Spring 2006, #2 Consider an initial value problem for the *Korteweg-deVries equation*

$$u_t + u_{xxx} + 6uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x,0) = \phi(x).$$

Show that the following are conserved quantities for the KdV equation (you may assume that the function $u(x,t)$ vanishes as $|x| \to \infty$, together with all of its derivatives):

- Mass:

$$\int_{-\infty}^{\infty} u(x,t)\ dx$$

- Momentum:

$$\int_{-\infty}^{\infty} u^2(x,t)\ dx$$

- Energy:

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}u_x(x,t)^2 - u(x,t)^3\right)\ dx$$

To show that each quantity is conserved, we simply demonstrate that it does not change in time (that is, when we differentiate with respect to t, we get that the quantity is zero). We start with mass:

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} u(x,t)\ dx &= \int_{\mathbb{R}} \partial_t u(x,t) \\
&= -\int_{\mathbb{R}} u_{xxx} + 6uu_x\ dx \\
&= -\int_{\mathbb{R}} u_{xxx} + 3(u^2)_x\ dx \\
&= u_{xx}|_{-\infty}^{\infty} + 3u^2|_{-\infty}^{\infty} = 0
\end{aligned}$$

Momentum:

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} u^2(x,t)\ dx &= \int_{\mathbb{R}} 2uu_t\ dx \\
&= -\int_{\mathbb{R}} 2uu_{xxx} + 6u^2u_x\ dx \\
&= \int_{\mathbb{R}} 2u_xu_{xx} - 2(u^3)_x\ dx - \cancelto{0}{2uu_{xx}|_{-\infty}^{\infty}} \\
&= \int_{\mathbb{R}} (u_x^2)_x - 2(u^3)_x\ dx \\
&= \left[u_x^2 - 2u^3\right]_{-\infty}^{\infty} = 0.
\end{aligned}$$

And finally, energy:

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}} \left(\frac{1}{2} u_x(x,t)^2 - u(x,t)^3 \right) \, dx &= \int_{\mathbb{R}} u_x u_{xt} - 3u^2 u_t \\
&= \int_{\mathbb{R}} u_x (u_t)_x - 3u^2(-u_{xxx} - 6uu_x) \, dx \\
&= \int_{\mathbb{R}} u_x(-u_{xxxx} - 6uu_{xx} - 6u_x^2) + 3u^2 u_{xxx} + 18u^3 u_x \, dx \\
&= \int_{\mathbb{R}} u_{xx} u_{xxx} - 6uu_x u_{xx} - 6u_x^3 + 3u^2 u_{xxx} + \frac{9}{2}(u^4)_x \, dx \\
&= \int_{\mathbb{R}} \left(\frac{1}{2} u_{xx}^2 \right)_x + 6(uu_x)_x u_x - 6u_x^3 - 3(u^2)_x u_{xx} + \frac{9}{2}(u^4)_x \, dx \\
&= \int_{\mathbb{R}} \left(\frac{1}{2} u_{xx}^2 \right)_x + \cancel{6u_x^3} + \cancel{6uu_x u_{xx}} - \cancel{6u_x^3} - \cancel{6uu_x u_{xx}} + \frac{9}{2}(u^4)_x \, dx \\
&= \left[\frac{1}{2} u_{xx}^2 + \frac{9}{2} u^4 \right]_{-\infty}^{\infty} \\
&= 0
\end{aligned}
$$

and hence all the quantities are conserved.

Spring 2006, #3 Let $0 < L < \infty$ and let $0 < p(x) \in C^\infty([0,L])$. Consider the following initial-boundary value problem on $(0,L)\times(0,\infty)$:

$$\begin{cases} \partial_t u = \partial_x(p(x)\partial_x u), & (x,t)\in(0,L)\times(0,\infty), \\ u(x,0)=\phi(x), \quad \partial_x u(0,t)=\partial_x u(L,t)=0, \end{cases}$$

with $\phi \in C^\infty([0,L])$. Compute the limit of $u(x,t)$ as $t\to\infty$.

Let's start by looking at the operator $L = \partial_x(p(x)\partial_x)$. We wish to demonstrate that this operator has a set of eigenfunctions that form a basis for $L^2([0,L])$. Let's check to see if L is self-adjoint in this space. For notational simplicity, let's call $[0,L]=D$. Note all boundary terms will vanish when integrating by parts since $u_x(0,t)=u_x(L,t)=0$.

$$\begin{aligned} (Lu,v) &= \int_D \partial_x(p\partial_x u)v \\ &= -\int_D p u_x v_x \\ &= \int_D \partial_x(p v_x)u \\ &= (u, Lv) \end{aligned}$$

Since L is self-adjoint, we have that the eigenfunctions of L form a basis for $L^2(D)$. So, let's write a solution to the PDE in the eigenfunctions, $\phi_n(x)$.

$$u(x,t)=\sum_{n=1}^\infty a_n(t)\phi_n(x)$$

Plugging into the PDE, we find that

$$\sum_{n=1}^\infty a_n'(t)\phi_n(x) - a_n(t)\partial_x(p(x)\partial_x\phi_n(x)) = \sum_{n=1}^\infty a_n'(t)\phi_n(x) - \lambda_n a_n(t)\phi_n(x) = 0$$

and hence

$$a_n'(t)+\lambda_n a_n(t)=0.$$

So, $a_n(t)$ satisfies the above ODE, which has solution

$$a_n(t)=ce^{\lambda_n t}$$

with c coming from the initial condition $u(x,0)=\phi(x)=\sum_{n=1}^\infty b_n\phi_n(x)$ with $b_n=\int_D \phi(x)\phi_n(x)$. So, we have that $a_n(0)=b_n=c$. So,

$$u(x,t)=\sum_{n=1}^\infty b_n e^{\lambda_n t}\phi_n(x).$$

In order to determine the behaviour of the solution, we need some more information on the eigenvalues. Consider,

$$\begin{aligned} (Lu-\lambda u, u) &= \int_D (pu_x)_x u - \lambda u^2 \\ &= \int_D -pu_x^2 - \lambda u^2 \\ &= 0 \end{aligned}$$

and so

$$\begin{aligned}
\lambda \int_D u^2 &= -\int_D p u_x^2 \\
\lambda &= \frac{-\int_D p u_x^2}{\int_D u^2} \\
&\leq 0.
\end{aligned}$$

So, we have that all of the eigenvalues are non-positive. Any term in the series with $\lambda_n < 0$ will go to zero as $t \to \infty$. We need to see whether 0 is an eigenvalue of T (we rename the operator here since L is now referring to too many different objects). Choose $\phi_1(x) = \frac{1}{L^{1/2}}$ (so that it has norm one on $L^2(D)$), then $(p(\phi_1)_x)_x = 0$ and hence $\phi_1(x)$ is an eigenfunction corresponding to $\lambda_1 = 0$. Then, we have that

$$\begin{aligned}
\lim_{t\to\infty} u(x,t) &= b_1 e^{\lambda_1 t}\phi_1(x) \\
&= \frac{b_1}{L^{1/2}} \\
&= \frac{\int_D \phi(x)}{L}.
\end{aligned}$$

Spring 2006, #4 Consider the initial value problem of the form

$$\frac{dy}{dt} = f(y), \quad y(0) = 0.$$

Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(y) = 0$ precisely when $y = 0$ and such that f does not satisfy the Lipschitz condition in any neighborhood of 0, while the uniqueness for the initial value problem holds.

Consider the function $f(y) = -|x|^{1/2}$. Clearly $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and is continuous. Also, we have that $f(y)$ is not Lipschitz at zero. Further, we have that $y = 0$ is a stationary point for the ODE and this is a stable fixed point. Given an initial condition of $y(0) = 0$ (starting at the stationary point), we have that the solution will never leave the stationary point, and hence we have uniqueness.

Spring 2006, #6 Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and connected set. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\Delta u + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + c(x)u = 0 \quad \text{ in } \Omega$$

where $a_k(x), 1 \leq k \leq n$, and $c(x)$ are continuous in $\overline{\Omega}$, with $c(x) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies that $u = 0$ in Ω.

To show this is true, we need to prove the maximum principle for elliptic operators. We will demonstrate this case by showing first that $\max u \leq 0$ and then $\min u \geq 0$. Suppose the maximum occurs on the interior of the domain, then we have an $x_0 \in \Omega$ such that $u(x_0) \geq u(x) \; \forall \; x \in \overline{\Omega}$ and hence that $u(x_0) \geq 0$. Furthermore, we also have that $\frac{\partial u}{\partial x_k} = 0 \; \forall \; k = 1, \ldots, n$ and $\Delta u \leq 0$ at x_0. In order for u to satisfy the PDE, we then must have that $c(x_0)u(x_0) \geq 0$ and since $c(x) < 0$ in Ω, we have that $\max u(x) \leq 0$.

Similarly, let's suppose the minimum occurs inside the domain, $x_0 \in \Omega$. Again we have that $\frac{\partial u}{\partial x_k} = 0 \; \forall \; k = 1, \ldots, n$, but now $\Delta u \geq 0$ at x_0. Now to satisfy the PDE we need that $c(x_0)u(x_0) \leq 0$ which gives us that $\min u(x) \geq 0$. Combining these two results, we then have that $u = 0$ in Ω.

Spring 2006, #7 Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $f \in C(\overline{\Omega})$. Find the minimum of the functional

$$E(u) = \int_\Omega \left(\frac{1}{2} \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k} \right)^2 - f(x)u(x) \right) \, dx$$

on the space of smooth functions in $\overline{\Omega}$, subject to the constraints

$$u|_{\partial\Omega} = 0, \qquad \int_\Omega u(x)dx = A,$$

where A is a given constant. You may assume that a smooth solution of this problem exists. You may also regard the solution of

$$\Delta w = h \text{ in } \Omega, \quad w|_{\partial\Omega} = 0$$

as known, for any $h \in C(\overline{\Omega})$.

We wish to minimize $E(u)$ subject to the constraint $G(u) = \int_\Omega u \; dx - A = 0$. The minimum u_0 must then satisfy

$$E'(u_0)v = \lambda G'(u_0)v$$

for some $\lambda \in \mathbb{R}$ and all v. So, let's start by computing the derivatives $E'(u)v$ and $G'(u)v$.

$$\begin{aligned}
E'(u)v &= \lim_{\epsilon \to 0} \frac{E(u+\epsilon v) - E(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \left(\frac{1}{2} \sum_{k=1}^n (\partial_{x_k}(u+\epsilon v))^2 - f(u+\epsilon v) \right) - \left(\frac{1}{2} \sum_{k=1}^n (\partial_{x_k} u)^2 - fu \right) \, dx \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \frac{1}{2} \sum_{k=1}^n \cancel{(\partial_{x_k} u)^2} + 2\epsilon \partial_{x_k} u \partial_{x_k} v + \epsilon^2 (\partial_{x_k} v)^2 - \cancel{(\partial_{x_k} u)^2} - \cancel{fu} - \epsilon f v + \cancel{fu} \, dx \\
&= \int_\Omega \sum_{k=1}^n \partial_{x_k} u \partial_{x_k} v - fv \\
&= \int_\Omega \nabla u \cdot \nabla v - fv \\
&= \int_\Omega -\Delta u v - fv + \cancelto{0}{\int_{\partial\Omega} v(n \cdot \nabla u)} \\
&= \int_\Omega (-\Delta u - f)v.
\end{aligned}$$

$$\begin{aligned}
G'(u)v &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega u + \epsilon v - A - u + A \\
&= \int_\Omega v
\end{aligned}$$

So, we then have that

$$E'(u)v - \lambda G'(u)v = \int_\Omega (-\Delta u - f - \lambda)v = 0$$

Taking $v = u$, we have that

$$\int_\Omega |\nabla u|^2 - fu - \lambda u = 0$$

and hence that

$$\begin{aligned}\lambda \int_\Omega u &= \int_\Omega -|\nabla u|^2 + fu \\ &= \frac{\int_\Omega -|\nabla u|^2 + fu}{A}.\end{aligned}$$

Taking $\lambda = \inf_u \frac{\int_\Omega -|\nabla u|^2 + fu}{A}$

Fall 2005, #2 Consider the two point boundary value operator L defined for $u = u(x)$ by

$$Lu = u'' + u' - a(1+x^2)u$$

defined on the interval $x \in [0,1]$ with boundary conditions

$$u(0) = u(1) = 0$$

and $a > 0$. Let λ_{a0} be the eigenvalue of smallest absolute value for L and let u_{a0} be the corresponding eigenfunction. Do the following:

(a) Find an inner product in terms of which L is self-adjoint.
(b) Show that $\lambda_{a0} < 0$.
(c) Show that $|\lambda_{a0}|$ is an increasing function of a; i.e., if $0 < a_1 < a_2$, then $|\lambda_{a_1 0}| < |\lambda_{a_2 0}|$.

(a) We will assume that L is self-adjoint in some weighted L^2 inner product space, and search for the weighting function, ϕ. So,

$$\begin{aligned}
(Lu, v)_\phi &= \int_0^1 (u'' + u' - a(1+x^2)u)v\phi \\
&= \int_0^1 u''(v\phi) + u'(v\phi) - a(1+x^2)u(v\phi) \\
&= -\int_0^1 u'(v\phi)' + \cancelto{0}{u'v\phi|_0^1} - \int_0^1 u(v\phi)' + \cancelto{0}{uv\phi|_0^1} - \int_0^1 a(1+x^2)uv\phi \\
&= \int_0^1 u(v\phi)'' - \cancelto{0}{u(v\phi)'|_0^1} - \int_0^1 u(v\phi)' - \int_0^1 a(1+x^2)uv\phi \\
&= \int_0^1 u(v'\phi + v\phi')' - u(v'\phi + v\phi') - a(1+x^2)uv\phi \\
&= \int_0^1 u(v''\phi + 2v'\phi' + v\phi'') - u(v'\phi + v\phi') - a(1+x^2)uv\phi \\
&= \int_0^1 \left(v'' + \left[2\frac{\phi'}{\phi} - 1\right] v' + \left[\frac{\phi''}{\phi} - \frac{\phi'}{\phi} - a(1+x^2)\right] v \right) u\phi
\end{aligned}$$

In order for L to be self-adjoint in this inner product space, we need that $2\frac{\phi'}{\phi} - 1 = 1$ and $\frac{\phi''}{\phi} - \frac{\phi'}{\phi} = 0$. These leads to the constraints: $\phi' = \phi$ and $\phi'' = \phi$ which we can satisfy by choosing $\phi = e^x$.

