

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We state two more corollaries of 5.2.5 that use duality in $MU$ to bound the degrees of generators of $MU_*(X)$ for a finite CW complex $X$. These are not of much interest to this exposition, and the proofs in [7] are clear, so we omit them here.

**Corollary 5.2.7.** Let $X$ be a finite CW complex that can be embedded into an almost complex manifold $M$ of dimension $n$. Then $MU_*(X)$ is generated as a $MU^*(*)$-module by elements of degree $\leq n$, and by elements of degree $< n$ if none of the components of $M$ are compact.

**Corollary 5.2.8.** If $X$ is a finite CW complex of dimension $r$, then $MU_*(X)$ is generated as an $MU^*(*)$-module by elements of degree $\leq 2r$.

### 5.3. Proof of Quillen’s Theorem.

With 5.2.5, we are already very close to proving Quillen’s theorem: we know the map $f : L \to \pi_*(MU)$ from 4.2.3 is a surjection. To prove it is an injection, we will need to know that the Lazard ring is torsion free. In fact we have the following purely algebraic result about the Lazard ring.

**Proposition 5.3.1.** (Lazard’s Theorem) The Lazard ring $L$ is a polynomial ring over $\mathbb{Z}$ with a generator in degree $q$ for each $q > 0$. [3]

The basic idea of our proof that $f$ is an injection will be to use our Landweber-Novikov operations to build a ring map $\pi_*(MU) \to R$ for an easier ring $R$ and show that the composition $L \to \pi_*(MU) \to R$ is an injection by showing it is an injection after tensoring with $Q$, and then using the fact that $L$ is torsion free.

As we noted in the construction of our total Chern classes $c_a$, since $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x]$ and $MU^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x]$, the construction of the $c_a$ can be carried out identically in $H^*$. In fact since by construction the ring map $\phi : MU \to H_2 \mathbb{Z}$ of 2.7 sends the class $x \in MU^2(\mathbb{CP}^\infty)$ to $x \in H_2 \mathbb{Z}(\mathbb{CP}^\infty)$, $\phi$ carries the classes $c_a^H$ to $c_a^H$. We thus have a ring map $\phi : U^*(X) \to H^*(X)$ for a manifold $X$ that preserves the $c_a$’s. We define the following map:

$$\beta : U^*(X) \xrightarrow{s_l} U^*(X)[t_1, t_2, \ldots] \xrightarrow{\phi} H^*(X)[t_1, t_2, \ldots]$$

where $s_l$ is the Landweber-Novikov operation. Then $\beta$ is a ring homomorphism since $s_l$ and $\phi$ are.

**Lemma 5.3.2.** If $L$ is a line complex line bundle, then

$$\beta(e^{\tilde{L}}(L)) = \sum_{j \geq 0} t_j(e^{\tilde{L}}(L))^{j+1}, \quad t_0 = 1$$

**Proof:** Let $i$ be the zero section of $L$, and let $\tilde{i}$ be homotopic and transverse to $i$. Then, as in 3.1.2, we may factor $i$ through itself, and then $s_l(i_*(1)) = i_*(c_l(v_i))$ and $v_i = L$, and similarly we may factor $e^{\tilde{L}}(L) = i^* i_*(1)$ through $\tilde{i}^* L$ and the normal bundle is just $\tilde{i}^* L$. We have

$$s_l(e^{\tilde{L}}(L)) = e^{\tilde{L}}(L) _* (c_l(\tilde{i}^* L))$$

$$= e^{\tilde{L}}(L) _* \left( \sum_{j \geq 0} t_j i^* ((e^{\tilde{L}}(L))^{j+1}) \right)$$

$$= \sum_{j \geq 0} t_j i^* (i^* ((e^{\tilde{L}}(L))^{j+1}))$$

$$= \sum_{j \geq 0} t_j i^* (i^* (1)) \cdot (e^{\tilde{L}}(L))^{j+1}$$

$$= \sum_{j \geq 0} t_j (e^{\tilde{L}}(L))^{j+1}$$

and $t_0 = 1$. In line 2 we used homotopy invariance, the fact that $i^*$ is a ring homomorphism, and the formula for the total Chern class of a line bundle given in 3.4.2. In line 3 we used that the $s_l$ are defined so that
\[e^U(L) = i_\ast i^\ast\] moves past the indeterminates, and line 4 is direct application of 3.1.6. Finally since \(\phi\) is a ring homomorphism and it sends \(c^U_a \mapsto c^H_a\), it therefore sends \(e^U \mapsto e^H\), the result follows. \[\square\]

