



Calculation of complex singular solutions to the 3D incompressible Euler equations

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ABSTRACT

This paper presents numerical computations of complex singular solutions to the 3D incompressible Euler equations. The Euler solutions found here consist of a complex valued velocity field $\mathbf{u}_+$ that contains all positive wavenumbers; $\mathbf{u}_+$ satisfies the usual Euler equations but with complex initial data. The real valued velocity $\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-$ (where $\mathbf{u}_- = \bar{\mathbf{u}}_+$) is an approximate singular solution to the Euler equations under Moore's approximation. The method for computing singular solutions is an extension of that in Caflisch (1993) [25] for axisymmetric flow with swirl, but with several improvements that prevent the extreme magnification of round-off error which affected previous computations. This enables the first clean analysis of the singular surface in three-dimensional complex space. We find singularities in the velocity field of the form $\mathbf{u}_+ \sim \xi^{\alpha-1}$ for α near $3/2$ and $\mathbf{u}_+ \sim \log \xi$, where $\xi = 0$ denotes the singularity surface. The logarithmic singular surface is related to the double exponential growth of vorticity observed in recent numerical simulations.

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1. Introduction

The question of finite time singularity formation from smooth initial data for solutions to the 3D incompressible Euler equations has been an important open problem for over 50 years. The physical significance of Euler singularities was first proposed by Onsager [1] who suggested that they could play a role in the cascade of energy from large to small scales, which is an essential feature of turbulence. Onsager further showed that inviscid energy dissipation could be produced by a sufficiently strong Euler singularity, i.e., one in which the Holder singularity exponent α for the velocity field is no larger than $1/3$. This result was rigorously derived in [2,3], and generalised in [4,5]. Reviews in [6–8] further discuss the physical implications of Euler singularities and their relation to turbulence.

The first important analytic result on singularities in fluid flows was that of Beale, Kato and Majda (BKM) [9] who showed that singularity formation at time t_* implies that $\int^{t_*} \sup_{\mathbf{x}} |\boldsymbol{\omega}(\mathbf{x}, t)| dt = \infty$ for vorticity $\boldsymbol{\omega}$. Constantin, Fefferman and Majda [10] and others [11,12] derived further necessary conditions for finite time singularity formation. More recently, Deng, Hou and Yu [13] have

derived a localised singularity criterion that is especially useful for testing candidate singularities in numerical calculations.

The numerical search for singularities was initiated by Brachet et al. [14,15] who performed direct numerical simulations for the high-symmetry Taylor–Green flow. As part of their investigation, Brachet and coworkers traced the location of complex space singularities (i.e., singularities in the complex analytic continuation of the space variables). A least squares fit was used to ascertain the exponential decay of the Fourier series and approximate the width $\delta(t)$ of the analyticity strip in complex space. It was found that $\delta(t)$ decays exponentially in time, which is evidence against finite time singularities in this flow (see also [16]). Similar results were obtained in [16] for Kida–Pelz flow, which is another high-symmetry flow. Further numerical studies of possible singularity formation in vortex tubes, axisymmetric flow with swirl, Boussinesq flow and other configurations have been carried out (e.g., in [17–22]). So far none of these numerical or analytic constructions has provided convincing evidence for development of Euler singularities from standard initial data.

The investigations of complex space singularities are partly motivated by the fact that real Euler singularities (if they exist) are necessarily preceded by the formation of complex-space singularities, which move onto the real-space domain. Li and Sinai [23] investigated complex singular solutions to the 3D Navier Stokes equations. Pauls et al. [24] numerically investigated complex space

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singularities to the 2D Euler equations in the short time asymptotic regime, when such singularities are far from the real domain. Although singularities in 2D Euler do not reach the real domain, their close presence generates small space scales. A drawback of Pauls et al. study is the need to use ultra high precision arithmetic – up to 100 digits – to accurately resolve the singularities due to their large distance from the real-space domain. Caflisch [25] and Caflisch and Siegel [26] constructed complex singular solutions to the Euler equations for axisymmetric flow with swirl. Their calculations also required ultra-high precision, although for a different reason than Pauls et al. The requirement of ultra high precision precludes accurate detection of singularities for fully 3D flow, since the computations are prohibitively expensive. A major aim of this paper is to resolve this difficulty.

In this paper, we compute complex singular solutions to the fully 3D Euler equations. The Euler solutions found here consist of a complex valued velocity field $\mathbf{u} = \mathbf{u}_+$ where

$$\mathbf{u}_+ = \mathbf{u}_+(x, y, z, t) = \sum_{k>0} \hat{\mathbf{u}}_k e^{ik \cdot \mathbf{x}}. \quad (1.1)$$

Here the notation $\mathbf{a} < \mathbf{b}$ for vector quantities $\mathbf{a}$ and $\mathbf{b}$ signifies that the inequality holds for at least one component, with the other components satisfying $a_i \leq b_i$. The motivation for considering complex-valued solutions is that they provide simple examples of singular solutions to the 3D Euler equations and may indicate generic singularity types (i.e., those which are stable with respect to perturbations). Additionally, the real velocity

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-,$$

where $\mathbf{u}_- = \bar{\mathbf{u}}_+$, is an approximate solution to the Euler equations under Moore's approximation. Moore's approximation was originally applied to fluid interface problems in [27] and extended as a general approximation for singularity formation in [25]. If we denote the Euler equations by

$$E[\mathbf{u}] = 0, \quad (1.2)$$

then Moore's approximation is to replace (1.2) by

$$E[\mathbf{u}_+] = 0 \quad (1.3)$$

$$E[\mathbf{u}_-] = 0. \quad (1.4)$$

Thus the complex velocities $\mathbf{u}_+$, $\mathbf{u}_-$ are exact solutions to the usual Euler equations, but with complex initial data. Moore's approximation neglects products of $+$ and $-$ terms, which is equivalent to neglecting energy backflow from high to low wavenumber modes. Since singularity formation is dominated by energy outflow to high wavenumber modes, we expect the approximation to at least qualitatively describe singularity formation.

In [26], a method is proposed in which a real solution to the 3D Euler equations is produced as a sum

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_- + \tilde{\mathbf{u}}, \quad (1.5)$$

where $\mathbf{u}_+$ and $\mathbf{u}_-$ satisfy (1.4) and $\tilde{\mathbf{u}}$ is a small real-valued remainder term. Since $\mathbf{u}_+$ and $\mathbf{u}_-$ are exact solutions of the Euler equations, $\tilde{\mathbf{u}}$ satisfies a system of equations in which the forcing terms are quadratic, i.e. products of $+$ and $-$ terms. The proposed construction of $\tilde{\mathbf{u}}$ is based on (rigorous) perturbation methods and requires $\mathbf{u}_+$ and $\mathbf{u}_-$ to satisfy certain properties. Specifically, the singular parts of $\mathbf{u}_+$ and $\mathbf{u}_-$ need to be of small amplitude $O(\epsilon)$. Additionally, we ask that the leading order real velocity $\mathbf{u}_+ + \mathbf{u}_-$ satisfy consistency conditions such as the Beale–Kato–Majda condition. Validity of the expansion (1.5) up to and including the singularity time requires an analysis showing that the singularity of $\tilde{\mathbf{u}}$ is the same or weaker than that of $\mathbf{u}_+$ and $\mathbf{u}_-$, but with smaller amplitude. This paper is not concerned with this analysis, but rather with the accurate and efficient calculation of $\mathbf{u}_+$ with the desired properties.

A major advantage of the complex Euler equations is that there exist special solutions of the form of a travelling wave, e.g.,

$$\mathbf{u}_+ = \mathbf{u}_+(x - i\sigma t, y, z) \quad (1.6)$$

which greatly reduces the degrees of freedom in computation of the solution. When σ is real, the solution consists of pure growing modes. The computation of complex travelling wave solutions was initiated in [25,26] for axisymmetric flow with swirl in a periodic annulus for which $0 < r_1 < r < r_2$ and $0 < z < 2\pi$, where (r, θ, z) is a cylindrical polar coordinate system. They constructed a complex, upper-analytic traveling wave solution of the form $\mathbf{u} = \bar{\mathbf{u}}(r) + \mathbf{u}_+(r, z, t)$ where

$$\mathbf{u}_+ = \sum_{k=1}^{\infty} \hat{\mathbf{u}}_k(r) \exp(ik(z - i\sigma t)) \quad (1.7)$$

and the travelling wave moves at speed σ in the imaginary z -direction. The construction of [25,26] depends on a steady background flow $\bar{\mathbf{u}} = (0, \bar{u}_\theta, \bar{u}_z)(r)$, chosen to satisfy the Rayleigh criterion for instability [28], and on an unstable eigenmode $\mathbf{u}_1 = \hat{\mathbf{u}}_1(r) e^{iz + \sigma t}$, in which $\hat{\mathbf{u}}_1 = (\hat{u}_{1r}, \hat{u}_{1\theta}, \hat{u}_{1z})$.