(b) To simplify notation, let $\lambda = \lambda_{a0}$ and $u = u_{a0}$. Consider

$$\begin{aligned}
\lambda \|u\|_\phi^2 &= (\lambda u, u)_\phi \\
&= (Lu, u)_\phi \\
&= \int_0^1 (u'' + u' - a(1+x^2)u)ue^x \\
&= -\int_0^1 u'(ue^x)' + \cancelto{0}{u'ue^x|_0^1} + \int_0^1 u'ue^x - a(1+x^2)u^2e^x \\
&= \int_0^1 -u'(u'e^x + ue^x) + u'ue^x - a(1+x^2)u^2e^x \\
&= -\int_0^1 (|u'|^2 + a(1+x^2)u^2)e^x \\
&< 0
\end{aligned}$$

and so we can conclude that $\lambda_{a0} < 0$.

(c) Recall that the eigenvalue λ_{a0} is given by

$$\lambda_{a0} = \inf \frac{(Lu, u)_\phi}{(u, u)_\phi}.$$

In part (b), we have shown that $(Lu, u)_\phi$ is decreasing with respect to a. So clearly for $a_2 > a_1 > 0$, we have that $\lambda_{a_2 0} < \lambda_{a_1 0} < 0$ and hence $|\lambda_{a_1 0}| < |\lambda_{a_2 0}|$.

Fall 2005, #7 Find the (entropy) solution for all time $t > 0$ of the inviscid Burgers equation $u_t + \frac{1}{2}\left(u^2\right)_x = 0$ with initial condition

$$u(x,0) = \begin{cases} 0, & x < -1 \\ x+1, & -1 < x < 0 \\ 1 - \frac{1}{2}x, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

Winter 2005, #2 Let $\phi \in C^1(\mathbb{R}^2)$. Solve the following Cauchy problem in $\mathbb{R}^3$,

$$\begin{cases} x_1 \partial_{x_1} u + 2x_2 \partial_{x_2} u + \partial_{x_3} u = 3u, \\ u(x_1, x_2, 0) = \phi(x_1, x_2). \end{cases}$$

The characteristic equations for this PDE are

$$\frac{dx_1}{d\tau} = x_1, \qquad \frac{dx_2}{d\tau} = x_2,$$
$$\frac{dx_3}{d\tau} = 1, \qquad \frac{du}{d\tau} = 3u,$$

with $\Gamma = (s_1, s_2, 0, \phi(s_1, s_2))$. These ODEs have solutions

$$x_1 = c_1 e^{\tau}, \qquad x_2 = c_2 e^{\tau}$$
$$x_3 = \tau + c_3, \qquad u = c_4 e^{3\tau}$$

where we can find the constants c_i from Γ: $c_1 = s_1$, $c_2 = s_2$, $c_3 = 0$, and $c_4 = \phi(s_1, s_2)$. From here, we see that $\tau = x_3$ and $s_i = x_i e^{-x_3}$ for $i = 1, 2$. Plugging these relationships into our equation for u, we find that the solution to the Cauchy problem is given by

$$u(x_1, x_2, x_3) = \phi(x_1 e^{-x_3}, x_2 e^{-x_3}) e^{3x_3}.$$

Winter 2005, #6 Find the Fourier transform of the integrable function $x \mapsto \sin^2 x/x^2$.

We start by looking at the Fourier transform of $f = \frac{1}{x}\sin x$ since $\widehat{f^2} = \hat{f} * \hat{f}$. We begin with the observation that $\sin x = \frac{1}{x} x \sin x$. So we start by computing the Fourier transform of $\sin x$.

$$\begin{aligned}
\mathcal{F}(\sin x) &= \int_{\mathbb{R}} \sin x e^{-2\pi i x\xi}\ dx \\
&= \int_{\mathbb{R}} \frac{1}{2i}(e^{ix} - e^{-ix})e^{-2\pi i x\xi}\ dx \\
&= \frac{1}{2i}\int_{\mathbb{R}} e^{ix-2\pi i x\xi} - e^{-ix-2\pi i x\xi}\ dx \\
&= \frac{1}{2i}\int_{\mathbb{R}} e^{-2\pi i x\left(\xi - \frac{1}{2\pi}\right)} - e^{-2\pi i x\left(\xi + \frac{1}{2\pi}\right)}\ dx \\
&= \frac{1}{2i}\left[\delta\left(\xi - \frac{1}{2\pi}\right) - \delta\left(\xi + \frac{1}{2\pi}\right)\right]
\end{aligned}$$

by the fact that $\mathcal{F}(1) = \delta(\xi)$. Also recall that $\mathcal{F}(xg(x)) = -\frac{1}{2\pi i}\frac{d}{d\xi}\hat{g}$. So, we have that

$$\begin{aligned}
\mathcal{F}(\sin x) &= \mathcal{F}\left(x\frac{1}{x}\sin x\right) \\
&= -\frac{1}{2\pi i}\frac{d}{d\xi}\mathcal{F}\left(\frac{1}{x}\sin x\right)
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{F}\left(\frac{1}{x}\sin x\right) &= -2\pi i \int_{\mathbb{R}} \mathcal{F}(\sin x)\ d\xi \\
&= -2\pi i \int_{\mathbb{R}} \frac{1}{2i}\left[\delta\left(\xi - \frac{1}{2\pi}\right) - \delta\left(\xi + \frac{1}{2\pi}\right)\right]\ d\xi \\
&= \pi \int_{\mathbb{R}} \delta\left(\xi + \frac{1}{2\pi}\right) - \delta\left(\xi - \frac{1}{2\pi}\right)\ d\xi \\
&= \begin{cases} \pi & -\frac{1}{2\pi} \le \xi \le \frac{1}{2\pi} \\ 0 & \text{otherwise} = \end{cases} \\
&= \hat{f}.
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
\mathcal{F}\left(\frac{\sin^2}{x^2}\right) &= \hat{f} * \hat{f} \\
&= \begin{cases} 2\pi^2\xi + 2\pi & -\frac{1}{\pi} \le \xi < 0 \\ -2\pi^2\xi + 2\pi & 0 \le \xi \le \frac{1}{\pi} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Winter 2005, #7 Consider an autonomous system in $\mathbb{R}^n$, $x'(t) = f(x(t))$ where $f = (f_1, f_2, \ldots, f_n)$ is a smooth vector field, such that

$$\sum_{k=1}^{n} x_k f_k(x) < 0 \quad \text{for } x \neq 0.$$

Show that $x(t) \to 0$ as $t \to \infty$ for each solution of the system, independently of the initial condition $x(0)$.

Let $V(x) = \frac{1}{2}|x|^2$. Then $V^*(x) = \nabla V \cdot f = x \cdot \nabla x < 0$. So, we have that $V(x)$ is positive definite and $V^*(x)$ is negative definite. Also we have that the set $\{x : V^*(x) = 0\} = \{0\}$ and hence 0 is the only invariant subset. Then, by Lyapunov's theorem, the zero solution of system is globally asymptotically stable and so solutions tend to zero for any initial condition as $t \to \infty$.

Fall 2004, #3 Let us consider a damped wave equation,

$$\begin{cases} (\partial_t^2 - \Delta + a(x)\partial_t)u = 0 & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \end{cases}$$

Here the damping coefficient $a \in C_0^\infty(\mathbb{R}^3)$ is a non-negative function and $u_0, u_1 \in C_0^\infty(\mathbb{R}^3)$. Show that the energy of the solution $u(x,t)$ at time t,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla_x u|^2 + |\partial_t u|^2) \, dx$$

is a decreasing function of $t \geq 0$.

We look at the time derivative of the energy. Note that since u_0 and u_1 are in $C_0^\infty(\mathbb{R}^3)$ we have that $u \in C_0^\infty(\mathbb{R}^3)$ at time t since the wave equation has a finite propogation speed. This then gives us that the boundary terms will vanish when we integrate by parts.

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla u_t + u_t u_{tt} \, dx \\ &= \int_{\mathbb{R}^3} -\Delta u u_t + u_t u_{tt} \, dx \\ &= \int_{\mathbb{R}^3} (u_{tt} - \Delta u) u_t \, dx \\ &= \int_{\mathbb{R}^3} (-a(x) u_t) u_t \, dx \\ &= -\int_{\mathbb{R}^3} a(x) u_t^2 \, dx \leq 0. \end{aligned}$$

Fall 2004, #4 Prove that each solution (except $x_1 = x_2 = 0$) of the autonomous system

$$\begin{cases} x_1' = x_2 + x_1(x_1^2 + x_2^2) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

blows up in finite time. What is the blow-up time for the solution which starts at the point (1,0) when $t = 0$?

Let $r = x_1^2 + x_2^2$. Then

$$\begin{aligned} r' &= (x_1^2 + x_2^2)' = 2(x_1 x_1' + x_2 x_2') \\ &= 2(x_2 x_1 + x_1^2(x_1^2 + x_2^2) - x_1 x_2 + x_2^2(x_1^2 + x_2^2)) \\ &= 2(x_1 + x_2)^2 \\ &= 2r^2. \end{aligned}$$

Now we can solve this ODE $r' = 2r^2$ by separation of variables.

$$\begin{aligned} \int \frac{dr}{r^2} &= \int 2dt \\ -\frac{1}{r} &= 2t + c \\ r &= -\frac{1}{2t + c} \end{aligned}$$

So we have that solutions blow up in finite time, specifically at $t = -\frac{c}{2}$. The solution which has $(x_1, x_2) = (1, 0)$ at $t = 0$ has $r = 1$ at $t = 0$. Plugging this into our solution, we find that $c = -1$ and hence the solution for this initial condition blows up at $t = \frac{1}{2}$.

Fall 2004, #7 Consider the partial differential equation

$$uu_x + u_t + u = 0, \quad (z, t) \in \mathbb{R}^2.$$

- Find the particular solution that satisfies the condition $u(0, t) = e^{-2t}$.
- Show that at the point $(z, t) = (1/9, \ln 2), u = 1/3$.

For this problem, the characteristic equations are given by

$$\frac{dz}{d\tau} = u, \quad \frac{dt}{d\tau} = 1, \quad \frac{du}{d\tau} = -u$$

with boundary condition given by the curve $\Gamma = (0, s, e^{-2s})$. We start by solving the two equations that are not coupled (those for t and u), finding that $t = \tau + c_1$ and $\ln u = -\tau + c_2$. Using our data, Γ, we find that $c_1 = s$ and $c_2 = -2s$, giving us

$$t = \tau + s \qquad u = e^{-\tau} e^{-2s}.$$

Plugging this solution for u into the ODE for z, we find $\frac{dz}{dt} = e^{-\tau} e^{-2s}$ has solution $x = -e^{-\tau} e^{-2s} + c_3$ with $c_3 = e^{-2s}$, so we have that

$$z = -e^{-\tau} e^{-2s} + e^{-2s}$$

from which we conclude that

$$e^{-s} = \frac{1}{2}\left(e^{-t} + \sqrt{e^{-2t} + 4z}\right)$$

and hence

$$\begin{aligned} u &= e^{-(\tau+s)} e^{-s} \\ &= e^{-t} e^{-s} \\ &= \frac{1}{2} e^{-t} \left(e^{-t} + \sqrt{e^{-2t} + 4z}\right). \end{aligned}$$

Now we evaluate the solution at $(z, t) = (1/9, \ln 2)$.

$$\begin{aligned} u(1/9, \ln 2) &= \frac{1}{2} e^{-\ln 2} \left(e^{-\ln 2} + \sqrt{e^{-2\ln 2} + 4(1/9)}\right) \\ &= \frac{1}{4}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4}{9}}\right) \\ &= \frac{1}{8} + \frac{1}{4}\sqrt{\frac{9}{36} + \frac{16}{36}} \\ &= \frac{1}{8} + \frac{5}{24} \\ &= \frac{8}{24} = \frac{1}{3}. \end{aligned}$$

Winter 2004, #2 Let $C^2(\overline{\Omega})$ be the space of all twice continuously differentiable functions in the bounded, smooth, and closed domain $\overline{\Omega} \subset \mathbb{R}^2$. Let $u_0(x,y)$ be the function that minimizes the functional

$$D(u) = \int\int_{\Omega} \left[\left(\frac{\partial u(x,y)}{\partial x} \right)^2 + \left(\frac{\partial u(x,y)}{\partial y} \right)^2 + f(x,y)u(x,y) \right] \, dxdy + \int_{\partial\Omega} a(s)u^2(x(s),y(s)) \, ds,$$

where $f(x,y)$ and $a(s)$ are given continuous functions and ds is the arclength element on $\partial\Omega$.

Find the differential equation and the boundary condition that u_0 satisfies.