For line bundles \(L_1, L_2\), setting \(\theta(x)\) to be the power series \(\sum_{j \geq 0} t_j x^{j+1}\), we plug in \(L_1 \otimes L_2\) into 5.3.2 and we have

\[
(\beta F)(\theta(e^H(L_1)), \theta(e^H(L_2))) = \sum_{ij} \beta(c_{ij}) \theta(e^H(L_1))^i \theta(e^H(L_2))^j
\]

\[
= \beta(F(e^U(L_1), e^U(L_2)))
\]

\[
= \beta(e^U(L_1 \otimes L_2))
\]

\[
= \theta(e^H(L_1 \otimes L_2))
\]

\[
= \theta(e^H(L_1) + e^H(L_2))
\]

using the fact that Euler classes of tensor products add in ordinary cohomology. Thus the formal group law \(\beta F\) satisfies \((\beta F)(\theta(x), \theta(y)) = \theta(x + y)\). Then since \(\theta\) has constant term zero and degree 1 term 1, it has a compositional inverse \(\theta^{-1}\), and we find

\[
(\beta F)(x, y) = (\beta F)(\theta(\theta^{-1}(x)), \theta(\theta^{-1}(y))) = \theta(\theta^{-1}(x) + \theta^{-1}(y))
\]

and for a formal group law \(G\) and a compositionally invertible power series \(f\), we define \(f^* G\), the conjugation of \(G\) by \(f\) to be \(f^{-1} G(f(x), f(y))\), so that \(\beta F = \theta^{-1} G_a\), where \(G_a(x, y) = x + y\) is the additive formal group law. We therefore have a ring homomorphism

\[
L \xrightarrow{f} U^*(\ast) \xrightarrow{\beta} H^*(\ast)[t_1, t_2, \ldots] \cong \mathbb{Z}[t_1, t_2, \ldots]
\]

\[
F_{\text{Univ}} \mapsto F \mapsto \theta^{-1} G_a
\]

where when we say a formal group law is sent to another, we mean that the coefficients of one are sent to those of the other.

**Theorem 5.3.3.** (Quillen’s Theorem) The above composition is an injection, and therefore \(f\) is an isomorphism. It follows also by Lazard’s theorem that \(\pi_*(MU)\) is a polynomial ring over \(\mathbb{Z}\) with one generator of degree \(-2q\) for each \(q > 0\), since \(f(L_q) \subset U^{-2q}(\ast)\) as \(|c_{ij}| = 2 - 2i - 2j\).

**Proof:** Since \(L\) is torsion free by 5.3.1, it suffices to show that \(Q \otimes (\beta f)\) is an injection. Consider the natural transformation induced by \(\beta f\):

\[
\text{Hom}_{CRing}(\mathbb{Z}[t_1, t_2, \ldots], -) \xrightarrow{(\beta f)^\ast} \text{Hom}_{CRing}(L, -)
\]

Plugging in some ring \(R\), we have that ring maps \(u : \mathbb{Z}[t_1, t_2, \ldots] \rightarrow R\) are in bijection with power series \(\theta_u(x) = \sum u(t_j)x^{j+1}\), and ring maps \(L \rightarrow R\) are in bijection with formal group laws over \(R\). Under these identifications and \((\beta f)^\ast\), \(\theta_u\) is sent to the formal group law over \(R\) given by \(\theta_u^{-1} G_a\). It is a general fact that for each formal group law \(G\) over a \(Q\)-algebra \(R\), there is a unique power series \(\log_G(x)\) over \(R\) - called the logarithm of \(G\) by analogy with the power series \(\log \) and \(\exp\) - such that \(G = \log_C G_a\) [3]. Hence this map is an isomorphism for \(Q\)-algebras \(R\). Tensoring up with \(Q\) gives the adjunction

\[
\text{Hom}_{Q-Alg}(Q[t_1, t_2, \ldots], R) \cong \text{Hom}_{CRing}(\mathbb{Z}[t_1, t_2, \ldots], R)
\]

when \(R\) is a \(Q\)-algebra, and we have a similar statement for \(L\). We thus have an isomorphism of functors

\[
\text{Hom}_{Q-Alg}(Q[t_1, t_2, \ldots], -) \rightarrow \text{Hom}_{Q-Alg}(Q \otimes L, -)
\]

and it is induced by \(Q \otimes (\beta f)\), hence by the Yoneda lemma, \(Q \otimes (\beta f)\) is an isomorphism. \[\square\]