The numerical construction of $\mathbf{u}_+$ was carried out in [25] for pure swirling background flow $\bar{\mathbf{u}} = (0, \bar{u}_\theta, 0)(r)$. The asymptotics of the Fourier components was fitted to obtain a singularity $\mathbf{u} \sim \mathbf{x}^{\alpha-1}$ with $\alpha = 2/3$, so that the velocity itself becomes infinite at the singularity. The amplitude of $\mathbf{u}_+$ was not small, however, so that there was no possibility to create a real singular solution by a perturbation construction.

Caflisch and Siegel [26] included axial flow in the background, and let the amount of swirl go to 0. They were motivated by a related perturbation approach to singularity formation on vortex sheets in [29,30]. In this limit, they found that the amplitude of the singularity goes to 0. However, the addition of axial flow caused the singularity exponent α to change to $1/2$, which does not satisfy the BKM criterion. Another major difficulty was that the numerical construction in [26,25] involves an iteration that significantly magnifies roundoff error, which was controlled using ultra-high precision (e.g. precision 10^{-100}) with the MPFUN package from David Bailey [31]. This increased the computational complexity by a factor of around 100. These difficulties are overcome by the method described in this paper.

In this paper we extend the method of [25,26] to compute strongly nonlinear, small amplitude travelling wave solutions to the full 3D Euler equations. The new calculations rely on several simplifications and improvements to the method which are found to prevent the extreme magnification of roundoff error. In particular, an analytic forcing term is employed in place of a background flow $\bar{\mathbf{u}}$ and unstable eigenmode $\hat{\mathbf{u}}_1$ to generate instability. The calculations with forcing are much simpler, lead to cleaner Fourier spectra and singularity fits, and are found to remove the need for extended precision arithmetic. The forcing also provides a simple way to control the singularity amplitude. The computations are devoid of truncation or aliasing error, in a sense which is explained in Section 2. We are therefore able to obtain the first clean analysis of the singular surface in the three dimensional complex space C^3 using standard double precision arithmetic, making the 3D calculations feasible.

After a rotation of coordinates, the singularity in the velocity field $\mathbf{u}$ is found to have a simple form. Define $\xi = x' - i\sigma t + i\mathbf{Y}'\mathbf{M}\mathbf{Y}'$, in which $\mathbf{Y}' = (y', z')^T$, $\mathbf{M}$ is a positive definite 2×2 matrix and x', y', z' are orthogonal spatial coordinates centred at the singularity. We find singularities of the form

$$\mathbf{u}_+ \sim \xi^{\alpha-1} \quad (1.8)$$

for α near $3/2$, and

$$\mathbf{u}_+ \sim \log \xi. \quad (1.9)$$

The simple form of these complex singularities suggests that they may be generic for 3D flow. Although the perturbation construction (1.5) of a real singular solution is beyond the scope of this paper, the complex singularities found here are not expected to yield real-valued, finite time singular solutions, as discussed in Section 5.

The rest of this paper is organised as follows. In Section 2 we derive equations satisfied by a complex travelling wave solution to the Euler equations. In Section 3 we describe the numerical method for computing the traveling wave and detecting complex singularities. Results of the Fourier fits are given in Section 4, with discussion in Section 5, where in particular the logarithmic singularity is related to double exponential growth of vorticity observed in numerical simulations [22]. Concluding remarks are presented in Section 6.

2. Complex travelling wave

In this section we derive equations satisfied by a complex travelling wave solution to the Euler equations. The starting point for the calculations is the incompressible Euler equations in a Cartesian coordinate system $\mathbf{x} = (x, y, z)$ with analytic forcing $\mathbf{F}(\mathbf{x})$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{F}, \quad (2.10)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2.11)$$

The velocity $\mathbf{u}(\mathbf{x})$, pressure $p(\mathbf{x})$ and forcing $\mathbf{F}(\mathbf{x})$ are assumed to be 2π periodic in x, y and z . We look for a travelling wave solution to (2.10), (2.11) which is complex valued and upper analytic in x, y and z , i.e., composed of Fourier modes with wavenumbers $\mathbf{k} = (k, l, m)$ that have nonnegative k, l and m . The travelling wave solution takes the form

$$\mathbf{u}_+ = \sum_{\mathbf{0} < \mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}} \exp i\mathbf{k} \cdot (\mathbf{x} - i\boldsymbol{\Sigma}t), \quad (2.12)$$

$$p_+ = \sum_{\mathbf{0} < \mathbf{k}} \hat{p}_{\mathbf{k}} \exp i\mathbf{k} \cdot (\mathbf{x} - i\boldsymbol{\Sigma}t), \quad (2.13)$$

$$\mathbf{F}_+ = \sum_{\mathbf{0} < \mathbf{k} < \mathbf{k}_{\max}} \hat{\mathbf{F}}_{\mathbf{k}} \exp i\mathbf{k} \cdot (\mathbf{x} - i\boldsymbol{\Sigma}t). \quad (2.14)$$

Here $\boldsymbol{\Sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the wave speed (for $\sigma_j \in \mathbb{C}$) and $\mathbf{k} = (k, l, m)$ the vector wavenumber. We choose a finite set of forcing modes with $\mathbf{k} < \mathbf{k}_{\max}$, so that the forcing is analytic in the finite complex plane. Note that there is no constant Fourier mode $\hat{\mathbf{u}}_0$ in (2.12). Such a mode is equivalent (via a Galilean transformation) to an imaginary component $i\hat{\mathbf{u}}_0$ of the wavespeed $\boldsymbol{\Sigma}$. To see this, suppose $\mathbf{u}_+, p_+$ is a travelling wave solution of the form (2.12) with speed $\boldsymbol{\Sigma} = i\hat{\mathbf{u}}_0 + \boldsymbol{\Sigma}'$. Let $\mathbf{x}' = \mathbf{x} + \hat{\mathbf{u}}_0 t$, $\mathbf{u}'_+ = \mathbf{u}_+ + \hat{\mathbf{u}}_0$, and $p'_+ = p_+$. Then substitution into (2.10), (2.11) shows that $\mathbf{u}'_+(\mathbf{x}', t)$, $p'_+(\mathbf{x}', t)$ satisfy the Euler equations with wavespeed $\boldsymbol{\Sigma}'$.

In contrast to [26,25] the base flow $\bar{\mathbf{u}}$ is taken to be zero, and instead the instability is generated by the choice of an unstable wavespeed (e.g., $\boldsymbol{\Sigma} = (1, 0, 0)$) and lowest wavenumber growing modes $\hat{\mathbf{u}}_{\mathbf{k}}$, which determine the forcing $\mathbf{F}_+$ and higher wavenumber modes as described in Section 3. This greatly simplifies the construction of the travelling wave. We emphasize that the forcing is analytic (consisting of a few low wavenumber modes) and not directly responsible for generating the singularity. In addition, perturbations of the (nonzero) forcing modes are not observed to affect the singularity type.

Several significant advantages have been gained by restricting attention to triply-periodic solutions with only non-negative wavenumbers $\mathbf{k} > 0$ and to travelling wave solutions:

(i) The dimensionality of the problem is reduced, by eliminating the time variable.

(ii) Because the sum (1.7) involves only positive wavenumbers, there is only a one-way coupling between wavenumbers, i.e., each

$\mathbf{k}$ is only influenced by $\mathbf{k}'$ with $\mathbf{k}' < \mathbf{k}$. This significantly simplifies the construction of $\mathbf{u}_+$, as the equation for $\hat{\mathbf{u}}_{\mathbf{k}}$ has the form

$$L_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}} = \hat{\mathbf{G}}_{\mathbf{k}}(\hat{\mathbf{u}}_{\mathbf{k}'}), \quad \text{for } \mathbf{0} < \mathbf{k}' < \mathbf{k}, \quad (2.15)$$

where $L_{\mathbf{k}}$ is a linear (matrix) operator and $\hat{\mathbf{G}}_{\mathbf{k}}$ is a nonlinear function of the $\hat{\mathbf{u}}_{\mathbf{k}}$. The one-way coupling between wavenumbers also means that there is no truncation error introduced by the restriction to finite $\mathbf{k}$. The nonlinear term $\hat{\mathbf{G}}_{\mathbf{k}}$ in (2.15) is computed using a pseudospectral method, and since there are only quadratic nonlinearities, aliasing error can be completely eliminated by padding with zeros.