To find the differential equation that u_0 satisfies, we look for the Euler-Lagrange equation for $D(u)$, given by $D'(u)v = 0$ for all $v \in C^2(\overline{\Omega})$. So,

$$\begin{aligned}
D'(u)v &= \lim_{\epsilon \to 0} \frac{D(u+\epsilon v) - D(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int\int_{\Omega} (\partial_x(u+\epsilon v))^2 + (\partial_y(u+\epsilon v))^2 + f(u+\epsilon v) - (\partial_x u)^2 - (\partial_y u)^2 - fu \; dxdy \\
&\quad + \frac{1}{\epsilon} \int_{\partial\Omega} a(u+\epsilon v)^2 - au^2 \; ds \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int\int_{\Omega} \cancel{(\partial_x u)^2} + 2\epsilon \partial_x u \partial_x v + \epsilon^2 (\partial_x v)^2 + \cancel{(\partial_y u)^2} + 2\epsilon \partial_y u \partial_y v + \epsilon^2 (\partial_y v)^2 + \cancel{fu} + \epsilon f v \\
&\quad - \cancel{(\partial_x u)^2} - \cancel{(\partial_y u)^2} - \cancel{fu} \; dxdy + \frac{1}{\epsilon} \int_{\partial\Omega} \cancel{au^2} + 2a\epsilon uv + \epsilon^2 av^2 - \cancel{au^2} \; ds \\
&= \lim_{\epsilon \to 0} \int\int_{\Omega} 2\partial_x u \partial_x v + 2\partial_y u \partial_y v + \epsilon (\partial_x v)^2 + \epsilon (\partial_y v)^2 + fv \; dxdy + \int_{\partial\Omega} 2auv + \epsilon av^2 \; ds \\
&= \int\int_{\Omega} 2\nabla u \cdot \nabla v + fv \; dxdy + \int_{\partial\Omega} 2auv \; ds = 0.
\end{aligned}$$

Integrating by parts once, we find

$$\begin{aligned}
D'(u)v &= \int\int_{\Omega} -2\Delta uv + fv \; dxdy + \int_{\partial\Omega} n \cdot \nabla uv + 2auv \; ds \\
&= \int\int_{\Omega} (-2\Delta u + f)v \; dxdy + \int_{\partial\Omega} (n \cdot \nabla u + 2au)v \; ds = 0.
\end{aligned}$$

This then tells us that u_0 satisfies the PDE

$$\begin{cases} -\Delta u = \frac{1}{2} f, & \text{in } \Omega \\ n \cdot \nabla u + 2au = 0 & \text{on } \partial\Omega. \end{cases}$$

Fall 2003, #2 (a) Let Ω_1 and Ω_2 be two smooth sets in $\mathbb{R}^2$ with Ω_1 a (strict) subset of Ω_2. Let $-\lambda_1$ and $-\lambda_2$ be the smallest (i.e. least negative) eigenvalues for the Dirichlet problem on Ω_1 and Ω_2, with eigenfunctions ϕ_1 and ϕ_2, respectively. That is

$$\begin{aligned} \Delta\phi_1 &= -\lambda_1\phi_1 \quad \text{in } \Omega_1 \\ \Delta\phi_2 &= -\lambda_2\phi_2 \quad \text{in } \Omega_2 \\ \phi_1 &= 0 \quad \text{on } \partial\Omega_1 \\ \phi_2 &= 0 \quad \text{on } \partial\Omega_2 \end{aligned}$$

Show that $\lambda_1 > \lambda_2 > 0$. Hint: Use the variational characterization of the smallest eigenvalue λ for a set Ω that $\lambda = \min_u \int_\Omega (\nabla u)^2 dxdy / \int_\Omega u^2 dxdy$.

(b) Suppose Ω is a smooth set in $\mathbb{R}^2$ with mirror symmetry about the y axis; i.e. if $(x, y) \in \Omega$ then $(-x, y) \in \Omega$. Let ϕ be the eigenfunction for the Dirichlet problem on Ω with the smallest eigenvalue. Use the result in (a) to show that $\phi(x) = \phi(-x)$.

Fall 2003, #3 The function

$$h(X,T) = (4\pi T)^{-\frac{1}{2}} \exp(-X^2/4T)$$

satisfies (you do not need to show this)

$$h_T = h_{XX}.$$

Using this result, verify that for any smooth function U

$$u(x,t) = \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h(x - t^2 - \xi, t)\ d\xi$$

satisfies

$$u_t + xu = u_{xx}.$$

Given that $U(x)$ is bounded and continuous everywhere on $-\infty \leq x \leq \infty$, establish that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} U(\xi) h(x - \xi, t)\ d\xi = U(x)$$

and show that $u(x,t) \to U(x)$ as $t \to 0$. (You may use the fact that $\int_0^\infty e^{-\xi^2}\ d\xi = \frac{1}{2}\sqrt{\pi}$)

We start by computing the appropriate derivatives of u, noting that h is a function of the variable $h(x,t)$.

$$u_t = (t^2 - x) \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h\ d\xi + \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) \left[(-2t)h_x + h_t\right]\ d\xi$$
$$u_x = -t \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h\ d\xi + \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h_x\ d\xi$$
$$u_{xx} = t^2 \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h\ d\xi - 2t \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h_x\ d\xi + \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h_{xx}\ d\xi$$

Let's condense notation a bit, by letting $g = \frac{1}{3}t^3 - xt$. Plugging all these functions into $u_t + xu - u_{xx} = 0$, we find that

$$\begin{aligned} &(t^2 - x)e^g \int_{\mathbb{R}} U(\xi) h\ d\xi - 2te^g \int_{\mathbb{R}} h_x\ d\xi + e^g \int_{\mathbb{R}} U(\xi) h_t\ d\xi + xe^g \int_{\mathbb{R}} U(\xi) h\ d\xi - t^2 e^g \int_{\mathbb{R}} U(\xi) h\ d\xi \\ &\qquad + 2te^g \int_{\mathbb{R}} U(\xi) h_x\ d\xi - e^g \int_{\mathbb{R}} U(\xi) h_{xx}\ d\xi \\ &= (t^2 - x + x - t^2) \int_{\mathbb{R}} U(\xi) h\ d\xi + (2t - 2t) \int_{\mathbb{R}} U(\xi) h_x\ d\xi + e^g \int_{\mathbb{R}} U(\xi) h_t d\xi - e^g \int_{\mathbb{R}} U(\xi) h_{xx}\ d\xi \\ &= 0 \end{aligned}$$

since $h_t = h_{xx}$ and hence $u(x,t)$ satisfies the PDE. For the second part of this problem, we have that this statement will be true if $h(x - \xi, t) \to \delta(\xi - x)$ as $t \to 0$ since $\int_{\mathbb{R}} U(\xi)\delta(\xi - x)\ d\xi = U(x)$. Let's verify that this is indeed the case.

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} U(\xi) \exp\left(-(x - \xi)^2/4t\right)\ d\xi$$

We now make the substitution $z = (x - \xi)/\sqrt{4t}$. Then $dz = -\frac{d\xi}{\sqrt{4t}}$ and $\xi = x - \sqrt{4t}z$. So we get that

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} U(\xi)\exp\left(-(x-\xi)^2/4t\right)\ d\xi = \frac{1}{\sqrt{4\pi t}}\int_{\infty}^{-\infty} -\sqrt{4t}U(x-\sqrt{4t}z)e^{-z^2}\ dz$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} U(x-\sqrt{4t}z)e^{-z^2}\ dz.$$

Now, taking the limit as $t \to 0$, we get

$$\begin{aligned}\lim_{t\to 0}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} U(x-\sqrt{4t}z)e^{-z^2}\ dz &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} U(x)e^{-z^2}\ dz \\ &= \frac{1}{\sqrt{\pi}}U(x)\int_{\mathbb{R}} e^{-z^2}\ dz \\ &= \frac{1}{\sqrt{\pi}}U(x)\sqrt{\pi} \\ &= U(x).\end{aligned}$$

Now we look at the expression for $u(x,t)$. Here, we use the change of variables $z = \frac{x-t^2-\xi}{\sqrt{4t}}$ which gives us that $dz = -\frac{d\xi}{\sqrt{4t}}$, as before, and $\xi = x - t^2 - \sqrt{4t}z$. So, as in the previous problem, we have that the limit of the integral quantity approaches $U(x)$ as $t \to 0$. We also have that $\lim_{t\to 0}\exp\left(\frac{1}{3}t^3 - xt\right) = 1$, so we can conclude that $u(x,t) \to U(x)$ as $t \to 0$.

Winter 2003, #1 For the ODE

$$u_{tt} = u^3 - u$$

find and analyze the type of the stationary points and draw the phase plane diagram. Identify any connections between stationary points and any regions of periodic orbits.

We start by letting $u_1 = u$ and $u_2 = u_t$. Then, we have that

$$\begin{aligned} u_1' &= u_2 = f_1(u_1, u_2) \\ u_2' &= u_1^3 - u_1 = u_1(u_1^2 - 1) = f_2(u_1, u_2) \end{aligned}$$

where the prime indicates differentiation with respect to t. This system has three stationary points at $(0,0)$, $(1,0)$, and $(-1,0)$. If ξ is our critical point, then the Taylor expansion has the form

$$\begin{aligned} u_1' &= f_1(\xi) + \partial_{u_1} f_1(\xi)(u_1 - \xi_1) + \partial_{u_2} f_1(\xi)(u_2 - \xi_2) + H.O.T. \\ u_2' &= f_2(\xi) + \partial_{u_1} f_2(\xi)(u_1 - \xi_1) + \partial_{u_2} f_2(\xi)(u_2 - \xi_2) + H.O.T. \end{aligned}$$

where $\partial_{u_1} f_1 = 0$, $\partial_{u_2} f_1 = 1$, $\partial_{u_1} f_2 = 3u_1^2 - 1$, and $\partial_{u_2} f_2 = 0$. This then gives us

$$\begin{aligned} u_1' &= (u_2 - \xi_2) + H.O.T. \\ u_2' &= (2\xi_1^2 - 1)(u_1 - \xi_1) + H.O.T. \end{aligned}$$

or

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 3\xi_1^2 - 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 - \xi_1 \\ u_2 - \xi_2 \end{pmatrix} + H.O.T.$$

The point $(0,0)$ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which was eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i,$$

indicating that $(0,0)$ is a center. Now, the points $(1,0)$ and $(-1,0)$ result in the same matrix, $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, which has eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0 \Rightarrow \pm\sqrt{2}$$

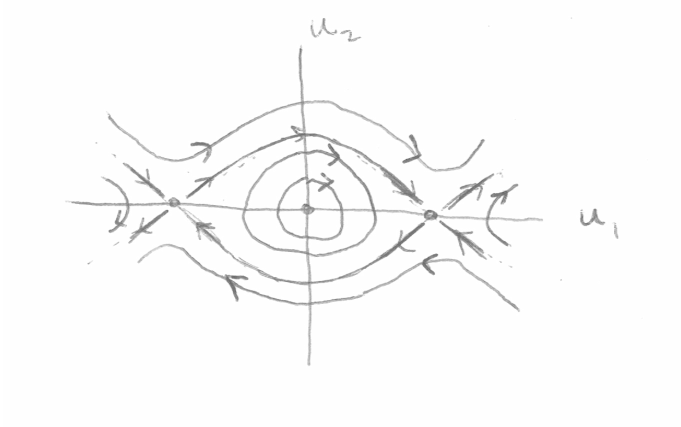
and hence $(1,0)$ and $(-1,0)$ are both saddle points. We will need the eigenvectors in order to sketch the phase plane, so for $\sqrt{2}$ we have

$$\begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -\sqrt{2}x_1 + x_2 = 0 \\ 2x_1 - \sqrt{2}x_2 = 0. \end{cases}$$

So $x_1 = \frac{\sqrt{2}}{2} x_2 \Rightarrow x_2 = 1$ and $x_1 = \frac{1}{\sqrt{2}}$ and hence the eigenvector (corresponding to the unstable manifold) is $\left(\frac{1}{\sqrt{2}}, 1\right)^t$. The eigenvector corresponding to the stable manifold can be found from

$$\begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} \sqrt{2}x_1 + x_2 = 0 \\ 2x_1 + \sqrt{2}x_2 = 0. \end{cases}$$

So we have that $x_1 = -\frac{\sqrt{2}}{2}x_2 \Rightarrow x_2 = 1$. Then $x_1 = -\frac{1}{\sqrt{2}}$ and the eigenvector is then $\left(-\frac{1}{\sqrt{2}}, 1\right)^t$.



As we can see in the phase diagram, the stationary points $(-1, 0)$ and $(1, 0)$ are connected. In addition, any value of u inside the outter two stationary points, that is $-1 < u < 1$, we have periodic orbits.

Winter 2003, #2 Let L be the second order differential operator $L = \Delta - a(x)$ in which $x = (x_1, x_2, x_3)$ is in the three-dimensional cube $C = \{0 < x_i < 1, i = 1, 2, 3\}$. Suppose $a > 0$ in C. Consider the eigenvalue problem

$$\begin{cases} Lu = \lambda u & \text{for } x \in C \\ u = 0 & \text{for } x \in \partial C. \end{cases}$$

(a) Show that all eigenvalues are negative.

(b) If u and v are eigenfunctions for distinct eigenvalues λ and μ, show that u and v are orthogonal in the appropriate inner product.

(c) If $a(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$, find an expression for the eigenvalues and eigenvectors of L in terms of the eigenvalues and eigenvectors of a set of one-dimensional problems.

(a) Let's take the inner product of the equation with its solution,

$$\begin{aligned} (Lu, u) &= (\lambda u, u) \\ \int_C \Delta u u - a(x) u u &= \lambda \int_C u^2 \\ -\int_C \nabla u \cdot \nabla u + \cancelto{0}{\int_{\partial C} u(n \cdot \nabla u)} - \int_C a(x) u^2 &= \lambda \int_C u^2. \end{aligned}$$

Now, we have that the eigenvalues are given by

$$\begin{aligned} \lambda &= \frac{-\int_C \nabla u \cdot \nabla u - \int_C a(x) u^2}{\int_C u^2} \\ &= -\frac{\int_C |\nabla u|^2 + \int_C a(x) u^2}{\int_C u^2} \end{aligned}$$

which are clearly negative.

(b) Let's look at solutions to the weak form of the PDE (here $w \in H_0^1(C)$),

$$b(u, w) = \int_C \nabla u \cdot \nabla w - a(x) u w = \lambda \int_C u w.$$

Since u and v are both eigenfunctions of the operator L, we have that the above weak form must be satisfied for each both u and v (that is, in the above equation we can let $u = v$ and $\lambda = \mu$). Further, since $u, v \in H_0^1(C)$, we have that the weak form is satisfied for both $w = u$ and $w = v$. Now we have that

$$\begin{aligned} b(u, v) &= \lambda \int_C u v \\ b(v, u) &= \mu \int_C v u. \end{aligned}$$

Subtracting these equations from each other, we have that

$$\begin{aligned} (\lambda - \mu) \int_C u v &= b(u, v) - b(v, u) \\ &= \int_C \nabla u \cdot \nabla v - a(x) u v - \int_C \nabla v \cdot \nabla u + a(x) u v \\ &= 0. \end{aligned}$$

However, $\lambda - \mu \neq 0$ which implies that $\int_C uv = 0$ and hence u and v are orthogonal.

(c)

Winter 2003, #3 Let Ω be a smooth domain in three dimensions and consider the initial-boundary value problem for the heat equation

$$\begin{cases} u_t = \Delta u + f & \text{for } x \in \Omega, \quad t > 0 \\ \partial u/\partial n = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u = u_0 & \text{for } x \in \Omega, \quad t = 0 \end{cases}$$

in which f and u_0 are known smooth functions with

$$\partial u_0/\partial n = 0 \quad \text{for } x \in \partial\Omega.$$

(a) Find an approximate formula for u as $t \to \infty$.
(b) If $u_0 \geq 0$ and $f > 0$, show that $u > 0$ for all $t > 0$.