**Remark 5.3.4.** This result is powerful because if \(E\) is a ring spectrum and there is a ring map \(\phi : MU \rightarrow E\) (i.e. \(E\) is a complex-oriented cohomology theory), then there is a formal group law over the ring \(\pi_*(E)\) coming from tensor products of line bundles, just as in \(\pi_*(MU)\), and the map \(\phi_* : \pi_*(MU) \rightarrow \pi_*(E)\) carries the formal group law over \(\pi_*(MU)\) to the one over \(\pi_*(E)\). Then 5.3.3 says that this map is just the
map from the Lazard ring corresponding to the formal group law on $\pi_\ast(E)$.

**Remark 5.3.5.** Quillen proves a similar result at the end of his paper for unoriented cobordism - i.e. $\pi_\ast(MO)$. In fact, $MO^\ast(X)$ can be identified with the ring of cobordism classes of smooth proper maps $f : Z \to X$ - there is no need to add any conditions to this since $f$ admits a stable normal bundle, and it doesn’t need to have any additional structure for Thom’s theorem to go through. Then the arguments of this paper can be carried out in identical fashion, except that they become much simpler because one only needs the power operations for $p = 2$ since everything is in characteristic 2. The proof of the main theorem then goes through unchanged, except that the $py$ term vanishes since the tensor square of a real line bundle is trivial, so that $[2]_F(x) = 0$, where $F$ is the analogous formal group law coming from tensor products of line bundles. Quillen then uses the fact that any formal group law in characteristic 2 with a vanishing 2 series has a canonical logarithm to prove the following result:

**Theorem 5.3.6.** If $\Lambda$ is the ring corepresenting formal group laws over commutative rings of characteristic 2 with a vanishing 2 series, the corresponding map $\Lambda \to \pi_\ast(MO)$ is a ring isomorphism, and moreover there is an isomorphism of $\pi_\ast(MO)$ algebras

$$\pi_\ast(MO) \otimes_{\mathbb{Z}/2} H^\ast(X; \mathbb{Z}/2) \to MO^\ast(X)$$

for $X$ a finite CW complex. □

**Appendix A. Comments on Quillen’s Paper**

Other than including details left out in Quillen’s paper, there are few places where this paper differs from Quillen’s, so that the reader who would like to understand Quillen’s paper may use this as a guide. There are, however, a few differences, which I think have made the arguments easier to understand. The key differences are as follows:

1. My definition of complex oriented map is much simpler in that it considers only factorizations through trivial bundles and all the isotopy information is wrapped up in saying that we choose a single class in $[Z, BU]$ for the normal bundle. These definitions are of course equivalent.

2. I do not define a virtual bundle $v_f$ as he does for the purposes of defining the Landweber-Novikov operations, because it is not necessary since we can always take a complex oriented map $f$ to factor through a trivial bundle, as in my definition of a complex orientation.

3. I distinguish between the theories $U^\ast$ on manifolds and $MU^\ast$ on spaces. For Quillen these are the same and it is understood that when applied to a manifold, one can use the geometric model.

4. Quillen uses $h^\ast$-theories, which could be understood as a geometric version of complex oriented cohomology theories. However, all the ones used in the paper are $U^\ast(-)$, $U^\ast(Q \times \mathbb{Z}/p -)$ and $U^\ast(B \times -)$, and I think it is easier to understand the arguments working with these directly, so I don’t use this terminology.

Quillen’s paper is very important for a number of reasons - it is much more than a novel proof of Quillen’s theorem. His geometric construction of complex cobordism in this paper has since been generalized to other cobordism theories and used in other contexts. The main theorem 5.2.5 was a new result that was totally unexpected at the time. The paper was ahead of its time with respect to power operations as well. The formula 4.2.6 can be shown to be of a standard form in the theory of power operations, following the method of defining these as in the introduction and looking at the $\mathbb{Z}/p$ Tate construction applied to $E$. The fact that Quillen derived this by hand is astounding, and his formula is also a significant improvement on this general formula in the case of $E = MU$, since he finds for instance that $m = 1$ in 5.2.5 (ii). We refer the reader to [8] for more details.