(iii) Since the velocity $\mathbf{u}(x - it, y, z)$ (for $\boldsymbol{\Sigma} = (1, 0, 0)$) is a travelling wave with imaginary wave speed, a singularity with imaginary component $\rho = -\text{Im } x$ at $t = 0$ will hit the real x line at $t = \rho$. The singularity positions also depend on y and z , i.e., $\rho = \rho(y, z)$, and the first singularity occurs for y, z that maximises ρ . Thus a singularity will occur at a real space point if the computed data $\mathbf{u}_+(x, y, z, t)$ has a singularity at any complex value of x .

(iv) The periodicity in x, y and z allows a purely spectral computation of $\hat{\mathbf{u}}_{\mathbf{k}}$, as opposed to the calculations for axisymmetric flow with swirl [25] in which finite differences were used in the radial direction. Another difference with [25] is the use here of forcing to generate the (unstable) travelling wave. These modifications are found to prevent the extreme magnification of round-off error that affected the earlier computations.

To derive the system of Eqs. (2.15), define

$$\mathbf{N}(\mathbf{u}) = -\mathbf{u} \cdot \nabla \mathbf{u} \quad (2.16)$$

and let $\hat{\mathbf{N}}_{\mathbf{k}}$ be the $\mathbf{k}$ Fourier coefficient of $\mathbf{N}$. Substituting the travelling wave (2.12) into the Euler equations (2.10) and (2.11) gives

$$(\boldsymbol{\Sigma} \cdot \mathbf{k}) \hat{\mathbf{u}}_{\mathbf{k}} = -ik\hat{p}_{\mathbf{k}} + \hat{\mathbf{F}}_{\mathbf{k}} + \hat{\mathbf{N}}_{\mathbf{k}} \quad (2.17)$$

$$\mathbf{k} \cdot \hat{\mathbf{u}}_{\mathbf{k}} = 0. \quad (2.18)$$

The pressure is eliminated by taking the cross product of $\mathbf{k}$ with (2.17) (this is equivalent to taking the curl of (2.10)), from which we obtain

$$(\boldsymbol{\Sigma} \cdot \mathbf{k}) (\hat{l}\hat{\mathbf{u}}_{\mathbf{k}} - k\hat{v}_{\mathbf{k}}) = i\hat{M}_{\mathbf{k}}^{(1)} - k\hat{M}_{\mathbf{k}}^{(2)} \quad (2.19)$$

$$(\boldsymbol{\Sigma} \cdot \mathbf{k}) (m\hat{\mathbf{u}}_{\mathbf{k}} - k\hat{w}_{\mathbf{k}}) = m\hat{M}_{\mathbf{k}}^{(1)} - k\hat{M}_{\mathbf{k}}^{(3)} \quad (2.20)$$

$$(\boldsymbol{\Sigma} \cdot \mathbf{k}) (m\hat{v}_{\mathbf{k}} - l\hat{w}_{\mathbf{k}}) = m\hat{M}_{\mathbf{k}}^{(2)} - l\hat{M}_{\mathbf{k}}^{(3)}, \quad (2.21)$$

where $(\hat{u}_{\mathbf{k}}, \hat{v}_{\mathbf{k}}, \hat{w}_{\mathbf{k}})$ are the components of $\hat{\mathbf{u}}_{\mathbf{k}}$. Here we have introduced

$$\hat{M}_{\mathbf{k}}^{(i)} = \hat{F}_{\mathbf{k}}^{(i)} + \hat{N}_{\mathbf{k}}^{(i)}, \quad i = 1, \dots, 3 \quad (2.22)$$

where the superscript i denotes the i th vector component. Note from (2.16) and (2.22) that

$$\hat{M}_{\mathbf{k}} = \hat{\mathbf{F}}_{\mathbf{k}} - \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \\ \mathbf{k}_1, \mathbf{k}_2 > 0}} i(\mathbf{k}_2 \cdot \hat{\mathbf{u}}_{\mathbf{k}_1}) \hat{\mathbf{u}}_{\mathbf{k}_2}. \quad (2.23)$$

Eqs. (2.18) and (2.19)–(2.23) provide a system of relations for the $\mathbf{k}$ th Fourier component of velocity in terms of Fourier components of the forcing and velocity terms with wavenumber $\mathbf{k}' < \mathbf{k}$. Under the assumption $\boldsymbol{\Sigma} \cdot \mathbf{k} \neq 0$, Eqs. (2.18)–(2.21) can be inverted to provide an expression for $(\hat{u}_{\mathbf{k}}, \hat{v}_{\mathbf{k}}, \hat{w}_{\mathbf{k}})$ in terms of known modes for $\mathbf{k}' < \mathbf{k}$. We first eliminate $\hat{w}_{\mathbf{k}}$ from the system (2.18)–(2.20) and solve for $\hat{u}_{\mathbf{k}}, \hat{v}_{\mathbf{k}}$ to obtain

$$\hat{u}_{\mathbf{k}} = a_{\mathbf{k}} \left[(l^2 + m^2) \hat{M}_{\mathbf{k}}^{(1)} - lk\hat{M}_{\mathbf{k}}^{(2)} - km\hat{M}_{\mathbf{k}}^{(3)} \right] \quad (2.24)$$

$$\hat{v}_{\mathbf{k}} = a_{\mathbf{k}} \left[-lk\hat{M}_{\mathbf{k}}^{(1)} + (m^2 + k^2) \hat{M}_{\mathbf{k}}^{(2)} - lm\hat{M}_{\mathbf{k}}^{(3)} \right], \quad (2.25)$$

where we have introduced

$$a_{\mathbf{k}} = [(\boldsymbol{\Sigma} \cdot \mathbf{k}) (\mathbf{k} \cdot \mathbf{k})]^{-1}. \quad (2.26)$$

When $k \neq 0$ we obtain an expression for $\hat{w}_{\mathbf{k}}$ from (2.20):

$$\hat{w}_{\mathbf{k}} = [k (\boldsymbol{\Sigma} \cdot \mathbf{k})]^{-1} \left[k \hat{M}_{\mathbf{k}}^{(3)} - m \hat{M}_{\mathbf{k}}^{(1)} \right] + m k^{-1} \hat{u}_{\mathbf{k}} \quad (2.27)$$

where $\hat{u}_{\mathbf{k}}$ is given by (2.24). In the case $k = 0$ but $l \neq 0$, we have from (2.21)

$$\hat{w}_{\mathbf{k}} = [l (\boldsymbol{\Sigma} \cdot \mathbf{k})]^{-1} \left[l \hat{M}_{\mathbf{k}}^{(3)} - m \hat{M}_{\mathbf{k}}^{(2)} \right] + m l^{-1} \hat{v}_{\mathbf{k}} \quad (2.28)$$

where $\hat{v}_{\mathbf{k}}$ is given by (2.25). If both $k = 0$ and $l = 0$ but $m \neq 0$, then

$$\hat{w}_{\mathbf{k}} = 0. \quad (2.29)$$

Eqs. (2.24)–(2.29) give the k th Fourier mode $\hat{\mathbf{u}}_{\mathbf{k}}$ in terms of known modes $\hat{\mathbf{u}}_{\mathbf{k}'}$ for $\mathbf{k}' < \mathbf{k}$, and are the main results of this section. The specification of data for the lowest wavenumber modes and determination of forcing are described in the next section.

3. Numerical method

We describe the numerical method for computing the travelling wave and detecting complex singularities. Unless otherwise noted, the travelling wave velocity is fixed by setting $\boldsymbol{\Sigma} = (1, 0, 0)$, which corresponds to an unstable (growing) wave.