(a) Let's look for a solution that is a linear combination of the eigenfunctions, $\phi_n(x)$, of the Laplacian,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Let's also expand the non-homogeneous part in the eigenfunctions, finding $f = \sum_{n=1}^{\infty} f_n\phi_n(x)$ where $f_n = \int_\Omega f(x)\phi_n(x)$. Plugging this form of the solution into the PDE, we find

$$\sum_{n=1}^{\infty} a_n'(t)\phi_n(x) - a_n(t)\Delta\phi_n(x) - f_n\phi_n(x) = 0$$

from which we have

$$a_n'(t) + \lambda_n a_n(t) - f_n = 0.$$

Solving this ODE with an integrating factor, we find that

$$\begin{aligned} a_n(t) &= e^{-\lambda_n t}\int e^{\lambda_n t} f_n \\ &= e^{-\lambda_n t}\left(\frac{f_n}{\lambda_n}e^{\lambda_n t} + c\right) \\ &= \frac{f_n}{\lambda_n} + ce^{-\lambda_n t} \end{aligned}$$

By also expanding the initial condition in the eigenfunctions, $u_0 = \sum_{n=1}^{\infty} u_n\phi_n(x)$, we find that

$$a_n(0) = u_n = \frac{f_n}{\lambda_n} + c$$

and hence that $c = u_n - \frac{f_n}{\lambda_n}$. So, we can now write our solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{f_n}{\lambda_n} + \left(u_n - \frac{f_n}{\lambda_n}\right)e^{-\lambda_n t}\right]\phi_n(x).$$

Now taking $t \to \infty$, we find

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty}\frac{f_n}{\lambda_n}\phi_n(x).$$

(b) Recall that for $u, v \in C^{2;1}(U_T) \cap C^1(\overline{U_T})$, where $U_T = \Omega \times (0,T)$, we have the Neumann comparison principle. That is, if the following hold

$$
\begin{gathered}
u_t - \Delta u - f \geq v_t - \Delta v - f \quad \text{for } x \in \Omega,\ t > 0 \\
\frac{\partial u}{\partial \nu} \geq \frac{\partial v}{\partial \nu} \quad \text{for } x \in \partial\Omega,\ t > 0 \\
u > v \quad \text{for } x \in \Omega,\ t = 0
\end{gathered}
$$

then we have that $u > v$ for all $(x,t) \in \overline{U_T}$. So, let's choose $v = 0$ for all t. Then $u_0 > 0 = v_0$ and $\partial_\nu u = 0 \geq 0 = \partial_\nu v$. So, we need that

$$u_t - \Delta u - f \geq -f$$

holds in order to use the comparison principle. But we have that

$$u_t - \Delta u - f = 0 \geq -f$$

so the Neumann comparison principle holds and we have that $u > 0$ for $t > 0$.

Winter 2003, #5 Find a solution to $xu_x + (x+y)u_y = 1$ which satisfies $u(1,y) = y$ for $0 \le y \le 1$. Find the region in $\{x \ge 0, y \ge 0\}$ where u is uniquely determined by these conditions.

The characteristic equations for this PDE are

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = x + y \qquad \frac{du}{dt} = 1$$

with $\Gamma = (1, s, s)$. We start by solving the decoupled equations (those for x and u):

$$x = c_1 e^t \qquad u = t + c_2.$$

Using our initial data, we find that $c_1 = 1$ and $c_2 = s$. Given that $x = e^t$, we now solve the equation $\frac{dy}{dt} = e^t + y$ using an integrating factor. In this case, the integrating factor is given by $e^{-\int 1 dt} = e^{-t}$, so we find that $e^{-t}y = t + c_3$. Our initial data then gives us that $c_3 = s$. So, we find that

$$x = e^t \qquad u = t + s \qquad y = (t+s)e^t.$$

From these equations, we see that $y = ux$ and so our solution is given by

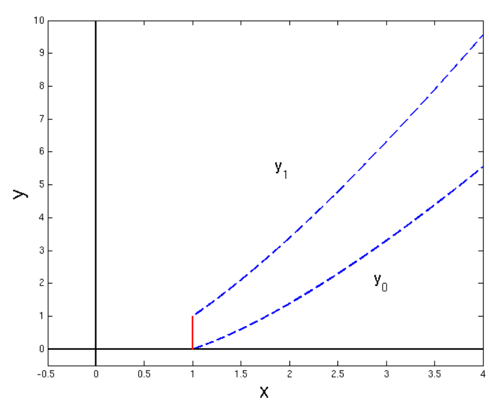
$$u = \frac{y}{x}.$$

To determine where the solution, u, is uniquely determined, let's look at the characteristics is the xy-plane. Above, we found that $y_s = te^t + se^t = x\ln x + sx$ for $s \in [0,1]$. Recall that our boundary data is given on $x = 1$ and $0 \le y \le 1$. This initial data then traces out a region in the xy-plane, according to the characteristics, in a manner as shown in the sketch below.

The boundaries of this region are given by $y_0 = x\ln x$ and $y_1 = x + x\ln$ x. Note that interior to this region, we can always find t and s in terms of x and y since

$$\begin{vmatrix} x_t & x_s \\ y_t & y_s \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t + te^t + se^t & e^t \end{vmatrix} = 2e^t \ne 0.$$

Further, the characteristic for u will only produce a singularity if $x = 0$, which can clearly not occur for $t \ge 0$, so we have that the solution is unique in the region defined by $\{(x,y) : x \ge 1 \text{ and } x\ln x \le y \le x + x\ln x\}$.



Winter 2003, #8 (a) Consider the damped wave equation for high-speed waves ($0 < \epsilon << 1$) in a bounded region D

$$\epsilon^2 u_{tt} + u_t = \Delta u$$

with the boundary condition $u(x,t) = 0$ on the boundary of D. Show that the energy functional

$$E(t) = \int_D \epsilon^2 u_t^2 + |\nabla u|^2 \, dx$$

is nonincreasing on solutions of the boundary value problem.

(b) Consider the solution to the boundary value problem in part (a) with initial data $u^\epsilon(x,0) = 0, u_t^\epsilon(x,0) = \epsilon^{-\alpha} f(x)$, where f does not depend on ϵ and $\alpha < 1$. Use part (a) to show that

$$\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$$

uniformly on $0 \le t \le T$ for any T as $\epsilon \to 0$.

(c) Show that the result in part (b) does not hold if $\alpha = 1$. To do this consider the case where f is an eigenfunction of the Laplacian, i.e. $\Delta f + \lambda f = 0$ in D and $f = 0$ on the boundary of D, and solve for u^ϵ explicitly.

(a)

$$\begin{aligned}
\partial_t E &= \int_D 2\epsilon^2 u_t u_{tt} + 2\nabla u \cdot \nabla u_t \, dx \\
&= 2\int_D \epsilon^2 u_t u_{tt} - \Delta u u_t \\
&= 2\int_D (\epsilon^2 u_{tt} - \Delta u) u_t \\
&= 2\int_D (-u_t) u_t \\
&= -2\int_D u_t^2 \le 0
\end{aligned}$$

So we have that the energy is non-increasing.

(b) Initially, we have

$$\begin{aligned}
E(0) &= \int_D \epsilon^2 \left(u_t^\epsilon(x,0)\right)^2 + |\nabla u^\epsilon(x,0)|^2 \, dx \\
&= \int_D \epsilon^2 \epsilon^{-2\alpha} f(x)^2 \\
&= \epsilon^{2(1-\alpha)} \int_D f(x)^2.
\end{aligned}$$

Note that $\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \le E(t)$. Since $\alpha < 1$, we have that $E(0) \to 0$ as $\epsilon \to 0$. Further, since $E(t)$ is non-increasing, we have that $E(t) \to 0$ uniformly on $0 \le t \le T$ as $\epsilon \to 0$. Since $\int_D |\nabla u^\epsilon(x,t)|^2 \, dx$ is bounded by $E(t)$, we also have that $\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$ uniformly as $\epsilon \to 0$.

(c) Let $\{\phi_n(x)\}$ be the eigenfunctions for the Laplacian. Let's write u^ϵ as $u^\epsilon(x,t) = \sum_{n=1}^\infty a_n(t)\phi_n(x)$. Plugging this into the PDE, we find

$$
\begin{aligned}
0 &= \epsilon^2 a_n''(t)\phi_n(x) + a_n'(t)\phi_n(x) - a_n(t)\Delta\phi_n(x) \\
&= \epsilon^2 a_n''(t)\phi_n(x) + a_n'(t)\phi_n(x) + \lambda_n a_n(t)\phi_n(x) \\
&= \epsilon^2 a_n''(t) + a_n'(t) + \lambda_n a_n(t).
\end{aligned}
$$

The characteristic equation for this ODE is $\epsilon^2 r^2 + r + \lambda_n = 0$ which has roots $r_{1,2} = \dfrac{-1 \pm \sqrt{1 - 4\epsilon^2\lambda_n}}{2\epsilon^2}$. Since $\epsilon << 1$, we shall assume both roots are real. So, we have that

$$
a_n(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.
$$

Consider the initial conditions, $u^\epsilon(x, 0) = \sum_{n=1}^\infty a_n(0)\phi_n(x) = 0$ which implies that $c_1 + c_2 = 0$. The second condition is given by $u_t^\epsilon(x, 0) = \sum_{n=1}^\infty a_n'(0)\phi_n(x) = \epsilon^{-1} f(x)$. Note that since $f(x)$ is an eigenfunction of the Laplacian, we have that $a_i'(0)\phi_i(x) = \epsilon^{-1} f(x)$ for some i and further that all the other $a_n'(0) = 0$ and hence $a_n(0) = 0$ for $n \neq i$. So, we have that $c_1 = -c_2$ which we shall simply call c and then

$$
\begin{aligned}
cr_1 - cr_2 &= \epsilon^{-1} \\
\Rightarrow c &= \frac{1}{\epsilon(r_1 - r_2)}.
\end{aligned}
$$

So, we have that

$$
u^\epsilon(x, t) = \left(\frac{\epsilon}{\sqrt{1 - 4\epsilon^2\lambda}} e^{\frac{-1+\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} - \frac{\epsilon}{\sqrt{1 - 4\epsilon^2\lambda}} e^{\frac{-1-\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} \right) f(x).
$$

Now we need to look at $\int_D |\nabla u^\epsilon(x, t)|^2 \, dx$,

$$
\begin{aligned}
\int_D |\nabla u^\epsilon(x, t)|^2 \, dx &= \int_D \left(\frac{\epsilon}{\sqrt{1 - 4\epsilon^2\lambda}} e^{\frac{-1+\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} - \frac{\epsilon}{\sqrt{1 - 4\epsilon^2\lambda}} e^{\frac{-1-\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} \right)^2 |\nabla f|^2 \, dx \\
&= \frac{\epsilon^2}{1 - 4\epsilon^2\lambda} \left(e^{\frac{-1+\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} - e^{\frac{-1-\sqrt{1-4\epsilon^2\lambda}}{2\epsilon^2} t} \right)^2 \int_D |\nabla f|^2 \, dx.
\end{aligned}
$$

Not sure where to go from here. Finding everything converging to zero as $\epsilon \to 0$.

Fall 2002, #3 Consider the first order system

$$u_t + u_x + v_x = 0$$
$$v_t + u_x - v_x = 0$$

on the domain $0 < t < \infty$ and $0 < x < 1$. Which of the following sets of initial boundary data are well posed for this system? Explain your answers.
(a) $u(x,0) = f(x), v(x,0) = g(x)$
(b) $u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(0,t) = k(x)$
(c) $u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(1,t) = k(x)$.

We are going to find the characteristic equations needed to solve this system and then use these to determine what data is need for well-posedness. We start be decoupling the system

$$w_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} w_x = 0$$

where $w = (u,v)^t$. The eigenvalues of the coefficient matrix, A, are given by

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 1 - 1 = \lambda^2 - 2 = (\lambda + \sqrt{2})(\lambda - \sqrt{2}) = 0$$

and hence $\lambda_{1,2} = \pm\sqrt{2}$ with corresponding eigenvectors given by

$$\begin{cases} (1-\lambda)x_1 + x_2 = 0 \\ x_1 - (1+\lambda)x_2 = 0 \end{cases} \Rightarrow \begin{cases} (1-\lambda)x_1 = -x_2 \\ x_1 = (1+\lambda)x_2 \end{cases} \Rightarrow v_{1,2} = (1 + \lambda_{1,2}, 1)^t.$$

If $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and we let $z = P^{-1}w$, we then have that our decoupled system is given by

$$z_t + \Lambda z_x = 0$$

with $\Lambda = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$ and $z_0 = P^{-1}w_0$. Let's look at the characteristic for $z = (z_1, z_2)^t$. We have $(z_1)_t + \sqrt{2}(z_1)_x = 0$ and $(z_2)_t - \sqrt{2}(z_2)_x = 0$. Here we see we have characteristics traveling in both the positive and negative x directions and forward in time, indicating that we need initial data specified along $t = 0$ for both z_1 and z_2 and boundary data on the left boundary for z_1 and the right boundary for z_2.

This results in both (a) and (b) being ill-posed, because they do not have boundary data specified along both boundaries. The conditions given in (c) give a well-posed problem since data is specified everywhere where we need data for the characteristics.

Fall 2002, #6 The temperature of a rod insulated at the ends with an exponentially decreasing heat source in it is a solution of the following boundary value problem:

$$\partial_t u = \partial_x^2 u + e^{-2t} g(x) \text{ for } (x,t) \in [0,1] \times \mathbb{R}^+,$$
$$\partial_x u(0,t) = \partial_x u(1,t) = 0, \text{ and } u(x,0) = f(x).$$

Find the solution to this problem by writing u as a cosine series,

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos n\pi x$$

and determine $\lim_{t\to\infty} u(x,t)$.