**Appendix B. The Proof of Thom’s Theorem**

We now restate and give the proof of Theorem 3.1.7. It closely follows the proof of Thom’s theorem, but requires a bit more detail to handle the complex structures.
Theorem 3.1.7. Regarding $MU^*$ and $U^*$ as functors from the category of smooth manifolds to the category of graded rings, there is a natural isomorphism $U^* \rightarrow MU^*$. For any manifold $X$, $U^*(X)$ has the structure of a graded algebra over $U^*(*)$, and similarly for $MU^*(X)$. Under the isomorphism $U^*(*) \rightarrow MU^*(*)$, the ring isomorphism $U^*(X) \rightarrow MU^*(X)$ is an isomorphism of graded $U^*(*)$-algebras. If $A$ is a strong deformation retract of an open neighborhood $U$ in $X$, we may similarly identify

$$U^*(X, X - A) := \{ \text{Complex-oriented maps } f : Z \rightarrow X : f(Z) \subset A \}$$

with $MU^*(X, X - A)$.

Proof: We first prove the result for $X$ a compact manifold, and explain briefly how this generalizes to an arbitrary manifold. When $X$ is compact, since manifolds are Hausdorff, a complex oriented map $f : Z \rightarrow X$ being proper just amounts to $Z$ and $X$ being compact and $f$ smooth.

(i) The Map: Let $X$ be a compact smooth manifold and $(f : Z \rightarrow X, v) \in U^2(X)$. Then $f$ factors through an embedding $i : Z \rightarrow X \times \mathbb{C}^n$ and $v_i$ has a complex structure so that the class of $v_i$ in $[Z, BU]$ is $v$. By the tubular neighborhood theorem, we have a diagram of embeddings

$$\begin{array}{c}
\nu_i \\
\downarrow \\
Z \hookrightarrow X \times \mathbb{C}^n
\end{array}$$

Using Thom’s method as in the case of a point, noticing that $(X \times \mathbb{C}^n)^+ \cong X_+ \cup S^{2n}$, we obtain a map $S^{2n} \wedge X_+ \rightarrow v_i^+ \cong \text{Thom}(v_i) \rightarrow \text{Thom}(EU(n + q)) = MU(n + q)$, and taking its class in $[\Sigma^\infty_+X, MU]_{-2q}$ gives us a map $\Phi_q : U^2(X) \rightarrow MU^2(X)$. We need to check that this map does not depend on the choice of $v_i \in v$ or the choice of representative of the cobordism class of $(f, v)$.

(ii) Well-Defined: If we chose a different factorization of $f$ through an embedding, say $j : Z \rightarrow X \times \mathbb{C}^m$, to say that $v_j \in [v_i] \in [Z, BU]$ is to say that the normal bundles $v_i$ and $v_j$ are stably isomorphic. Therefore, there exist $k_1, k_2$ with $v_i \oplus k_1 \cong v_j \oplus k_2$ and we set $N := n + k_1 = m + k_2$. We note that the normal bundle of the embedding

$$l_1 : Z \rightarrow X \times \mathbb{C}^n \rightarrow (X \times \mathbb{C}^n) \times \mathbb{C}^{k_1}$$

is $v_i \oplus k_1$ and similarly for $j$ and $k_2$, we may define $l_2$. If we apply $\Phi_q$ to $(f, v)$ where we represent $(f, v)$ with the factorization $l_1$, the tubular neighborhood map $v_i \oplus k_1 \rightarrow X \times \mathbb{C}^n \times \mathbb{C}^{k_1}$ is just the one from before on the left and the identity on the right, and the resulting map $S^{2k_1} \wedge S^{2n} \wedge X_+ \rightarrow S^{2k_1} \wedge \text{Thom}(v_i)$ is just the $2k_1$-fold suspension of the one from before, hence they produce the same class in $[\Sigma^\infty_+X, MU]_{-2q}$. We can make the same remarks about $j$ and $l_2$, and it thus suffices to show that the factorizations of $f$ through $l_1$ and $l_2$, respectively, determine the same class in $[\Sigma^\infty_+X, MU]$. But since $v_i$ and $v_j$ are isomorphic as vector bundles, the corresponding maps $Z \rightarrow BU(N)$ are homotopic, hence so are the maps $\text{Thom}(v_i) \rightarrow MU(N)$. It would thus suffice to know that the two tubular neighborhood maps $v_i \oplus k \cong v_j \oplus k \rightarrow X \times \mathbb{C}^N$ are isotopic, but we can arrange this by taking $N$ as large as necessary.