Define $|\mathbf{k}| = k + l + m$. The travelling wave is computed by iteratively solving (2.24)–(2.29) for wavenumbers with increasing $|\mathbf{k}|$, with forcing determined as discussed below. The numerical method is initialised by prescribing ‘data’ for the lowest wavenumber modes, i.e., we specify $\hat{\mathbf{u}}_{\mathbf{k}}$ for $|\mathbf{k}| \leq N$ and some positive integer N . The data is subject to the constraint of incompressibility, so that

$$k \hat{u}_{\mathbf{k}} + l \hat{v}_{\mathbf{k}} + m \hat{w}_{\mathbf{k}} = 0. \quad (3.30)$$

For example, the most general data for $N = 2$ is

$$\begin{aligned} \mathbf{u} = & \hat{\mathbf{u}}_{100} e^{ix+t} + \hat{\mathbf{u}}_{010} e^{iy} + \hat{\mathbf{u}}_{001} e^{iz} + \hat{\mathbf{u}}_{110} e^{i(x+y)+t} + \hat{\mathbf{u}}_{101} e^{i(x+z)+t} \\ & + \hat{\mathbf{u}}_{011} e^{i(y+z)} + \hat{\mathbf{u}}_{200} e^{2ix+2t} + \hat{\mathbf{u}}_{020} e^{2iy} + \hat{\mathbf{u}}_{002} e^{2iz}, \end{aligned} \quad (3.31)$$

where $\boldsymbol{\Sigma} = (1, 0, 0)$ and the subscripts on the Fourier coefficients denote the wavenumber. Note that the modes without x dependence will not grow. Incompressibility requires that $\hat{u}_{100} = \hat{v}_{010} = \hat{w}_{001} = 0$ and that $\hat{v}_{110} = -\hat{u}_{110}$, $\hat{w}_{101} = -\hat{u}_{101}$, $\hat{v}_{011} = -\hat{u}_{011}$ and $\hat{u}_{200} = \hat{v}_{020} = \hat{w}_{002} = 0$. Otherwise the Fourier coefficients in (3.31) can be chosen arbitrarily. For the numerical simulations reported in this paper we use data of the form (3.31), sometimes supplemented by $|\mathbf{k}| = 3$ mode

$$\hat{\mathbf{u}}_{111} e^{i(x+y+z)+t} \quad \text{with } \hat{u}_{111} + \hat{v}_{111} + \hat{w}_{111} = 0. \quad (3.32)$$

We have also tried an upper analytic analogue of Kida–Pelz data, which is discussed in Section 4.1.3.

As mentioned earlier, [26,25] generate an unstable travelling wave for axisymmetric flow with swirl by imposing a steady background flow corresponding to a smoothed unstable vortex sheet. In contrast, we generate instability by specific choice of wavespeed, unstable data (e.g., (3.31)) and forcing. Force coefficients $\hat{\mathbf{F}}_{\mathbf{k}}$ for $|\mathbf{k}| \leq N$ are determined by the requirement that the imposed (unstable) data be a solution of (2.24)–(2.29), and for $|\mathbf{k}| > N$ we set $\hat{\mathbf{F}}_{\mathbf{k}} = 0$. A slightly different choice of forcing that leads to a small amplitude singularity is described in Section 3.2.

Higher wavenumber modes are generated by products of Fourier modes in the nonlinear term $\mathbf{N}(\mathbf{u})$. We solve for the higher wavenumber modes using a pseudospectral method, and note that since $\mathbf{N}$ is quadratic, padding with enough zeros completely eliminates aliasing error from the pseudospectral part of the calculation.

Knowledge of $\hat{\mathbf{M}}_{\mathbf{k}} = \hat{\mathbf{F}}_{\mathbf{k}} + \hat{\mathbf{N}}_{\mathbf{k}}$ in (2.24) is used to calculate $\hat{\mathbf{u}}_{\mathbf{k}}$ for $|\mathbf{k}| = M + 1$ in (2.29). Note there is no truncation error in the calculation. For most of the computations presented here, the total number of modes M_T before dealiasing (i.e., maximum value of M) is $M_T = 420$.

The method was validated by comparing the numerically computed $\hat{\mathbf{u}}_{\mathbf{k}}$ with a hand calculation of some of the Fourier modes for small $|\mathbf{k}|$.

3.1. Numerical detection of singularities

The singularity in $\mathbf{u}_+$ occurs at a position $\mathbf{x} = \mathbf{x}_0(y, z)$, where

$$\mathbf{x}_0(y, z) = \delta_R(y, z) - i\delta_I(y, z). \quad (3.33)$$

Near this singularity the structure of the velocity $\mathbf{u}_+$ (and similarly $\mathbf{v}_+$, $\mathbf{w}_+$) is sought in the form

$$\mathbf{u}_+ \approx c(\mathbf{x} - \mathbf{x}_0)^{\alpha-1} \ln^{\beta+1}(\mathbf{x} - \mathbf{x}_0) \quad (3.34)$$

for $\mathbf{x}$ near $\mathbf{x}_0$, where

$$c = c_R + i c_I, \quad \alpha = \alpha_R + i \alpha_I, \quad \beta = \beta_R + i \beta_I$$

are complex parameters. Following [32], singularities in $\mathbf{u}_+$ are numerically detected through asymptotics of the Fourier coefficients. It is shown in Appendix that the local structure (3.34) implies the Fourier coefficients $\hat{u}_{\mathbf{k}}$ have the asymptotic form

$$\hat{u}_{\mathbf{k}} = [c_1(\alpha)(-\ln k)^{\beta+1} + c_2(\alpha, \beta)(-\ln k)^{\beta}] k^{-\alpha} e^{-ik\alpha_0}, \quad (3.35)$$

at leading order in $k \gg 1$, where

$$c_1(\alpha) = -2c \sin[\pi(\alpha - 1)] e^{\frac{\pi i}{2}(\alpha-1)} \Gamma(\alpha), \quad (3.36)$$

$$\begin{aligned} c_2(\alpha, \beta) = & -\frac{c\pi(\beta+1)}{2} e^{-\frac{\pi i}{2}(\alpha-1)} (1 + 3e^{\pi i(\alpha-1)}) \Gamma(\alpha) \\ & - 2c(\beta+1) \sin[\pi(\alpha - 1)] e^{\frac{\pi i}{2}(\alpha-1)} \Gamma_1(\alpha). \end{aligned} \quad (3.37)$$

Here $\Gamma(\alpha)$ is the Gamma function and $\Gamma_1(\alpha) = \int_0^\infty e^{-r} r^{\alpha-1} \ln r \, dr$. Eq. (3.35) for $\beta = -1$ is given in [33,32].

The parameters c , α , α_0 , β depend on y and z , i.e.,

$$(c, \alpha, \alpha_0, \beta) = (c, \alpha, \alpha_0, \beta)(y, z) \quad (3.38)$$

and following [25,32] are determined by a sliding fit. We start by taking $\beta = -1$ (i.e., neglecting the logarithmic correction) and calculating (c, α, α_0) at fixed (y, z) using a three point fit, i.e., for each k the parameters are chosen to exactly fit the three values $\hat{u}_k, \hat{u}_{k+1}, \hat{u}_{k+2}$. The asymptotic fit is successful if (c, α, α_0) are nearly independent of the starting index k . A value $\alpha \approx 1$ is indicative of a purely logarithmic singularity at leading order, in which case we implement a four parameter fit using $\hat{u}_k, \hat{u}_{k+1}, \hat{u}_{k+2}, \hat{u}_{k+3}$ to determine β in addition to c , α , and α_0 .

Prior to fitting, we locate the maximum of the derivative u_{+x} . This gives an approximate location (y_0, z_0) of the closest singularity to the real- x line, i.e., that which maximises $\delta_I(y, z)$. Fits to the Fourier coefficients are performed for (y, z) in a neighbourhood of (y_0, z_0) in order to determine the dependence of parameters $(c, \alpha, \alpha_0, \beta)$ on (y, z) .

The magnitude of the data (i.e., size of $|\hat{\mathbf{u}}_{\mathbf{k}}|$ in (3.31)) controls the imaginary- x component of the singularity location. More precisely, for the travelling wave with $\boldsymbol{\Sigma} = (\sigma, 0, 0)$, a shift in t by an amount t_0 is equivalent to either multiplication of the k th Fourier mode by a factor of $e^{\sigma k t_0}$ or a shift in imaginary component of the singularity location $\mathbf{x}_0$ by an amount σt_0 . We choose this shift so $\text{Im}(\mathbf{x}_0)$ is near zero at $t = 0$. There are no other adjustable parameters in the data to influence the amplitude of the singularity c .