Plugging the form of the solution into the PDE and letting $\phi_n(x) = \cos n\pi x$, we find that we have

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) - a_n(t)\partial_x^2\phi_n(x) - e^{-2t} b_n \phi_n(x) = 0.$$

Note that since $\phi_n(x)$ are eigenfunctions of the Laplacian (satisfying the appropriate boundary conditions), we have that we can expand g in the eigenfunctions and for λ_n being the eigenvalues corresponding to $\phi_n(x)$, we have that

$$a_n'(t) + \lambda_n a_n(t) - e^{-2t} b_n = 0$$

for each eigenfunction, $\phi_n(x)$, where $b_n = \int_{[0,1]} g(x)\phi_n(x)\,dx$. We can integrate this ODE using an integrating factor, in this case given by $e^{\lambda_n t}$, so the solution is given by

$$\begin{aligned}
a_n(t) &= e^{-\lambda_n t} \int e^{\lambda_n t} e^{-2t} b_n \; dt \\
&= e^{-\lambda_n t} \int b_n e^{(\lambda_n - 2)t} \; dt \\
&= e^{-\lambda_n t} \left(\frac{b_n}{\lambda_n - 2} e^{(\lambda_n - 2)t} + c \right) \\
&= \frac{b_n}{\lambda_n - 2} e^{-\lambda_n t} e^{\lambda_n t} e^{-2t} + c e^{-\lambda_n t} \\
&= \frac{b_n}{\lambda_n - 2} e^{-2t} + c e^{-\lambda_n t}.
\end{aligned}$$

In order to find c, we look at the initial condition, $u(x,0) = f(x)$. We can expand $f(x)$ in the eigenfunctions, finding that $u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ where $c_n = \int_{[0,1]} f(x)\phi_n(x)\,dx$. So, we then have that

$$a_n(0) = \frac{b_n}{\lambda_n - 2} + c = c_n$$

giving us that

$$c = c_n - \frac{b_n}{\lambda_n - 2}$$

and finally, we have that

$$a_n(t) = \frac{b_n}{\lambda_n - 2} e^{-2t} + \left(c_n - \frac{b_n}{\lambda_n - 2} \right) e^{-\lambda_n t}$$

with $\lambda_n = n^2\pi^2$. The solution is then given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{b_n}{\lambda_n - 2} e^{-2t} + \left(c_n - \frac{b_n}{\lambda_n - 2} \right) e^{-\lambda_n t} \right) \cos n\pi x.$$

Note that $\lim_{t\to\infty} u(x,t) = 0$.

Fall 2002, #8 Let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. Show that a solution of the boundary value problem

$$\Delta^2 u = f \quad \text{in } D, u = \Delta u = 0 \quad \text{on } \partial D$$

must be unique.

Let u and v be two solutions to the biharmonic equation given in the problem statement. Let $w = u - v$. Then w satisfies

$$\begin{cases} \Delta^2 w = 0 & \text{in } D \\ w = \Delta w = 0 & \text{on } \partial D. \end{cases}$$

We now demonstrate uniqueness using an energy method. Consider

$$\begin{aligned} 0 &= \int_D \Delta^2 w w \\ &= -\int_D \nabla \Delta w \cdot \nabla w + \cancelto{0}{\int_{\partial D} w(\nu \cdot \nabla \Delta w)} \\ &= \int_D |\Delta w|^2 - \cancelto{0}{\int_{\partial D} \nabla w \cdot (\nu \cdot \Delta w)} \\ &= \int_D |\Delta w|^2. \end{aligned}$$

This last equality tells us that $\Delta w \equiv 0$ in D. So, we now have that w is harmonic in D and in fact, w satisfies the equation

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w = 0 & \text{on } \partial D. \end{cases}$$

So, proceeding as before, we have

$$\begin{aligned} 0 &= \int_D \Delta w w \\ &= -\int_D |\nabla w|^2 + \cancelto{0}{\int_{\partial D} w(\nu \cdot \nabla w)} \\ &= -\int_D |\nabla w|^2. \end{aligned}$$

Now we also have that $\nabla w \equiv 0$ in D, which allows us to conclude that w is constant. Further, we know $w = 0$ on ∂D, so we have that $w \equiv 0$ and hence $u = v$ and so solutions to the biharmonic equation are unique.

Spring 2002, #1 (a) Find a radially symmetric solution, u, to the equation in $\mathbb{R}^2$,

$$\Delta u = \frac{1}{2\pi} \log |x|,$$

and show that u is a fundamental solution for Δ^2, i.e. show

$$\phi(0) = \int_{\mathbb{R}^2} u \Delta^2 \phi \, dx$$

for any smooth ϕ which vanishes for $|x|$ large.

(b) Explain how to construct the Green's function for the following boundary value in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D

$$w = 0 \text{ and } \frac{\partial w}{\partial n} = 0 \text{ on } \partial D, \;\; \Delta^2 w = f \text{ in } D,$$

where ∂ / ∂_n denotes the normal derivative.

Spring 2002, #2 (a) Given a continuous function f on $\mathbb{R}$ which vanishes for $|x| > R$, solve the initial value problem $u_{tt} - u_{xx} = f(x)\cos t, u(x,0) = u_t(x,0) = 0, -\infty < x < \infty, 0 \le t < \infty$ by first finding a particular solution by separation of variables and then adding the appropriate solution of the homogeneous PDE.

(b) Since the particular solution is not unique, it will not be obvious that the solution to the initial value problem that you have found in part (a) is unique. Prove that it is unique.

(a) We are asked to write our solution as $u(x,t) = u_h(x,t) + u_p(x,t)$ where $u_h(x,t)$ is the solution to the homogeneous problem and $u_p(x,t)$ is a particular solution. We start by noting that $u_h(x,t) \equiv 0$, from D'Alembert's formula,

$$u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2}\int_{x-ct}^{x+ct} h(\xi)\ d\xi$$

where $g(x) = h(x) = 0$, so we find that $u(x,t) = c$, a constant. However, we have that $u(x,0) = 0 \Rightarrow u(x,t) = 0$. To find a particular solution, let's use Duhamel's principle, which tells us that $u_p(x,t) = \int_0^t U(x, t-s, s)\ ds$ where $U(x,t,s)$ satisfies

$$\begin{cases} U_{tt} - U_{xx} = 0 \\ U(x,0,s) = 0 \\ U_t(x,0,s) = f(x)\cos s. \end{cases}$$

For separation of variables, we assume that $U(x,t,s)$ has a solution of the form $U(x,t,s) = X(x)\Phi(t)\Psi(s)$. Plugging into the PDE, we find that

$$\Phi''(t)X(x)\Psi(s) - X''(x)\Phi(t)\Psi(s) = 0$$
$$\Phi'(0)X(x)\Psi(s) = f(x)\cos s$$

Now we have that $\frac{\Phi''(t)}{\Phi(t)} = \frac{X''(x)}{X(x)} = -\lambda$. The solutions to both of these ODE's are then given by linear combinations of sine and cosine functions,

$$\Phi(t) = \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n}t) + b_n \sin(\sqrt{\lambda_n}t)$$
$$X(x) = \sum_{n=1}^{\infty} c_n \cos(\sqrt{\lambda_n}x) + d_n \sin(\sqrt{\lambda_n}x)$$

The initial conditions tell us that $\Phi(0) = 0 \Rightarrow a_n = 0$ and that $\Phi'(0) = 1 \Rightarrow b_n\sqrt{\lambda_n} = 1 \Rightarrow b_n = \frac{1}{\sqrt{\lambda_n}}$. We also have that $X(x) = f(x) \Rightarrow c_n = \int_{\mathbb{R}} f(x)\cos(\sqrt{\lambda_n}x)\ dx$ and $d_n = \int_{\mathbb{R}} f(x)\sin(\sqrt{\lambda_n}x)\ dx$. So, we now have that

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}t)$$
$$X(x) = \sum_{n=1}^{\infty} c_n \cos(\sqrt{\lambda_n}x) + d_n \sin(\sqrt{\lambda_n}x)$$
$$\Psi(s) = \cos s$$

with c_n and d_n defined as above. Let's consider using Duhamel's principle to integrate this summation term by term. Then for a fixed n, we get

$$
\begin{aligned}
(u_n)_p(x,t) &= \Big(c_n\cos(\sqrt{\lambda_n}x) + d_n\sin(\sqrt{\lambda_n}x)\Big)\int_0^t \frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}(t-s))\cos s\ ds \\
&= \Big(c_n\cos(\sqrt{\lambda_n}x) + d_n\sin(\sqrt{\lambda_n}x)\Big)\frac{1}{\sqrt{\lambda_n}}\int_0^t (\sin\sqrt{\lambda_n}t\cos\sqrt{\lambda_n}s - \cos\sqrt{\lambda_n}t\sin\sqrt{\lambda_n}s)\cos s
\end{aligned}
$$

This is getting quite messy and no good. Easier way to approach problem, but I'm not going to go back and fix it.

(b) Let u_1 and u_2 be two solutions to the PDE and define the energy as follows,

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2 + u_x^2\ dx.$$

Now let $w = u_1 - u_2$, then w satisfies the PDE

$$\begin{cases} w_{tt} - w_{xx} = 0 \\ w(x,0) = w_t(x,0) = 0. \end{cases}$$

Then (note here boundary terms vanish when we integrate by parts since the initial data are zero),

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\mathbb{R}} w_t w_{tt} + w_x w_{xt}\ dx \\
&= \int_{\mathbb{R}} w_t w_{tt} - w_{xx} w_t\ dx \\
&= \int_{\mathbb{R}} w_t(w_{tt} - w_{xx})\ dx \\
&= \int_{\mathbb{R}} 0\ dx
\end{aligned}
$$

which implies that $E(t) = c$, a constant. However, we clearly have that $E(0) = 0 \Rightarrow E(t) = 0$ and hence $w \equiv 0$. So we have that $u_1 = u_2$ and solutions to the PDE are unique.

Spring 2002, #4 Use the Fourier transform on $L^2(\mathbb{R})$ to show that

$$\frac{du}{dx} + cu(x) + u(x-1) = f$$

has a unique solution $u \in L^2(\mathbb{R})$ for each $f \in L^2(\mathbb{R})$ when $|c| > 1$, $c \in \mathbb{R}$.

The Fourier transform of the equation gives us

$$2\pi i\xi \hat{u} + c\hat{u} + e^{-2\pi i\xi}\hat{u} = \hat{f}.$$

Solving for $\hat{u}$, we find that

$$\hat{u} = \frac{\hat{f}}{2\pi i\xi + c + e^{-2\pi i\xi}}.$$

Let

$$\hat{g} = \frac{1}{2\pi i\xi + c + e^{-2\pi i\xi}},$$

then we clearly see that $\hat{u} = \hat{f}\hat{g} = \widehat{f * g}$. So, we see that we will obtain a unique solution if the inverse transforms of $\hat{f}$ and $\hat{g}$ are unique. This is guaranteed if we can indeed inverse transform these functions. Clearly this is true for f since $f \in L^2(\mathbb{R})$. So, now we need to verify that $\hat{g} \in L^2(\mathbb{R})$. So, let's consider

$$\int |\hat{g}|^2 = \int \frac{1}{|2\pi i\xi + c + e^{-2\pi i\xi}|^2}.$$

Just considering the denominator, we have that

$$\begin{aligned}
|2\pi i\xi + c + e^{-2\pi i\xi}|^2 &= (c + \cos 2\pi\xi)^2 + (2\pi\xi - \sin(2\pi\xi))^2 \\
&= (c + \cos 2\pi\xi)^2 + \sin^2 2\pi\xi + 4\pi(\pi\xi^2 - \xi \sin 2\pi\xi) \\
&\geq k + 4\pi(\pi\xi^2 - \xi)
\end{aligned}$$

where $k > 0$ is a constant. So, we have that $\hat{g} \in L^2(\mathbb{R})$ and hence the equation has a unique solution.

Spring 2002, #5 The following equation (Fisher's equation) arises in the study of population genetics: $u_t = u(1-u) + u_{xx}$ on $-\infty < x < \infty$, $t > 0$. The solutions of physical interest satisfy $0 \leq u \leq 1$ and

$$\lim_{x\to-\infty} u(x,t) = 0 \qquad \lim_{x\to\infty} u(x,t) = 1.$$

One class of solutions is the set of "wavefront" solutions. These have the form $u(x,t) = \phi(x+ct), c \geq 0$.

Determine the ordinary differential equation and boundary conditions which ϕ must satisfy (to be of physical interest). Carry out a phase plane analysis of this equation, and show that physically interesting wavefront solutions are possible if $c \geq 2$, but not if $0 \leq c < 2$.

Let's start by substituting $u(x,t) = \phi(x+ct)$ into the PDE.

$$\phi'' - c\phi' + \phi - \phi^2 = 0$$

with the conditions that $\lim_{\xi\to-\infty} \phi(\xi) = 0$ and $\lim_{\xi\to\infty} \phi(\xi) = 1$ and $0 \leq \phi \leq 1$. In order to carry out a phase plane analysis of this problem, we need to convert the second order ODE into a system, so let $\phi_1 = \phi$ and $\phi_2 = \phi'$. Then we have that

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \phi_2 \\ c\phi_2 - \phi_1(1-\phi_1). \end{pmatrix}$$

This system has stationary points at the points $(\phi_1, \phi_2) = (1,0)$ and $(0,0)$. To classify the points, let's linearize the system about the fixed point a.