Now let $(f_0, v_0), (f_1, v_1) \in U^2(X)$ be cobordant, then we have a complex oriented map $h : W \rightarrow X \times [0, 1]$ with the maps $\epsilon_i : X \rightarrow X \times [0, 1]$ both transverse to $h$, so that the respective pullbacks yield $(f_0, v_0)$ and $(f_1, v_1)$. Fix an embedding $W \rightarrow X \times [0, 1] \times \mathbb{C}^N$, and let $v$ be its normal bundle. Then $v_i$ is the pullback of $v$ along $Z_i \rightarrow W$ coming from pulling back $h$ along $\epsilon_i$, hence we can pull back the tubular neighborhood diagram for $v$ along $\epsilon_i$ since the above isotopy argument shows that there is no dependence
on the choice of tubular neighborhood

\[
\begin{array}{cc}
W & X \times [0,1] \times \mathbb{C}^N \\
\downarrow \quad \downarrow \\
Z_i & X \times \mathbb{C}^N
\end{array}
\]

Applying compactification to the back right square gives us a commutative diagram

\[
\begin{array}{ccc}
S^{2n} \land I_+ \land X_+ & \longrightarrow & \text{Thom}(v) \\
\downarrow \quad \downarrow \\
S^{2n} \land * \sqcup \{i\} \land X_+ & \longrightarrow & \text{Thom}(v_i) \\
\end{array}
\]

The left square commutes because the back right square in the above diagram commutes, and the right square commutes because \(v_i\) is the pullback of \(v\). The commutativity of this diagram tells us that we have a map \(S^{2n} \land I_+ \land X_+ \to \text{MU}(N+q)\) that agrees on the endpoint \(i\) with the map we obtain from the tubular neighborhood for \(v_i\). This data amounts to a pointed homotopy between the two maps obtained from \(\Phi_q\).

(iii) Natural: Let \((f,v) \in U^q(X)\) and \(g : Y \to X\) with \(g\) transverse to \(f\), we may form the pullback \(g^*(f,v) \in U^q(Y)\). Represent \(v\) by the normal bundle of a factorization of \(f\) as \(Z \hookrightarrow X \times \mathbb{C}^n \to X\), then pulling everything back along \(g\) gives us the pullback map of \(f\) along \(g\) and \(v\) as the pullback of \(v\), representing the class of \(g^*(f,v)\). We would like to say that the following diagram commutes

\[
\begin{array}{ccc}
Y_+ \land S^{2n} & \longrightarrow & \text{Thom}(v') \\
\downarrow \quad \downarrow \\
X_+ \land S^{2n} & \longrightarrow & \text{Thom}(v) \\
\end{array}
\]

The right square commutes since \(v'\) is the pullback of \(v\), it thus suffices to know the left square commutes, and for that it suffices to know the square

\[
\begin{array}{ccc}
v' & \longrightarrow & v \\
\downarrow \quad \downarrow \\
Y \times \mathbb{C}^n & \xrightarrow{g \times 1} & X \times \mathbb{C}^n
\end{array}
\]

after replacing the vertical arrows with ones that are isotopic. But we showed that the choice of tubular neighborhood only depends on the isotopy class after taking \(n\) sufficiently large, so as above we just pull back any tubular neighborhood for \(v\), and the above square commutes.

(iv) Additive: Recall that the sum of \((f,v),(f',v') \in U^q(X)\) is defined as \((f \sqcup f',v \sqcup v')\). We factor \(f\) and \(f'\) through embeddings \(i : Z \hookrightarrow X \times \mathbb{C}^n\) and \(i' : Z' \hookrightarrow X \times \mathbb{C}^{n'}\), and take their sum \(Z \sqcup Z' \hookrightarrow X \times \mathbb{C}^{n+n'}\), the normal bundle of which is \((v_i \oplus n_i') \sqcup (v_i' \oplus n_i')\). The one-point compactification of this space is \((S^{2n'} \land v_i^+) \lor (S^{2n} \land v_i^+)\), hence the map given by \(\Phi_q\) is

\[
S^{2n+2n'} \land X_+ \to (S^{2n'} \land v_i^+) \lor (S^{2n} \land v_i^+) \to \text{MU}(n+n' + q)
\]

but this is the definition of \(\Phi_q(f,v) + \Phi_q(f',v')\) using the H-cogroup structure on spheres to define addition in \([\Sigma^\infty N, \text{MU}]_{-2q}\).