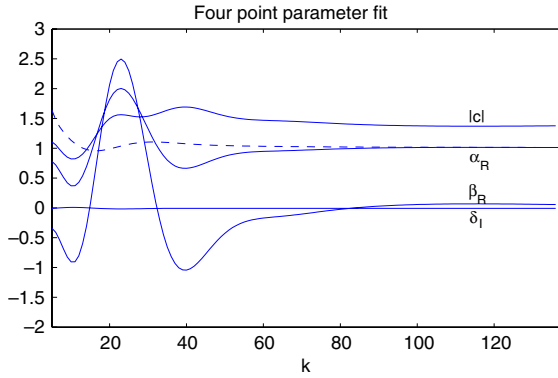


Fig. 1. Results from sliding fit for the Fourier coefficients at $(y_0, z_0) = (0, 0)$ as a function of k . The graph shows δ_I , β_R , α_R and $|c|$ determined by a 4 point Fourier fit to (3.34) ($\alpha_I \approx \beta_I \approx 0$ are not shown). The dashed line is the three point fit to $c(x - x_0)^{\alpha-1}$ for α_R . The log exponent β_R is decreasing at $k = 140$.

3.2. Small amplitude singularity

A travelling wave solution with a small amplitude singularity is generated by a particular choice of forcing. Let $\hat{\mathbf{F}}_{\mathbf{k}}$ for $|\mathbf{k}| \leq 2N$ be forcing modes such that the data (3.31) is an exact solution of the Euler equations, i.e., no higher wavenumber modes are generated (this requires $2N$ forcing modes due to quadratic nonlinearity of $\mathbf{N}$). Equivalently, $\hat{\mathbf{F}}_{\mathbf{k}}$ is such that $\hat{\mathbf{M}}_{\mathbf{k}} = 0$ for all $\mathbf{k}$. Consider the perturbed forcing

$$\hat{\mathbf{F}}_{\mathbf{k}}^\epsilon = \begin{cases} \hat{\mathbf{F}}_{\mathbf{k}} & \text{for } |\mathbf{k}| \leq N, \\ (1 - \epsilon)\hat{\mathbf{F}}_{\mathbf{k}} & \text{for } N < |\mathbf{k}| \leq 2N, \\ 0 & \text{for } 2N < |\mathbf{k}| \end{cases} \quad (3.39)$$

with $\epsilon \ll 1$. For this forcing the data (3.31) is no longer an exact solution to the Euler equations, but instead generates modes for $|\mathbf{k}| > N$. By construction, these modes have amplitude c (cf. (3.34)) of size ϵ . This results in a singular part $\mathbf{u}_+^s$ of the travelling wave $\mathbf{u}_+$ that has size $O(\epsilon)$. There is also a regular part $\mathbf{u}_+^r$ that has size $O(1)$. These assertions are verified numerically in Section 4.1.2.

4. Results

We searched initial data of the form (3.31) and (3.32) to look for different singularity types, and also implemented an upper analytic analogue of Kida–Pelz data. Our search produced the singularity forms described in (1.8) and (1.9). The singularity types and geometry of the singular surface are described in this section.

4.1. Logarithmic singularity

Logarithmic singularities were found for all tested data that did not contain a $|\mathbf{k}| = 1$ mode. For example, Figs. 1 and 2 show the Fourier fits for (c, α, x_0, β) corresponding to data of the form (3.31) and (3.32) with nonzero modes

$$\begin{aligned} \hat{u}_{101} &= -\hat{w}_{101} = a_0 i \\ \hat{u}_{110} &= -\hat{v}_{110} = a_0 i \\ \hat{u}_{111} &= -2\hat{v}_{111} = -2\hat{w}_{111} = 0.5a_0 i, \end{aligned} \quad (4.40)$$

where $a_0 = 0.9$. For this data to be a solution of (2.24)–(2.29) the forcing modes $\hat{\mathbf{F}}_{101}$ and $\hat{\mathbf{F}}_{111}$ must satisfy the constraint

$$\begin{aligned} \hat{F}_{101}^{(1)} - \hat{F}_{101}^{(3)} &= 2\hat{u}_{101}, & \hat{F}_{110}^{(1)} - \hat{F}_{110}^{(2)} &= 2\hat{u}_{110}, \\ \hat{F}_{111}^{(2)} - \hat{F}_{111}^{(3)} &= \hat{v}_{111} - \hat{w}_{111}, \\ \hat{F}_{111}^{(1)} - \hat{F}_{111}^{(3)} &= \hat{u}_{111} - \hat{w}_{111}, \end{aligned} \quad (4.41)$$

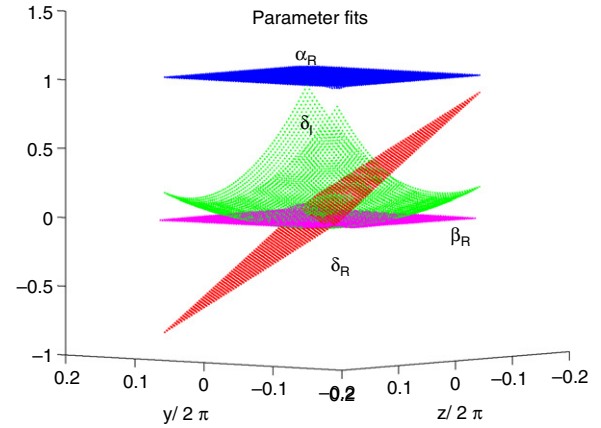


Fig. 2. Singularity parameters as a function of (y, z) from the sliding fit. Shown are α_R , β_R , δ_R and $10\delta_I$. Note that α_R and β_R are independent of (y, z) .

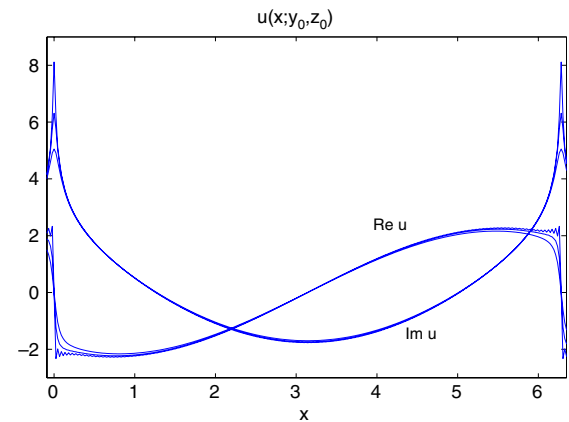


Fig. 3. The real and imaginary components of u_+ for $a_0 = 0.85, 0.875, 0.89$. The oscillations in the $a_0 = 0.89$ profile are an artifact of the truncation in Fourier space, which does not lead to error in the numerical method but does affect the real space representation of solution.

but are otherwise arbitrary (although $\hat{\mathbf{F}}_{\mathbf{k}} - i\mathbf{k}\hat{\mathbf{p}}_{\mathbf{k}}$ is uniquely determined, in view of (2.17)). All other forcing modes were set to 0. A logarithmic singularity is detected by using 3 and 4 point fits, as described in Section 3.1. We show the fit to u_+ , although the results for v_+ , w_+ are very similar.

Fig. 1 shows α_R , β_R , δ_I and $|c|$ as a function of k at the point $(y_0, z_0) = (0, 0)$, where u_{+x} has a maximum. Also plotted (dashed line) is the three point fit to the singularity exponent α_R . The figure shows a successful fit with parameter values that are roughly independent of k for sufficiently large k . The fit is equally good for other nearby values of (y, z) . The three point fit shows that α_R approaches 1 for large k , which from (3.35) is indicative of a leading order log singularity. The four point fit verifies this conclusion, and also shows that $\alpha_I \approx \beta_I \approx 0$ for all k and that β_R is near 0 for large k , which according to (3.34) suggests a leading order singularity of the form

$$u_+ \approx c \log(x - it - x_0(y, z)). \quad (4.42)$$

Perturbations of the data (4.40) and forcing that do not contain a $|\mathbf{k}| = 1$ mode give similar results.

Fig. 3 presents a plot of the real and imaginary components of $u_+(x, y_0, z_0)$ and provides additional evidence for log singularities. The figure shows that $\text{Im } u_+$ blows up as a_0 increases, i.e., as the singularity approaches the real line, while $\text{Re } u_+$ approaches a jump discontinuity, consistent with a log singularity.