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+2\phi_1 & c \end{pmatrix} \begin{pmatrix} (x_1 - a_1) \\ (x_2 - a_2) \end{pmatrix}$$

Now, to classify the points we need to find the eigenvalues. Let's start with $(0,0)$.

$$\begin{vmatrix} -\lambda & 1 \\ -1 & c-\lambda \end{vmatrix} = -\lambda(c-\lambda) + 1 = \lambda^2 - c\lambda + 1$$

so the eigenvalues are given by $\lambda_\pm = \dfrac{c \pm \sqrt{c^2-4}}{2}$. This gives us three possible cases:

1. $c > 2$

$$\lambda_\pm = \frac{c \pm \sqrt{c^2-4}}{2} > 0$$

   Unstable, improper node

2. $c = 2$

$$\lambda_\pm = 1$$

   Unstable, improper node

3. $0 \leq c < 2$

$$\lambda_\pm = \frac{c \pm i\sqrt{4-c^2}}{2}, \quad \text{with } \sqrt{4-c^2} > 0$$

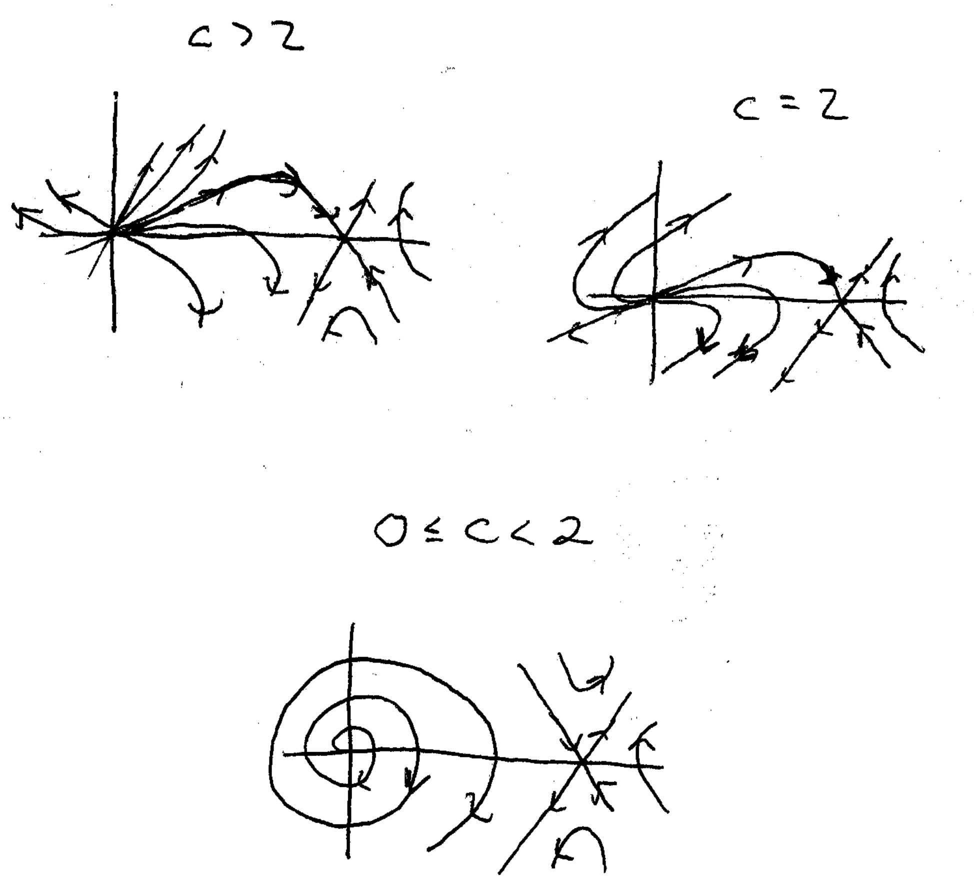
   Unstable spiral

The eigenvectors for this problem (when $\lambda_\pm \in \mathbb{R}$) are given by $\left(1, \dfrac{c+\sqrt{c^2-4}}{2}\right)^t$ and $\left(1, \dfrac{c-\sqrt{c^2-4}}{2}\right)^t$. Note that when $c = 2$, the eigenspace contains only one eigenvector, $(1,1)$.

Now we look at $(1,0)$,

$$\begin{vmatrix} -\lambda & 1 \\ 1 & c-\lambda \end{vmatrix} = \lambda^2 - c\lambda - 1 \Rightarrow \lambda_{\pm} = \frac{c \pm \sqrt{c^2+4}}{2}$$

giving us that $\lambda_+ > 0$ and $\lambda_- < 0$, so we have that $(1,0)$ is a saddle point with corresponding eigenvectors $\left(1, \frac{c+\sqrt{c^2+4}}{2}\right)^t$ and $\left(1, \frac{c-\sqrt{c^2+4}}{2}\right)^t$. We sketch the three different cases below.



Spring 2002, #8 (a) Assume that D is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D and outer unit normal ν. Find a variational formula for the lowest eigenvalue of $-\Delta u$ in D with the boundary condition $\frac{\partial u}{\partial \nu} + au = 0$ on ∂D, and show that the lowest eigenvalue will be positive or negative depending on the sign of a.

(b) For the values of a which make the lowest eigenvalue positive, derive the following estimate for the solution u of the boundary value problem $-\Delta u + k^2 u = 0$ in D with $\frac{\partial u}{\partial \nu} + au = g$ on ∂D:

$$\max_D |u| \leq C_a \max_{\partial D} |g|,$$

where C_a does not depend on k. Use maximum principle arguments.

(a) We begin by considering the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D \\ \frac{\partial u}{\partial \nu} + au = 0 & \text{on } \partial D. \end{cases}$$

Now, let's look at the equivalent weak form of the problem. Take $v \in H^1_{\partial D}(D)$ where $H^1_{\partial D}(D)$ is the space of functions in $H^1(D)$ that satisfy the boundary condition above. Then,

$$\int_D -\Delta u v = \int_D \lambda u v$$
$$\int_D \nabla u \cdot \nabla v - \int_{\partial D} v(\nu \cdot \nabla u) = \lambda \int_D uv$$
$$\int_D \nabla u \cdot \nabla v + \int_{\partial D} auv = \lambda \int_D uv.$$

Clearly the solution, u, is in $H^1_{\partial D}(D)$, so we let $v = u$ and find

$$\int_D |\nabla u|^2 + a \int_{\partial D} u^2 = \lambda \int_D u^2$$

and hence

$$\lambda = \frac{\int_D |\nabla u|^2 + a \int_{\partial D} u^2}{\int_D u^2}.$$

In particular, the lowest eigenvalue is then given by

$$\lambda_1 = \inf_{u \in H^1_{\partial D}(D)} \frac{\int_D |\nabla u|^2 + a \int_{\partial D} u^2}{\int_D u^2}.$$

Note that clearly if $a > 0$, then $\lambda_1 > 0$. If $a < 0$, we are looking for a function that will minimize $\int_D |\nabla u|^2$ term, which will result in the boundary term driving the entire numerator negative, so we find that $\lambda_1 < 0$.

(b) We start by writing the problem as

$$\begin{cases} \Delta u - k^2 u = 0 & \text{in } D \\ \frac{\partial u}{\partial \nu} + au = g & \text{on } \partial D. \end{cases}$$

Since the function multiplying the term u (in this case $-k^2$) is non-positive, we have that

$$\max_{x \in D} u(x) \leq \max_{x \in \partial D} u^+(x)$$

where $u^+(x) = \max(u(x), 0)$. Further, we see that $\max_{x \in \partial D} u^+(x) = \max_{x \in \partial D} \left(\frac{1}{a} g - \frac{1}{a} \frac{\partial u}{\partial \nu}, 0\right) \leq \max_{x \in \partial D} \left(\frac{1}{a} g, 0\right)$. Note the last inequality is clearly true if $u(x)$ is a constant. If $u(x)$ is not a constant, then we have that at the point where the maximum is attained on the boundary, x_0, $\frac{\partial u}{\partial \nu}(x_0) > 0$ (strong maximum principle). So, we have that

$$\max_{x\in D} u(x) \leq \max_{x\in\partial D}\left(\frac{1}{a}g, 0\right).$$

Now, we make the same argument on a solution v such that $v = -u$. The steps are exactly the same, only we now obtain that

$$\max_{x\in D} -u(x) \leq \max_{x\in\partial D}\left(-\frac{1}{a}g, 0\right)$$

which allows us to conclude that

$$\max_{x\in D} |u| \leq \frac{1}{a}\max_{x\in\partial D} |g|.$$

Spring 2001, #2 (a) Find the solution $u = (u_1(x,t), u_2(x,t))$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, to the (strictly) hyperbolic equation

$$u_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} u_x = 0,$$

satisfying $(u_1(x,0), u_2(x,0)) = (\exp(ixa), 0),\ a \in \mathbb{R}$.

(b) Write the solution of the initial value problem in part (a) for general initial data

$$(u_1(x,0), u_2(x,0)) = (f(x), 0)$$

as an inverse Fourier transform. You may assume that f is smooth and rapidly decreasing as $|x| \to \infty$.

(a) We begin by diagonalizing the matrix $A = \begin{pmatrix} -1 & 0 \\ -5 & -3 \end{pmatrix}$.

$$\begin{vmatrix} -1-\lambda & 0 \\ -5 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) = (\lambda+1)(\lambda+3) = 0$$

So we have that $\lambda_{1,2} = -1, -3$ with corresponding eigenvectors

$$\begin{pmatrix} 0 & 0 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -5x_1 = 2x_2 \Rightarrow v_1 = \left(1, -\frac{5}{2}\right)^t$$

and

$$\begin{pmatrix} 2 & 0 \\ -5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow v_2 = (0,1)^t.$$

Now we have $u_t + P\Lambda P^{-1} u_x = 0$ where $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Setting $v = P^{-1}u = (w,z)^t$, we now have a decoupled set of equations,

$$w_t - w_x = 0$$
$$z_t - 3z_x = 0$$

with initial condition $v_0 = P^{-1}u_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix} u_0 = \left(\frac{1}{2}e^{ixa}, \frac{5}{2}e^{ixa}\right)$.

For w, we get the following characteristic equations

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = -1, \quad \frac{dw}{d\tau} = 0$$

with $\Gamma = \left(0, s, \frac{1}{2}e^{isa}\right)$. So, we find that $t = \tau$, $x = -\tau + s$, and $w = \frac{1}{2}e^{isa}$. So, $s = x + t$ and we have that $w = \frac{1}{2}e^{i(x+t)a}$.

We have a very similar set of equations for z,

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = -3, \quad \frac{dz}{d\tau} = 0$$

with $\Gamma = \left(0, s, \frac{5}{2}e^{isa}\right)$. This then gives us that $t = \tau$, $x = -3\tau + s$, and $z = \frac{5}{2}e^{isa}$. Here we find that $s = x + 3t$ so we get that $z = \frac{5}{2}e^{i(x+3t)a}$. Our solution is then given by $u = Pv$,

$$\begin{aligned} u_1(x,t) &= e^{i(x+t)a} \\ u_2(x,t) &= -\frac{5}{2}e^{i(x+t)a} + \frac{5}{2}e^{i(x+3t)a} \\ &= \frac{5}{2}e^{ixa}\left(e^{i3ta} - e^{ita}\right). \end{aligned}$$

(b) For this initial condition, let's start by taking the Fourier transform of the PDE,

$$\hat{u}_t - A2\pi i\xi\hat{u} = 0$$

with $\hat{u}_0 = (\hat{f}, 0)$. Then, we have that the solution is given by

$$\begin{aligned} \hat{u}_t &= A2\pi i\xi\hat{u} \\ \hat{u} &= \hat{u}_0 e^{2\pi iA\xi t} \end{aligned}$$

and so our solution is given by

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} e^{2\pi ix\xi}\hat{u}_0 e^{2\pi iA\xi t}\ d\xi \\ &= \int_{\mathbb{R}} \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} e^{2\pi i\xi} e^{x+At}\ d\xi. \end{aligned}$$

Spring 2001, #4 The "Poincaré Inequality" states that for any bounded domain D in $\mathbb{R}^n$ there is a constant C such that

$$\int_D |u|^2 dx \leq C \int_D |\nabla u|^2 dx$$

for all smooth functions u which vanish on the boundary of D.

(a) Find a formula for the "best" constant (that means the smallest one that works) for the domain D in terms of the eigenvalues of the Laplacian on D.

(b) Give the best constant for the rectangular domain in $\mathbb{R}^2$:

$$D = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}.$$

Recall that the eigenvalues of the Laplacian satisfy

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

By ϕ_i, we denote the corresponding eigenfunctions, which satisfy

$$\begin{cases} \Delta\phi_i + \lambda_i\phi_i = 0 & \text{in } D \\ \phi_i = 0 & \text{on } \partial D. \end{cases}$$

Further, we note that for any smooth function vanishing on ∂D, we can write $u = \sum_{i=1}^{\infty} a_i\phi_i$. Now, any eigenfunction of the Laplacian will also satisfy the weak form of the above PDE, that is

$$\begin{aligned} 0 &= \int_D \Delta\phi_i\phi_i + \lambda_i\phi_i\phi_i \\ &= -\int_D \nabla\phi_i \cdot \nabla\phi_i + \lambda_i|\phi_i|^2 + \cancelto{0}{\int_{\partial D} \phi_i n \cdot \nabla\phi_i} \\ \int_D |\nabla\phi_i|^2 &= \lambda_i \int_D |\phi_i|^2. \end{aligned}$$

So, in particular

$$\int_D |\phi_i|^2 = \frac{1}{\lambda_i}\int_D |\nabla\phi_i|^2$$

and further

$$\int_D |\phi_i|^2 \leq \frac{1}{\lambda_1}\int_D |\nabla\phi_i|^2$$

for all i. Now, consider our function $u = \sum_{i=1}^{\infty} a_i\phi_i$. We then have

$$\begin{aligned}
\int_D |u|^2 &= \int_D \left| \sum_{i=1}^{\infty} a_i \phi_i \right|^2 \\
&\leq \int_D \sum_{i=1}^{\infty} |a_i|^2 |\phi_i|^2 \\
&= \sum_{i=1}^{\infty} |a_i|^2 \int_D |\phi_i|^2 \\
&\leq \sum_{i=1}^{\infty} |a_i|^2 \frac{1}{\lambda_1} \int_D |\nabla \phi_i|^2 \\
&= \frac{1}{\lambda_1} \int_D \sum_{i=1}^{\infty} |a_i|^2 |\nabla \phi_i|^2 \\
&= \frac{1}{\lambda_1} \int_D \sum_{i=1}^{\infty} |\nabla a_i \phi_i|^2
\end{aligned}$$

Note here that we can make the claim $\sum_{i=1}^{\infty} |\nabla a_i \phi_i|^2 = |\sum_{i=1}^{\infty} \nabla a_i \phi_i|$ since the eigenfunctions are orthogonal. So, this then leads to,

$$\int_D |u|^2 \leq \frac{1}{\lambda_1} \int_D \sum_{i=1}^{\infty} |\nabla a_i \phi_i|^2 = \frac{1}{\lambda_1} \int_D |\nabla u|^2.$$

(b) In this case, we know that the eigenfunctions are given by

$$\phi_{mn} = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

with corresponding eigenvalues

$$\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

So, the smallest eigenvalue is clearly λ_{11} which yields a constant of

$$C = \frac{1}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

for the Poincaré Inequality.