(v) Multiplicative and \(U^q(*)\)-linear: Let \((f : Z \to X,v) \in U^q(X)\) and \((g : Y \to X,\tau) \in U^q(X)\), their product is given by \(\Delta^q(f \times g, v \times \tau)\). Since \(\Phi_q\) is natural, it suffices to show that \(\mu_{\text{MU}} \circ (\Phi_q(f,v) \land \Phi_q(g,\tau)) = \Phi_q(f \times g,v \times \tau)\), where \(\mu_{\text{MU}} : \text{MU} \times \text{MU} \to \text{MU}\) is the ring map. This is almost tautological
as the class $v \times \pi$ is given by $Z \times Y \xrightarrow{v \times \pi} BU \times BU \to BU$, where the last map is induced on the colimit $BU = \colim_{n \to \infty} BU(n)$ by the direct sum maps $BU(m) \times BU(n) \to BU(n + m)$, and the ring map $MU \times MU \to MU$ is induced by the maps $MU(n) \wedge MU(n) \to MU(n + m)$ obtained from Thomifying the direct sum maps. The multiplicative unit in $[\Sigma^\infty_+ X, MU]$ is the Thomification of the map classifying the trivial bundle over $X$, so it is clear that $\Phi_q$ is unital.

$U^*(\ast)$-linearity is similarly tautological: the product of $(f, v) \in U^{2p}(\ast) \times U^{2q}(X)$ is $(f \times g, v \times \tau)$, which $\Phi_q$ sends to $S^{2(n+m)} \wedge (\ast \times X)_+ \cong S^{2(n+m)} \wedge S^0 \wedge X_+ \to \Thom(v) \wedge \Thom(\tau) \to MU(n + p) \wedge MU(m + q) \to MU(m + n + p + q)$ where we have factored $f$ through an embedding into $\ast \times C^n$ and $g$ through an embedding into $X \times C^m$. But the product of the classes of the maps $S^{2n} \wedge (\ast)_+ = S^{2n} \wedge S^0 \to \Thom(v) \to MU(n + p) \in [S, MU]_{-2p}$ and $S^{2m} \wedge X_+ \to \Thom(\tau) \to MU(m + q) \in [\Sigma^\infty_+ X, MU]_{-2q}$ is given by smashing the maps together and identifying $S \wedge \Sigma^n X \cong \Sigma^n X$, then using the ring structure on $MU$.

(iv) The Inverse: Let $f \in MU^{2q}(X) = \colim_{n \to \infty} [S^{2n} \wedge X_+, MU(n + q)]$ and represent $f$ by a map $f : S^{2n} \wedge X_+ \to MU(n + q)$. Since the space $S^{2n} \wedge X_+$ is compact, it lands in some $(\mathbb{I}^N_{n+q})^+$ where $\mathbb{I}^N_{n+q} \to \text{Gr}_{n+q}(C^N)$ is the tautological bundle. $(\mathbb{I}^N_{n+q})^+$ is a smooth manifold away from the point at infinity and the zero section $\text{Gr}_{n+q}(C^N) \to \mathbb{I}^N_{n+q}$ is proper. In particular, we can replace $f$ with a map homotopic to $f$ that is transverse to the zero section $\text{Gr}_{n+q}(C^N) \hookrightarrow \mathbb{I}^N_{n+q} \hookrightarrow (\mathbb{I}^N_{n+q})^+$, which stays away from the point at infinity [4]. The pullback square

\[
\begin{array}{ccc}
Z & \rightarrow & \text{Gr}_{n+q}(C^N) \\
\downarrow & & \downarrow \\
S^{2n} \wedge X_+ & \xrightarrow{f} & (\mathbb{I}^N_{n+q})^+
\end{array}
\]

thus defines a smooth proper map $f : Z \to X$. Since the zero section on the right stays away from the point at infinity, so does the map from $Z$, hence we may regard it as a map $Z \to X \times C^n$, and composing with projection to $X$, we obtain a complex oriented map. We check that it is well-defined with respect to suspension and homotopy.