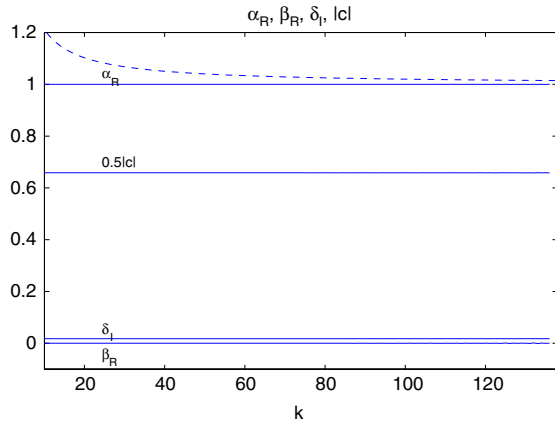


Fig. 4. Results of a sliding fit to the Fourier coefficients at $(y_0, z_0) = (0.06\pi, 0.04\pi)$ as a function of k , for the data (4.45). The graph shows $\delta_I, \beta_R, \alpha_R$ and $|c|$ determined by a 4 point Fourier fit ($\alpha_I \approx \beta_I \approx 0$ are not shown), and the three point fit (dashed line) to $c(x - x_0)^{\alpha_R - 1}$ for α_R .

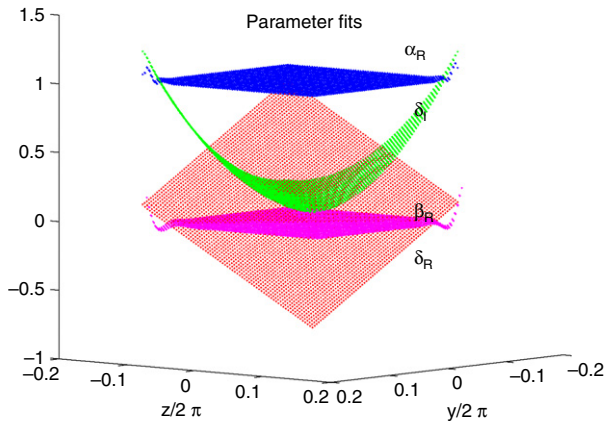


Fig. 5. Singularity parameters as a function of (y, z) for the data in (4.45). The singular surface $\delta_I(y, z)$ is a cylindrical parabola.

4.1.1. Geometry of the singular surface

The (y, z) dependence of the parameters $\alpha_R, \beta_R, \delta_I$, and δ_R is illustrated in Fig. 2. The figure shows the fits to these four parameters at the largest value of k ($k_{max} = 140$), as a function of $y - y_0$ and $z - z_0$, where $(y_0, z_0) = (0, 0)$. The singularity strength α_R and log exponent β_R are found to be independent of (y, z) , and the figure shows a paraboloidal dependence on y and z for the imaginary singularity location δ_I . The real singularity location δ_R exhibits a linear dependence on (y, z) . The imaginary components β_I and α_I (not shown) are $\sim 10^{-3}$ or less over the entire range of (y, z) in Fig. 2. These results show that the singularity surface is paraboloidal in a local neighbourhood of $\mathbf{x}_0 = (x_0, y_0, z_0)$, i.e., after a rotation of variables the surface can be described as $\xi = 0$ with

$$\xi = \mathbf{x}' - \mathbf{A} \cdot \mathbf{Y}' + i\mathbf{Y}'\mathbf{M}\mathbf{Y}' + H \quad (4.43)$$

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_0, \quad \mathbf{Y}' = (y - y_0, z - z_0)^T \quad (4.44)$$

for $\mathbf{A}$ and $\mathbf{Y}'$ real vectors and $\mathbf{M}$ a self-adjoint and positive definite 2×2 matrix. Here H contains higher order terms, including possibly real quadratic factors. Eq. (4.43) says that there is a singularity at $\mathbf{x}_0$, and that as (y, z) varies away from (y_0, z_0) the imaginary part of the singularity position grows quadratically in the negative direction. The real part of the singularity position is given by $\mathbf{A} \cdot \mathbf{Y}'$ and varies linearly with $y - y_0$ and $z - z_0$. The physical implications of this singular surface geometry are discussed in Section 5.

It is interesting to consider the degenerate case for which the singular surface is two dimensional, i.e., with no variation of the

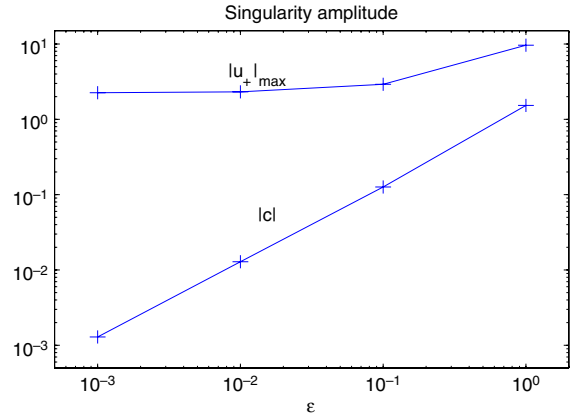


Fig. 6. Log-log plot of the maximum of $|u_+|$ and the singularity amplitude $|c|$ versus ϵ .

surface in one direction. We find the Fourier fits for this case to be particularly clean. For example, Fig. 4 shows parameter fits corresponding to the data

$$\begin{aligned} \hat{u}_{101} &= -\hat{w}_{101} = a_0 i \\ \hat{u}_{110} &= -\hat{v}_{110} = a_0 i \\ \hat{v}_{101} &= 0.1 \hat{w}_{110} = 0.2. \end{aligned} \quad (4.45)$$

Although the flow is fully three dimensional, $\hat{v}_{101}$ and $\hat{w}_{110}$ do not contribute to the generation of higher $\mathbf{k}$ modes and thus the singular surface only depends on the two independent variables $\tilde{x} = x + y$, $\tilde{y} = x + z$, for any a_0 . The remarkably clean fits in Fig. 4 show a logarithmic singularity of the form (4.42). The singular surface is depicted in Fig. 5 and $\delta_I(y, z)$ locally has the shape of a cylindrical parabola. As before, the singularity exponents α_R and β_R do not depend on (y, z) , and $\alpha_I \approx \beta_I \approx 0$ (not shown). Also, to leading order δ_R exhibits a linear dependence on y and z .

4.1.2. Small amplitude singularity

The method for generating small amplitude singularities is illustrated in Fig. 6. The figure displays a log-log plot of the maximum of $|u_+|$ and the singularity amplitude $|c|$ versus ϵ (cf. (3.39)), for the data (4.40) with $a_0 = 0.9$. The plot verifies that the regular part of the velocity u_+^r has size $O(1)$, while the singular part u_+^s has size $O(\epsilon)$. Thus the singularity amplitude can be made small, which is a requirement for the perturbative construction of a real singular solution proposed in [26].

4.1.3. Kida–Pelz flow

Kida–Pelz flow [34,35] is a high symmetry flow that has attracted interest as a candidate for finite time singularity formation. The symmetry of this flow greatly reduces operation counts and memory requirements during numerical simulation.

The initial conditions for Kida–Pelz flow are given by

$$\begin{aligned} u(x, y, z, t = 0) &= \sin x (\cos 3y \cos z - \cos y \cos 3z) \\ v(x, y, z, t = 0) &= \sin y (\cos 3z \cos x - \cos z \cos 3x) \\ w(x, y, z, t = 0) &= \sin z (\cos 3x \cos y - \cos x \cos 3y). \end{aligned} \quad (4.46)$$

The symmetries for the (real valued) flow are 2π periodicity, a bilateral symmetry through the planes $n\pi$ for integer n , a $\pi/2$ rotational symmetry about the axis $(\pi/2, \pi/2)$, and a permutation symmetry of the velocity components, $u(x, y, z) = v(y, z, x) = w(z, x, y)$. A detailed description of the symmetries is given in [34], where the flow was originally proposed.