Spring 2001, #5 (a) Show that the solution of the heat equation

$$u_t = u_{xx}, \qquad -\infty < x < \infty$$

with square-integrable initial data $u(x,0) = f(x)$, decays in time, and there is a constant α independent of f and t such that for all $t > 0$

$$\max_x |u_x(x,t)| \le \alpha t^{-3/4} \left(\int_x |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

(b) Consider the solution ρ of the transport equation $\rho_t + u\rho_x = 0$ with square-integrable initial data $\rho(x,0) = \rho_0(x)$ and the velocity u from part (a). Show that $\rho(x,t)$ remains square-integrable for all finite time

$$\int_{\mathbb{R}} |\rho(x,t)|^2 \, dx \le e^{Ct^{1/4}} \int_{\mathbb{R}} |\rho_0(x)|^2 \, dx$$

where C does not depend on ρ_0.

Let's start by looking at the solution to the heat equation, given by

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-|x-y|^2/4t} f(y) \, dy.$$

Now we check to see whether the solution is bounded

$$\begin{aligned}
|u(x,t)| &= \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-|x-y|^2/4t} f(y) \, dy \right| \\
&\le \frac{1}{\sqrt{4\pi t}} \left(\int_{\mathbb{R}} e^{-|x-y|^2/4t} \right)^{1/2} \|f\|_{L^2} \\
&= \frac{1}{\sqrt{4\pi t}} \left(\sqrt{4\pi t} \right)^{1/2} \|f\|_{L^2} \\
&= \frac{1}{(4\pi t)^{1/4}} \|f\|_{L^2}.
\end{aligned}$$

Note that as t increases, we have the bound on $|u(x,t)|$ gets smaller and smaller (and indeed goes to 0 as $t \to \infty$). Now let's demonstrate the bound on $|u_x(x,t)|$.

Spring 2001, #7 Consider the following system of PDEs:

$$f_t + f_x = g^2 - f^2$$
$$g_t - g_x = f^2 - g$$

(a) Find a system of ODEs that describes traveling wave solutions of the PDE system; i.e. for solutions of the form $f(x,t) = f(x - st)$ and $g(x,t) = g(x - st)$.

(b) Analyze the stationary points and draw the phase plane for this ODE system in the standing wave case $s = 0$.

(a) Plugging in solutions of the form $f(x,t) = f(x - st)$ and $g(x,t) = g(x - st)$, we find that

$$\begin{aligned} -sf' + f' &= g^2 - f^2 \\ f' &= \frac{g^2 - f^2}{1 - s} \\ -sg' - g' &= f^2 - g \\ g' &= \frac{f^2 - g}{-1 - s}. \end{aligned}$$

(b) When $s = 0$, we have

$$\begin{aligned} f' &= g^2 - f^2 \\ g' &= g - f^2. \end{aligned}$$

This system has stationary points when $f^2 = g^2$ and $g = f^2$. So, at $(f,g) = (0,0)$, $(1,1)$ and $(-1,1)$. We may come back and carry out the phase plane analysis later.

Fall 2000, #1 Consider the Dirichlet problem in a bounded domain $\mathcal{D} \subset \mathbb{R}^n$ with smooth boundary S,

$$\begin{cases} \Delta u + a(x)u = f(x) & x \in \mathcal{D} \\ u = 0 & x \in S. \end{cases}$$

(a) Assuming that $|a(x)|$ is small enough, prove the uniqueness of the classical solution.
(b) Prove the existence of the solution in the Sobolev space $H^1(\mathcal{D})$ assuming that $f \in L_2(\mathcal{D})$.

(a) Here we look at an energy argument to demonstrate uniqueness of the classical solution. Let u and v be two solutions to the equation and set $w = u - v$. Then w satisfies

$$\begin{cases} \Delta w + a(x)w = 0 & x \in \mathcal{D} \\ w = 0 & x \in S. \end{cases}$$

Then we have

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \Delta w w + a(x)w^2 \\ &= -\int_{\mathcal{D}} |\nabla w|^2 + \int_{\mathcal{D}} a(x)w^2 \end{aligned}$$

and so

$$\begin{aligned} \int_{\mathcal{D}} |\nabla w|^2 &= \int_{\mathcal{D}} a(x)w^2 \\ &\leq \|a(x)\|_\infty \int_{\mathcal{D}} w^2 \\ &\leq \|a(x)\|_\infty c \int_{\mathcal{D}} |\nabla w|^2 \end{aligned}$$

where c is some constant coming from the Poincaré inequality. So, if $\|a(x)\|_\infty < \frac{1}{c}$ we must have that $\int_{\mathcal{D}} |\nabla w|^2 = 0$ and since $w = 0$ for $x \in S$, we have that $w \equiv 0$ and so the classical solution is unique.

(b) In order to demonstrate the existence of the solution in $H^1(\mathcal{D})$, we first need to find the weak formulation of the problem. We start by defining the space $H_0^1(\mathcal{D}) = \{u \in H^1(\mathcal{D}) : u = 0 \text{ on } S\}$. Take $v \in H_0^1(\mathcal{D})$, then

$$\int_{\mathcal{D}} \Delta uv + a(x)uv = \int_{\mathcal{D}} fv$$

$$\int_{\mathcal{D}} -\nabla u \cdot \nabla v + a(x)uv + \underbrace{\int_S v(n \cdot \nabla u)}_{\to 0} = \int_{\mathcal{D}} fv.$$

The Lax-Milgram Lemma gives us well-posedness of the weak formulation (and hence existence of the solution in $H^1(\mathcal{D})$). So, we need to show that the bilinear form is coercive and bounded and that the linear form is also bounded. We begin by multiplying both sides of the weak form by -1 (needed to demonstrate coercivity). We start by demonstrating boundedness of the bilinear form:

$$\begin{aligned} \left| \int_{\mathcal{D}} \nabla u \cdot \nabla v - a(x)uv \right| &\leq \int_{\mathcal{D}} |\nabla u \cdot \nabla v| + \|a(x)\|_\infty \int_{\mathcal{D}} |uv| \\ &\leq c_1 \|\nabla u\|_{L^2(\mathcal{D})} \|\nabla v\|_{L^2(\mathcal{D})} + \|a\|_\infty c_2 \|u\|_{L^2(\mathcal{D})} \|v\|_{L^2(\mathcal{D})} \\ &\leq d \|u\|_{H^1(\mathcal{D})} \|v\|_{H^1(\mathcal{D})} \end{aligned}$$

Now we consider coercivity,

$$\int_{\mathcal{D}} \nabla v \cdot \nabla v - a(x)v^2 \geq \int_{\mathcal{D}} |\nabla v|^2 + \int_{\mathcal{D}} a(x)v^2$$

Note that in part (a) we demonstrated that

$$\int_{\mathcal{D}} |\nabla v|^2 \geq \frac{1}{c} \int_{\mathcal{D}} v^2$$

where c is the constant from the Poincaré inequality, giving us

$$\begin{aligned} \int_{\mathcal{D}} \nabla v \cdot \nabla v - a(x)v^2 &\geq \frac{1}{c} \int_{\mathcal{D}} v^2 - \int_{\mathcal{D}} a(x)v^2 \\ &= \int_{\mathcal{D}} \left(\frac{1}{c} - a(x) \right) v^2 \end{aligned}$$

hence giving us coercivity of the bilinear form if $|a(x)|$ is small enough, specifically we need $\frac{1}{c} - \gamma \geq 0$ where $\gamma = \sup_{x \in D} |a(x)|$. Finally, we can demonstrate boundedness of the linear form,

$$\begin{aligned} \left| \int_{\mathcal{D}} -fv \right| &\leq \|f\|_{L^2(\mathcal{D})} \|v\|_{L^2(\mathcal{D})} \\ &\leq c\|v\|_{H^1(\mathcal{D})} \end{aligned}$$

and so the conditions of the Lax-Milgram Lemma are satisfied and hence we have the existence of a solution.

Fall 2000, #2 Consider the Cauchy problem

$$\frac{\partial u}{\partial t} - \Delta u + u^2(x,t) = f(x,t) \quad x \in \mathbb{R}^n,\ 0 < t < T$$
$$u(x,0) = 0.$$

Prove the uniqueness of the classical bounded solution assuming that T is small enough.

Let u_1 and u_2 be two solutions to the PDE and let $w = u_1 - u_2$. Then w satisfies

$$\begin{aligned} &w_t - \Delta w + u_1^2 - u_2^2 = 0, \quad x \in \mathbb{R}^n,\ 0 < t < T \\ &w(x,0) = 0. \end{aligned}$$

Let's start by noting that we can also write the PDE as $w_t - \Delta w + w(u_1 + u_2) = 0$. Since the solutions are bounded, we have that $\lim_{\|x\| \to \infty} w(x,t) = 0$. This will result in boundary terms of w canceling when we integrate by parts. Now we define the energy $E(t) = \frac{1}{2}\int_{\mathbb{R}^n} w^2\ dx$. Now,

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^n} w w_t\ dx \\ &= \int_{\mathbb{R}^n} w(\Delta w - w(u_1 + u_2))\ dx \\ &= \int_{\mathbb{R}^n} -|\nabla w|^2 - w^2(u_1 + u_2)\ dx \\ &\leq -\int_{\mathbb{R}^n} w^2(u_1 + u_2)\ dx \\ &\leq \|u_1 + u_2\|_{L^\infty} \int_{\mathbb{R}^n} w^2\ dx. \end{aligned}$$

Now, Gronwall's inequality gives us that

$$E(t) \leq E(0) e^{\int \|u_1 + u_2\|\ ds}$$

but we have that $E(0) = 0$, so $E(t) \leq 0$ and hence $w \equiv 0$ and so $u_1 = u_2$ and solutions to the PDE are unique.

Fall 2000, #4 Consider the following functional

$$F(v) = \int\int\int_D \left[\sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2 + \alpha \left(\sum_{j=1}^{3} v_j^2(x) - 1 \right)^2 \right] \, dx,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $v(x) = (v_1(x), v_2(x), v_3(x))$, D is a bounded domain in $\mathbb{R}^3$ with a smooth boundary S, and $\alpha > 0$ is a constant. Let $u(x) = (u_1(x), u_2(x), u_3(x))$ be the minimizer of $F(v)$ among all smooth functions satisfying the Dirichlet condition, $u_k(x) = \phi_k(x)$, $k = 1, 2, 3$. Derive the system of differential equations that $u(x)$ satisfies.

We look at

$$\begin{aligned}
F'(v)u &= \lim_{\epsilon \to 0} \frac{F(v + \epsilon u) - F(v)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int\int\int_D \left[\sum_{j,k=1}^{3} \left(\frac{\partial (v + \epsilon u)_j}{\partial x_k} \right)^2 + \alpha \left(|v + \epsilon u|^2 - 1 \right)^2 \right] \\
&\quad - \left[\sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2 + \alpha \left(|v|^2 - 1 \right)^2 \right] \, dx \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int\int\int_D \left[\sum_{j,k=1}^{3} \cancel{(\partial_{x_k} v_j)^2} + 2\epsilon \partial_{x_k} v_j \partial_{x_k} u_j + \epsilon^2 (\partial_{x_k} u_j)^2 + \alpha \left(|v + \epsilon u|^4 - 2|v + \epsilon u|^2 + 1 \right) \right] \\
&\quad - \left[\cancel{\sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2} + \alpha \left(|v|^4 - 2|v|^2 + 1 \right) \right] \, dx \\
&= \lim_{\epsilon \to 0} \int\int\int_D \left[\sum_{j,k=1}^{3} 2 \partial_{x_k} v_j \partial_{x_k} u_j + \frac{\alpha}{\epsilon} \left(|v + \epsilon u|^4 - 2|v + \epsilon u|^2 - |v|^4 + 2|v|^2 \right) \right] \, dx
\end{aligned}$$

Fall 2000, #5 Consider the eigenvalue problem on the interval $[0,1]$,

$$\begin{aligned} y''(t) + p(t)y(t) &= \lambda y(t), \\ y(0) = y(1) &= 0. \end{aligned}$$

(a) Prove that all eigenvalues λ are simple.
(b) Prove that there is at most a finite number of negative eigenvalues.

(a) Suppose the eigenvalues are not simple. Then, for some λ there exists two eigenfunctions, f and g, such that for $Ty = y'' + py - \lambda y$, we have $Tf = 0$ and $Tg = 0$. So, we can now write that

$$\begin{aligned} 0 &= gTf - fTg \\ &= g(f'' + pf - \lambda f) - f(g'' + pg - \lambda g) \\ &= gf'' - fg'' \\ &= (gf' - fg')'. \end{aligned}$$

Integrating both sides we find that $gf' - fg' = c$, where c is a constant. However, our boundary condition gives us that $(gf' - fg')(1) = 0 = (gf' - fg')(0) \Rightarrow c = 0$ yielding $gf' = fg'$. Proceeding along the lines of separation of variables, we see that we can write $\frac{f'}{f} = \frac{g'}{g} = 0$. Integrating this equation gives us, $\ln f = \ln g + c$ and hence $f = cg$. So we have that there is actually only one eigenfunction and hence the eigenvalues must be simple.

Another argument (that I prefer) for the first part of part (a). Take two eigenfunctions f and g and use the fact that T is self-adjoint. Now consider $(Tf, g) - (f, Tg) = 0$. Carrying through similar arguments as above, we will find that $gf' - fg' = 0$.

(b) Let's check whether the operator $Ly = y'' + py$ is self-adjoint.

$$\begin{aligned} (Ly, z) &= \int (y'' + py)z \\ &= -\int y'z' + pyz \\ &= \int yz'' + ypz \\ &= (y, Lz). \end{aligned}$$

Since the operator is self-adjoint, we can conclude that the set of eigenvalues has no finite cluster points. Now we need to demonstrate that the set of eigenvalues is bounded below. Suppose that the eigenvalues are not bounded below. Then clearly we can find a λ such that $\lambda < p(t)$ for $t \in [0,1]$. Consider

$$\begin{aligned} 0 &= (Ly - \lambda y, y) \\ &= \int (y'' + py - \lambda y)y \\ &= \int -|y'|^2 + (p - \lambda)y^2 \end{aligned}$$

NOTE Online version has a typo. The operator should be $L = -y'' + py$ (not this does not affect any of the conclusions previously drawn). In this case we have $0 = (Ly - \lambda y, y) = \int |y'|^2 + (p - \lambda)y^2$. This equality can only be satisfied if $y \equiv 0$, in which case y is not an eigenvector, and so we have that the set of eigenvalues must be bounded below.