If we suspend $f$, we have the following diagram

\[
\begin{array}{cccccc}
& & & & \text{Gr}_{n+q}(C^N) & \rightarrow & \text{Gr}_{n+q+1}(C^N) \\
& & & & \downarrow & & \downarrow \\
S^2 \wedge (\mathbb{I}^N_{n+q})^+ & \rightarrow & (\mathbb{I}^N_{n+q+1})^+ \\
& & & \searrow & & \searrow \\
S^2 \wedge S^{2n} \wedge X_+ & \rightarrow & S^2 \wedge MU(n + q) & \rightarrow & MU(n + q + 1) \\
\end{array}
\]

using the definition of the structure maps in the spectrum $MU$, identifying $\text{Thom}(\mathbb{I}^N_{n+q+1}) \cong S^2 \wedge (\mathbb{I}^N_{n+q})^+$. All of the squares in the commutative cube are pullbacks, hence pulling back from $\text{Gr}_{n+q+1}(C^N) \hookrightarrow (\mathbb{I}^N_{n+q+1})^+$ will be the same as pulling back from $\text{Gr}_{n+q}(C^N) \hookrightarrow S^2 \wedge (\mathbb{I}^N_{n+q+1})^+$. Since the map $S^2 \wedge S^{2n} \wedge X_+ \rightarrow S^2 \wedge (\mathbb{I}^N_{n+q})^+$ is the identity on the left factor, we can choose the same map as before and its suspension will give us a transverse map. But then the pullback is easily seen to be $Z \to S^{2n} \wedge X_+ \hookrightarrow S^2 \wedge S^{2n} \wedge X_+$, hence we have the same map, with the complex orientation given by the direct sum of the normal bundle with a trivial bundle, which gives the same complex orientation. By similar reasoning, we see that our map does not depend on the choice of $N$.

If $f_i : S^{2n} \wedge X_+ \to MU(n + q)$ for $i = 0, 1$ are homotopic, we have a map $H : S^{2n} \wedge X_+ \wedge I_+ \to MU(n + q)$ which lands in some $\mathbb{I}^N_{n+q}$ such that on the $i$-th endpoint it is equal to $f_i$. We can replace each $f_i$ by a map that is transverse to the zero section $\text{Gr}_{n+k}(C^N) \hookrightarrow \mathbb{I}^N_{n+q}$, and then choose a map homotopic to $H$. The inverse sends $f$ to $\Theta(f) := \sum_i H_i$, where $H_i$ is the pullback of $H$ over $I_i$.
that is transverse to the zero section and matches the transverse replacements of \( f_i \) on the endpoints. This follows from the general fact that if \( M \) is a compact manifold and \( f : M \to N \) is a map with \( \partial f \) transverse to some map \( Z \to N \), then one can replace \( f \) up to homotopy by a map that is transverse to \( Z \to N \) that agrees with \( f \) on the boundary [4]. We thus form the pullback square

\[
\begin{array}{ccc}
W & \longrightarrow & \text{Gr}_n(C^N) \\
\downarrow & & \downarrow \\
S^{2n} \times X_+ \cap I_+ & \longrightarrow & \text{L}_{n+q}^N \\
H & \longrightarrow & \\
\end{array}
\]

and obtain a map \( W \leftrightarrow X \times C^n \times I \to X \times I \), and the fact that \( H \) agrees with the \( f_i \) on the endpoints means that when we pull back \( W \to X \times I \) along \( \epsilon_i \) we get the complex oriented map obtained from \( f_i \). This shows that the map is well-defined with respect to homotopy and does not depend on the choice of transverse replacement of \( f \), since all such maps are homotopic.

The map we have described is an inverse to \( \Phi_q \) because if we take \( f : S^{2n} \times X_+ \to MU(n + q) \) and restrict to \( S^{2n} \times X_+ \to \text{L}_{n+q}^N \) for some \( N \), pulling back \( \text{Gr}_n(C^N) \to \text{L}_{n+q}^N \) gives a map \( Z \leftrightarrow X \times C^N \to X \), and we take the tubular neighborhood diagram for the normal bundle of \( Z \leftrightarrow X \times C^N \) to be the one pulled back from the one for the zero section, which we can take to be

\[
\begin{array}{ccc}
\text{L}_{n+q}^N & \longrightarrow & \\
\downarrow & \searrow & \\
\text{Gr}_n(C^N) & \longrightarrow & \text{L}_{n+q}^N \\
& & \\
\end{array}
\]

since taking the normal bundle of a zero section always recovers the vector bundle. We thus have a diagram

\[
\begin{array}{ccc}
& & \\
& \searrow & \\
& X \times C^n & \longrightarrow & \text{L}_{n+q}^N \\
\downarrow & \searrow & \\
& & \\
\end{array}
\]

where the vertical arrow is the pulled-back tubular neighborhood, the diagonal arrow is the upstairs map in the pullback square classifying \( \nu \), and the horizontal arrow is the restriction of \( f \) to the complements of the points at infinity. But then since \( \Phi_q \) sends this data to the compactification of the vertical arrow, followed by the Thomification of the diagonal arrow, the commutativity of the diagram tells us we recover \( f \).