Pseudospectral calculations of viscous Kida–Pelz flow were performed in [35]. The simulations show dramatic intensification of peak vorticity over an interval of time, followed by a sharp decrease

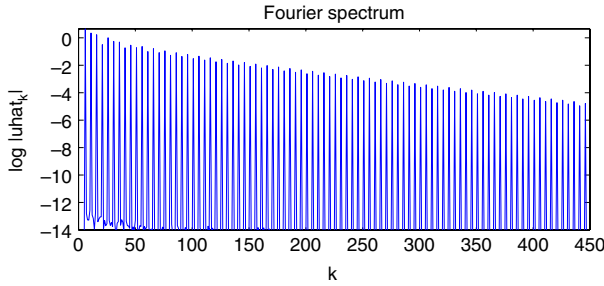


Fig. 7. Plot of $\log \hat{u}_{+k}$ versus $\tilde{k}$. The Fourier coefficients are nonzero for $\tilde{k} = 5\tilde{k}'$ where $\tilde{k}'$ is a positive integer. Note there are two sets of smoothly decaying modes.

that is likely due to viscous dissipation. During the time interval of vorticity intensification, the width of the analyticity strip $\delta(t)$ was found to approach zero at a rate faster than exponential, before reaching a minimum value and starting to increase. An analysis in [20] using power series in time and Padé approximants finds evidence for a finite-time singularity.

A complex travelling wave analogue of Kida–Pelz flow is generated, say, by choosing the lowest wavenumber Fourier coefficients of the travelling wave equal to the upper analytic part of the Kida–Pelz initial data. For $\mathbf{k} > 0$ we choose $\Sigma = (1, 1, 1)$ so that the travelling wave preserves the permutation symmetry of the velocity. This gives the data

$$\begin{aligned} u_+ &= c_0 e^{i(x-it)} (e^{3i(y-it)} e^{i(z-it)} - e^{i(y-it)} e^{3i(z-it)}) \\ v_+ &= c_0 e^{i(y-it)} (e^{3i(z-it)} e^{i(x-it)} - e^{i(z-it)} e^{3i(x-it)}) \\ w_+ &= c_0 e^{i(z-it)} (e^{3i(x-it)} e^{i(y-it)} - e^{i(x-it)} e^{3i(y-it)}) \end{aligned} \quad (4.47)$$

where c_0 is a pure imaginary complex constant. Unless otherwise noted, we set $c_0 = -2.25i$ so that the maximum imaginary component of the singular surface location is near 0 at $t = 0$. The forcing is constrained by (2.24)–(2.29) and the data (4.47), for example, the $\mathbf{k} = (1, 3, 1)$ mode satisfies

$$\begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \hat{F}_{131}^{(1)} \\ \hat{F}_{131}^{(2)} \\ \hat{F}_{131}^{(3)} \end{pmatrix} = 4 \begin{pmatrix} 3\hat{u}_{131} \\ \hat{u}_{131} - \hat{w}_{131} \\ -3\hat{w}_{131} \end{pmatrix}, \quad (4.48)$$

with similar relations holding for the $\mathbf{k} = (3, 1, 1)$ and $(1, 1, 3)$ modes. We set $\hat{\mathbf{F}}_{\mathbf{k}}$ to zero for all other positive wavenumbers.

The permutation symmetry of the velocity implies that

$$\hat{u}_{k,l,m} = \hat{v}_{m,k,l} = \hat{w}_{l,m,k} \quad (4.49)$$

and that

$$\begin{aligned} \hat{N}_{k,l,m}^{(2)} &= \hat{N}_{l,m,k}^{(1)} \\ \hat{N}_{k,l,m}^{(3)} &= \hat{N}_{m,k,l}^{(1)} \end{aligned} \quad (4.50)$$

where $\mathbf{N}$ is the nonlinear term in the momentum equation (cf. (2.16)). If we assume the forcing also exhibits the symmetry (4.50), then the system (2.24)–(2.29) simplifies to the single equation

$$\hat{u}_{k,l,m} = a_{\mathbf{k}} \left[(l^2 + m^2) \hat{M}_{k,l,m}^{(1)} - lk \hat{M}_{l,m,k}^{(1)} - km \hat{M}_{m,k,l}^{(1)} \right] \quad (4.51)$$

with the other velocity Fourier coefficients determined using (4.49). We use the symmetries (4.49) and (4.50) to reduce operation counts and memory requirements in our calculation by a factor of three.

The Kida–Pelz data (4.47) generates a travelling wave moving in the direction $i(1, 1, 1)$. We rotate variables so that this direction is the imaginary component of a single variable $\tilde{x}$. Set

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M} \mathbf{x}, \quad (4.52)$$

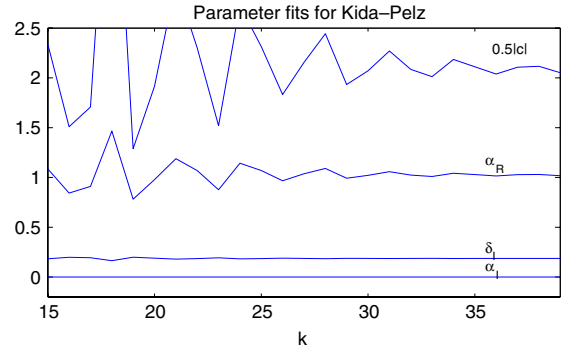


Fig. 8. Results from sliding fit for the Fourier coefficients at $(\tilde{y}_0, \tilde{z}_0) = (0, \pi)$ as a function of k . Plotted are δ_l , α_R , α_l , and $0.5|c_l|$.

so that

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 2 & 3 & 1 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \mathbf{M}^{-1} \tilde{\mathbf{x}}, \quad (4.53)$$

where $\mathbf{M}$ is an orthogonal transformation. We calculate $\tilde{\mathbf{k}}$ such that $\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}} = \mathbf{k} \cdot \mathbf{x}$, i.e., $\tilde{\mathbf{k}} = \mathbf{k} \cdot \mathbf{M}^{-1}$. Thus

$$\tilde{\mathbf{k}} = \begin{pmatrix} \tilde{k} \\ \tilde{l} \\ \tilde{m} \end{pmatrix} = \begin{pmatrix} k + l + m \\ \frac{3}{2}(k - l) \\ \frac{1}{2}(k + l - 2m) \end{pmatrix}. \quad (4.54)$$

For the Kida–Pelz travelling wave data (4.47), higher wavenumber modes generated by nonlinear interactions have all even or all odd (k, l, m) . Therefore, $(\tilde{k}, \tilde{l}, \tilde{m})$ run through the integers. We write

$$\begin{aligned} \tilde{\mathbf{u}}_+(\tilde{\mathbf{x}}) &= \mathbf{u}_+(\mathbf{x}) \\ &= \sum_{k,l,m} \hat{\mathbf{u}}(k, l, m) e^{i\mathbf{k} \cdot \mathbf{x}} \\ &= \sum_{k,l,m} \hat{\mathbf{u}}(k, l, m) e^{i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}}} \\ &= \sum_{\tilde{k}} \sum_{l,m} \hat{\mathbf{u}}(\tilde{k}, \tilde{l}, \tilde{m}) e^{i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{x}}}, \end{aligned} \quad (4.55)$$

where

$$\hat{\mathbf{u}}(\tilde{k}, \tilde{l}, \tilde{m}) = \sum_{l,m} \hat{\mathbf{u}}(\tilde{k} = k + l + m, l, m) e^{i(\tilde{l}\tilde{y} + \tilde{m}\tilde{z})}, \quad (4.56)$$

and the time dependence has been incorporated into $\hat{\mathbf{u}}$. Formulas (4.55) and (4.56) give $\mathbf{u}_+$ as a Fourier expansion in the variable $\tilde{\mathbf{x}}$ with fixed $\tilde{y}$ and $\tilde{z}$, as desired.

We detect singularities in $\tilde{\mathbf{u}}_+(\tilde{\mathbf{x}})$ using the fitting procedure described in Section 3.1. After rotating, $\tilde{u}_+$ is found to have nonzero modes for $\tilde{k} = 5\tilde{k}'$ where $\tilde{k}'$ is a positive integer (note that the data (4.47) corresponds to $\tilde{k} = 5$). As shown in Fig. 7, the Fourier coefficients divide into two sets of smoothly decaying modes with slightly different amplitude. Fig. 8 shows the fits to $\hat{u}_{\tilde{k}'}$ using modes from one such set. The fits show some oscillation, but eventually decay to a constant value. The plot of α_R tends toward 1 which indicates a log singularity. Further evidence for a log singularity comes from a plot of the real and imaginary components of $\tilde{u}_+(\tilde{x}, \tilde{y}_0, \tilde{z}_0)$ versus $\tilde{x}$ for fixed $(\tilde{y}_0, \tilde{z}_0) = (0, \pi)$. The plot shows that $|\text{Im } \tilde{u}_+|$ is large, while $\text{Re } \tilde{u}_+$ is close to a jump discontinuity, consistent with a log singularity (Fig. 9).