Fall 2000, #6 Consider the initial boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au(x,t) = 0 & t > 0, x > 0, \\ u(x,0) = 0, & x > 0 \\ u(0,t) = g(t), & t > 0. \end{cases}$$

where $g(t)$ is a continuous function with compact support and a is a constant. Find the explicit solution of this problem.

Let's take the Laplace transform of the equation,

$$\begin{aligned} \mathcal{L}(u_t - \partial_x^2 u + au) &= s\overline{u} - \overline{u}'' + a\overline{u} \\ &= \overline{u}'' - (s+a)u \\ &= 0. \end{aligned}$$

This ODE has characteristic equation $r^2 - (s+a) = 0$ and hence $r_{1,2} = \pm\sqrt{s+a}$, and so the ODE has solution given by

$$\overline{u}(x,s) = c_1 e^{\sqrt{s+a}x} + c_2 e^{-\sqrt{s+a}x}.$$

Note that if $a \geq 0$, we have that $r_{1,2} \in \mathbb{R}$. Note that we have a second order equation, but only one boundary condition, $\overline{u}(0,s) = \overline{g}s$. We need another constraint on our system in order to find both c_1 and c_2. We choose that we want $\overline{u}$ to be bounded, which requires that $c_1 = 0$ when $r_{1,2} \in \mathbb{R}$. Then, $\overline{u}(0,s) = \overline{g}(s) = c_2$. So, we then have that

$$\overline{u}(x,s) = \overline{g}(s) e^{-\sqrt{s+a}\ x}$$

and so

$$u(x,t) = \mathcal{L}^{-1}\left(\overline{g}(s) e^{-\sqrt{s+a}\ x}\right) = g * \mathcal{L}^{-1}\left(e^{-\sqrt{s+a}\ x}\right).$$

Fall 2000, #7 Consider the following system of ODEs

$$u_t = au - buv$$
$$v_t = -cv + duv$$

in which a, b, c, d are constants. For the phase plane region $R^{2+} = \{(u,v) : u > 0, v > 0\}$, do the following
a) Find all stationary points.
b) Analyze their type.
c) Draw a global picture of the solution set.
d) Show that R^{2+} is an invariant set for this flow.

---

a) We write the set of equations as

$$u_t = u(a - bv)$$
$$v_t = v(-c + du).$$

Here we see that the system has the following stationary points: $(0,0)$ and $(c/d, a/b)$.

b) Now we find the linear approximation to the system about the two stationary points. Let $f_1(u,v) = au - buv$ and $f_2(u,v) = -cv + duv$. Then we have that $\partial_u f_1 = a - bv$, $\partial_v f_1 = -bu$ and $\partial_u f_2 = dv$, $\partial_v f_2 = -c + du$. So, Taylor expanding about the critical points, $\xi \in \mathbb{R}^2$, we have,

$$u_t = f_1(\xi) + \partial_u f_1(\xi)(u - \xi_1) + \partial_v f_1(\xi)(v - \xi_2) + H.O.T.$$
$$v_t = f_2(\xi) + \partial_u f_2(\xi)(u - \xi_1) + \partial_v f_2(\xi)(v - \xi_2) + H.O.T.$$

When $\xi = (0,0)$ we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + H.O.T.$$

So, for the stationary point $(0,0)$ we have eigenvalues of a and $-c$ with eigenvectors $(1,0)^t$ and $(0,1)^t$. If $a(-c) < 0$, then $(0,0)$ is a saddle point. If $a = -c$ then $(0,0)$ is a proper node. In all other cases, $(0,0)$ is an improper node. We assume that $a \neq 0 \neq c$.

Now let's look at the point $\xi = (c/d, a/b)$. Here we are assuming $d \neq 0 \neq b$. Note that as before, we must have that both a and b are non-zero. From the Taylor expansion, we find

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix} \begin{pmatrix} u - \frac{c}{d} \\ v - \frac{a}{b} \end{pmatrix} + H.O.T. \tag{1}$$

The eigenvalues of this matrix are given by

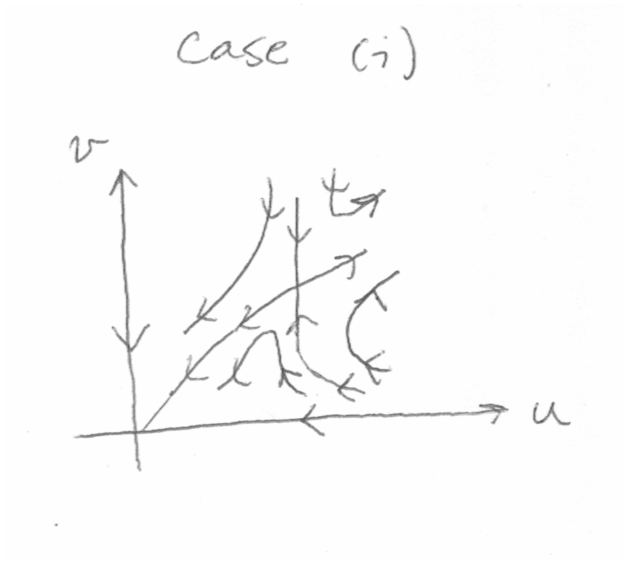
$$\begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{ad}{b} & -\lambda \end{vmatrix} = \lambda^2 + \frac{bc}{d}\frac{ad}{b} = 0 \Rightarrow \lambda = \pm\sqrt{-ac}.$$

If $ac > 0$, then we have that the eigenvalues are $\pm i\sqrt{ac}$ and the critical point is a center. If $ac < 0$, then the eigenvalues are given by $\pm\sqrt{ac}$, indicating that the stationary point $(c/d, a/b)$ is a saddle. In the case that the stationary point is a saddle, its corresponding eigenvectors are $(1, \frac{\sqrt{ad}}{\sqrt{cb}})^t$ and $(-\frac{b\sqrt{c}}{\sqrt{ad}}, 1)^t$.

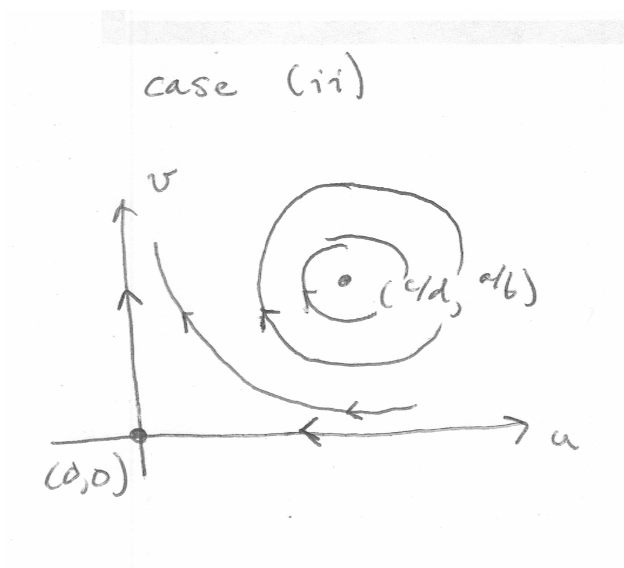
c) There are two main cases for possible diagrams for the phase plane.

(i) $(0,0)$ is a node and $(c/d, a/b)$ is a saddle.
(ii) $(0,0)$ is a saddle and $(c/d, a/b)$ is a center

In case (i), the node may be proper or improper and may be stable or unstable and the saddle point will change accordingly. We cannot know exactly how the saddle point is oriented without having values for a, b, c and d, so we approximate it in the sketch below (assuming the node is stable). If the node were unstable, we would need to reverse all the arrows on the diagram.



In case (ii) either manifold may be stable or unstable for the saddle point. We only illustrate one case (as the other simply involves changing the direction of the solution curves).



(d) Note that any solution starting on one of the axes of the phase portrait must remain on that axis. Further, uniqueness of solutions tells us that solution trajectories may not cross. For this reason, no solution in the region R^{2+} may ever leave the set and hence it is invariant.

Spring 2000, #3 Consider the initial-boundary value problem for $u = u(x,y,t)$

$$u_t = \Delta u - u$$

for $(x,y) \in [0,2\pi]^2$, with periodic boundary conditions and with $u(x,y,0) = u_0(x,y)$ in which u_0 is periodic. Find an asymptotic expansion for u for t large with terms tending to zero increasingly rapidly as $t \to \infty$.

Taking the Fourier transform of the problem, we find that

$$\sum_{k \in \mathbb{Z}^2} e^{ik\cdot x} \hat{u}_t(k) = \sum_{k \in \mathbb{Z}^2} -4\pi^2 |k|^2 e^{ik\cdot x} \hat{u}(k) - e^{ik\cdot x}\hat{u}(k)$$

and for a fixed $k \in \mathbb{Z}^2$, we then have that

$$\hat{u}_t(k) = -(1 + 4\pi^2|k|^2)\hat{u}(k).$$

So, we find that

$$\hat{u}(k) = \hat{u_0}(k) e^{-(1+4\pi^2|k|^2)t},$$

giving us that

$$u(x,t) = \sum_{k \in \mathbb{Z}^2} e^{ik\cdot x} \hat{u}_0(k) e^{-(1+4\pi^2|k|^2)t}.$$

Spring 2000, #5 Look for a traveling wave solution to the PDE

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

of the form $u(x,t) = v(x - ct)$. In particular, you should find an ODE for v. Under the assumption that v goes to a constant as $|x| \to \infty$, describe the form of the solution.

Plug the form of the solution, $u(x,t) = v(x - ct)$, into the PDE,

$$c^2 v'' + (v^2)'' = -v^{(4)}$$

We can integrate this ODE twice, finding

$$\begin{aligned} c^2 v' + (v^2)' &= -v''' + c_1 \\ c^2 v + v^2 &= -v'' + c_1 \xi + c_2 \end{aligned}$$

where v is a function of ξ, $v(\xi)$. Now, we have that $v \to d$ as $|x| \to \infty$. Using this condition, we find that $c_1 = 0$ and $c^2 d + d^2 = c_2$. This leaves us with the ODE

$$v'' + c^2 v + v^2 = c^2 d + d^2.$$

In order to determine the behaviour of the solutions to this ODE, lets conduct a phase plane analysis. We start by letting $v_1 = v$ and $v_2 = v'$. Then we have that

$$\begin{cases} v_1' = v_2 \\ v_2' = -c^2 v_1 - v_1^2 + c^2 d + d^2 \end{cases}$$

which has stationary points at $(v_1, v_2) = (d, 0)$ (there is most likely one more). Phase plane analysis may be done later.

Fall 1999, #7 Consider the differential operator

$$L = (d/dx)^2 + 2(d/dx)$$

with $x \in [0,1]$ and $u(0) = u(1) = 0$.

(i) Find a function $\phi = \phi(x)$ for which L is self-adjoint in the norm

$$\|u\|^2 = \int_0^1 u^2 \phi \; dx.$$

(ii) If $a < 0$, show that $L + aI$ is invertible.

(iii) Find a value of a so that $(L + aI)u = 0$ has a nontrivial solution.

(i)

$$\begin{aligned}
(Lu, v) &= \int_0^1 (u'' + 2u')v\phi \\
&= \int_0^1 u''(v\phi) + 2u'(v\phi) \\
&= -\int_0^1 u'(v\phi)' + \cancelto{0}{u'v\phi|_0^1} - \int_0^1 2u(v\phi)' + \cancelto{0}{uv\phi|_0^1} \\
&= \int_0^1 u(v\phi)'' - \cancelto{0}{u(v\phi)'|_0^1} - \int_0^1 2u(v\phi)' \\
&= \int_0^1 u(v'\phi + v\phi') - 2u(v'\phi + v\phi') \\
&= \int_0^1 u(v''\phi + v'\phi' + v'\phi' + v\phi'') - 2uv'\phi - 2uv\phi' \\
&= \int_0^1 uv''\phi + 2uv'\phi' + uv\phi'' - 2uv'\phi - 2uv\phi' \\
&= \int_0^1 \left(v'' + 2v'\frac{\phi'}{\phi} + v\frac{\phi''}{\phi} - 2v' - 2v\frac{\phi'}{\phi} \right) u\phi \\
&= \int_0^1 \left(v'' + \left[2\frac{\phi'}{\phi} - 2 \right] v' + \left[\frac{\phi''}{\phi} - 2\frac{\phi'}{\phi} \right] v \right) u\phi
\end{aligned}$$

In order to have $(Lu, v) = (u, Lv)$, we need that

$$\begin{aligned}
2\frac{\phi'}{\phi} - 2 &= 2 \\
\frac{\phi''}{\phi} - 2\frac{\phi'}{\phi} &= 0
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\phi' &= 2\phi \\
\phi'' &= 2\phi'.
\end{aligned}$$

Clearly, both of the conditions are satisfied if $\phi(x) = e^{2x}$ and hence L is self-adjoint under this norm.

(ii) For $L + aI$ to be invertible, we need the operator to have a trivial null space. Suppose it is not, that is there is a $u \neq 0$ such that $(L + aI)u = 0$. Then $(Lu + au, u)_\phi = 0$ and so

$$\begin{aligned}
0 &= (Lu + au, u)_\phi \\
&= \int (u'' + 2u' + au)ue^{2x} \\
&= \int u''ue^{2x} + 2u'ue^{2x} + au^2e^{2x} \\
&= \int -u'(ue^{2x})' + 2u'ue^{2x} + au^2e^{2x} \\
&= \int -u'(u'e^{2x} + 2ue^{2x}) + 2u'ue^{2x} + au^2e^{2x} \\
&= \int (-|u'|^2 + au^2)e^{2x}.
\end{aligned}$$

Note that if $a < 0$, then we must have that $u = 0$ and hence the nullspace is trivial. This gives uniqueness to the eigenvalue problem. Existence comes from the fact that the operator is self-adjoint in the weighted inner product space and hence the operator can be inverted.

(iii) We are looking for an eigenvalue/eigenvector pair to the operator L. The obvious choice for the eigenvector is $u = e^x$. We find the corresponding eigenvalue by applying the differential operator,

$$Lu = e^x + 2e^x = 3e^x.$$

So, if $a = -3$ then $(L + aI)u = 0$ has a nontrivial solution, $u = e^x$.