Going the other way, \( \Phi_q(f : Z \to X, \nu) \) is the map \( S^{2n} \times X_+ \to \nu^+ \to \text{L}_{n+q}^N \), but in the diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & Z \longrightarrow \text{Gr}_n(C^N) \\
\downarrow & \downarrow & \downarrow \\
S^{2n} \times X_+ & \longrightarrow & \nu^+ \longrightarrow \text{L}_{n+q}^N \\
\end{array}
\]

the bottom maps are already transverse to the corresponding vertical maps, and the right square is a pullback because the zero section pulls back to the zero section, and the left is a pullback because it is obtained from the tubular neighborhood commutative diagram. Hence the complex oriented map obtained from \( \Phi_q(f, \nu) \) is just the left vertical arrow, which is \( (f, \nu) \).

(v) **Odd-dimensional Maps**: All of our remarks about complex oriented maps of even dimension carry over to maps of odd dimension, since by definition, a complex orientation on a map \( Z \to X \) of odd dimension is a complex map on the map of even dimension \( Z \times X \to \mathbb{R} \).

(vi) **Non-compact Manifolds**: The above proof can be modified to a noncompact manifold \( X \) by noting that we can write noncompact \( X = \text{colim}_a X_a \) as a colimit of compact subspaces \( X_a \). Then since \( f \) is assumed proper, we set \( Z_a := f^{-1}(X_a) \subset Z \), which is compact and hence the Thom space of \( \nu_a := \nu|Z_a \) is \( \nu_a^+ \). By restricting the tubular neighborhood corresponding to \( \nu \) to \( \nu_a \), we get a map \( S^{2n} \times X_{a+} \to \text{Thom}(\nu_a) \) for each \( a \). Then since \( \text{Thom}(\nu) = \text{colim}_a \text{Thom}(\nu_a) \), we get a map between the colimits. The only other place
where compactness is used is to claim that any map \( \phi : S^{2n} \wedge X_+ \to MU(n + q) \) lands in some \( (L^{N}_{n+q})^+ \). This claim is modified similarly - write \( X \) as the colimit of the \( X_\alpha \)'s, this time choosing the \( X_\alpha \)'s to be an ascending union of compact submanifolds, then we choose a map homotopic to \( \phi \) whose restriction \( \phi|_{S^{2n} \wedge X_\alpha} \) is transverse to the zero section of the \( L^{N}_{n+q} \) that it lands in, for each \( \alpha \). Then taking the union of the preimages gives a manifold with a closed embedding into \( X \times C^n \).

(vii) The Relative Case: When \( A \hookrightarrow X \) is an open inclusion, the functoriality of one-point compactification comes from the fact that one has a homeomorphism \( X^+ / (X^+ - A) \to A^+ \). Thus, if \( Z \to X \) is a complex oriented map whose image is contained in such an \( A \), the tubular neighborhood embedding factors as \( \nu \hookrightarrow A \times C^n \hookrightarrow X \times C^n \), and when we apply one-point compactification, we get a map \( X^+ / (X^+ - A) \to \nu^+ \) and following that map with the usual one into \( MU \), we obtain an element of \( MU^*(X, X - A) \). Running through the above argument with slight modifications shows that we may identify \( MU^*(X, X - A) \) with the set of cobordism classes of complex oriented maps into \( X \) whose images are contained in \( A \), which we will denote \( U^*(X, X - A) \).

If \( A \) is a strong deformation retract of an open neighborhood \( U \) in \( X \), then we may replace any complex oriented map into \( X \) whose image is contained in \( A \), with the one whose image is contained in \( U \), and the homotopy invariance of \( U^* \) tells us these will be isomorphic. We conclude that we may identify \( MU^*(X, X - A) \) and \( U^*(X, X - A) \) when \( A \) has this property. \( \square \)

References

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Currier House, Harvard College, Cambridge, MA 02138

E-mail address: christiancarrick@college.harvard.edu