The upper analytic travelling wave solution with data (4.47) does not exhibit all the symmetries of Kida–Pelz flow, since it omits

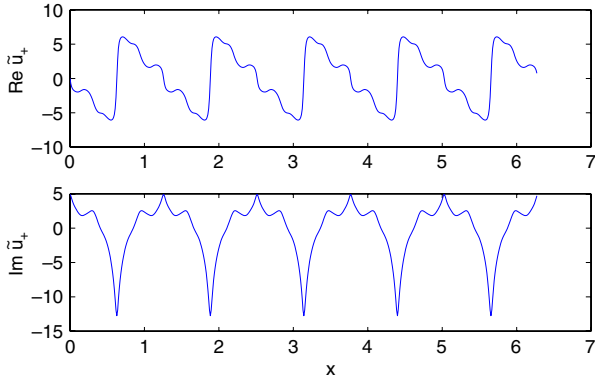


Fig. 9. Plot of the real and imaginary components of $\tilde{u}_+(\tilde{x}, \tilde{y}_0, \tilde{z}_0)$ versus $\tilde{x}$ for fixed $(\tilde{y}_0, \tilde{z}_0) = (0, \pi)$.

negative wavenumber components. Travelling wave solutions corresponding to the negative wavenumber components in (4.46), for example $\mathbf{u}_{-++} = \sum_{k<0, l>0, m>0} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x} - i\mathbf{\Sigma}t)}$ where $\mathbf{\Sigma} = (\text{sgn}(k) \cdot 1, \text{sgn}(l) \cdot 1, \text{sgn}(m) \cdot 1)$, are readily computed using the relations

$$\begin{aligned} \hat{u}_{k,l,m} &= \text{sgn}(k) \hat{u}_{|k|,|l|,|m|}, & \hat{v}_{k,l,m} &= \text{sgn}(l) \hat{v}_{|k|,|l|,|m|}, \\ \hat{w}_{k,l,m} &= \text{sgn}(m) \hat{w}_{|k|,|l|,|m|}, \end{aligned} \quad (4.57)$$

which follow from symmetry considerations. The real valued velocity

$$\mathbf{u} = \text{Re}(\mathbf{u}_+ + \mathbf{u}_{+--} + \mathbf{u}_{-+-} + \mathbf{u}_{--+}), \quad (4.58)$$

where $\mathbf{u}_+ \equiv \mathbf{u}_{+++}$, is an approximate solution to the Euler equations under Moore's approximation, in which the lowest

wavenumber modes at $t = 0$ match the data (4.46) (for $c_0 = 1$), and can be shown to satisfy all the usual symmetries of Kida–Pelz flow. Vorticity magnitude isosurfaces for the flow (4.58) are plotted at different times in the evolution in Figs. 10 and 11. These show some qualitative similarities to the viscous flow calculations in Boratov and Pelz [35] (compare Fig. 10 with Figure 2 in [35] and the left plot in Fig. 11 with Figure 1 in that paper). Fig. 11 (right plot) depicts the vorticity magnitude isosurface at a later time, and shows that the region of large vorticity is compressed or flattened into thin layers (cf. Section 5).

4.2. Algebraic singularity

Algebraic singularities were found for all data containing a $|\mathbf{k}| = 1$ mode. For example, Fig. 12 shows the Fourier fits to (c, α, x_0) for the data

$$\begin{aligned} \hat{u}_{010} &= \frac{1+i}{2} a_0, & \hat{v}_{100} &= 1.2a_0 & \hat{w}_{100} &= 0.9a_0 \\ \hat{u}_{001} &= 1.1a_0 & \hat{v}_{001} &= \frac{a_0}{2}(1-i) & \hat{w}_{010} &= 0.8a_0 \end{aligned} \quad (4.59)$$

where $a_0 = 0.37$. The fits were calculated at $(y_0, z_0) = (0.86\pi, 1.03\pi)$. The travelling wavespeed in this calculation was $\mathbf{\Sigma} = (1, i, i)$ corresponding to a (growing) wave which translates with constant real speed in the $(0, 1, 1)$ direction. The motivation for this $\mathbf{\Sigma}$ is that it allows more flexibility in the choice of data compared with $\mathbf{\Sigma} = (1, 0, 0)$, since $\mathbf{\Sigma} \cdot \mathbf{k} \neq 0$ and hence (2.18)–(2.21) are invertible for any $\mathbf{k} > 0$. The singularity exponent α_R quickly decays to a constant value $\alpha_R \approx 1.5$ which indicates a singularity of the form

$$u \approx c\xi^{1/2} \quad (4.60)$$

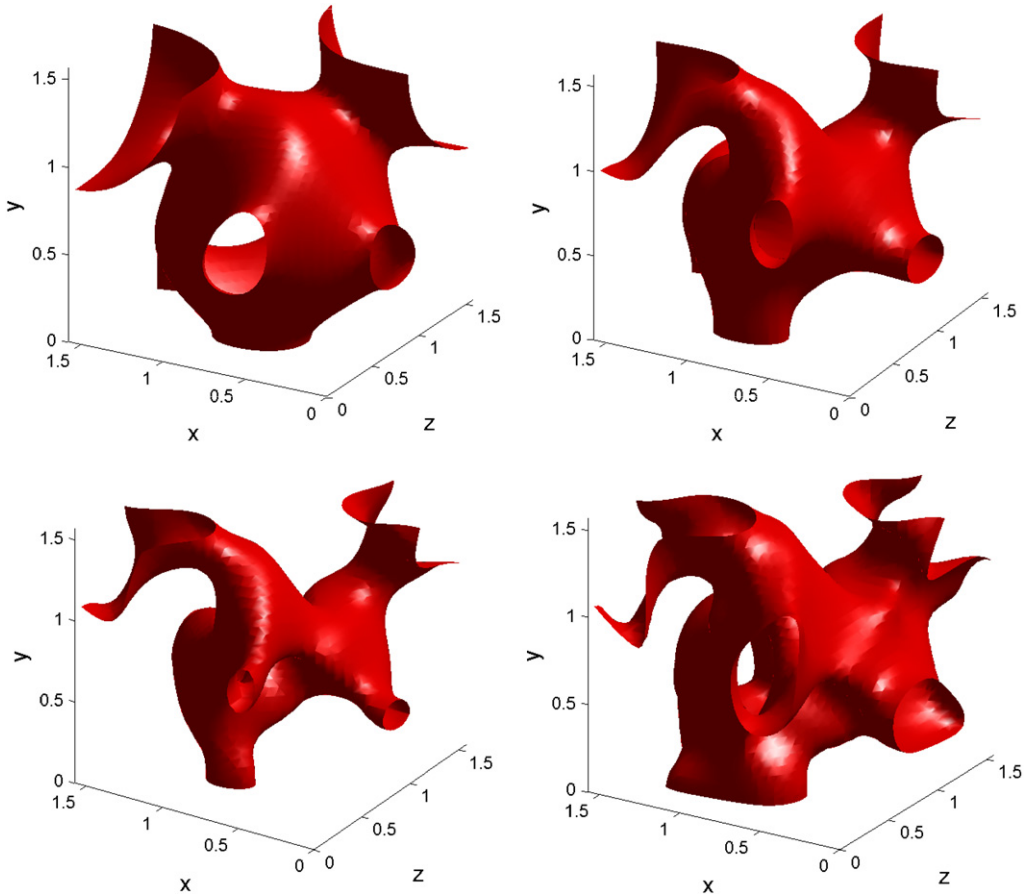


Fig. 10. Vorticity magnitude isosurfaces for the flow given by (4.58), plotted for $\mathbf{x} \in [0, \pi/2]^3$. Top left: $t = 0$, $|\omega| = 0.3\omega_0$, where $\omega_0 = 7.97$ is the maximum initial vorticity magnitude. Top right: $t = 0.36$, $|\omega| = 2.75\omega_0$. Bottom left: $t = 0.42$, $|\omega| = 4.5\omega_0$. Bottom right: $t = 0.46$, $|\omega| = 4.85\omega_0$.

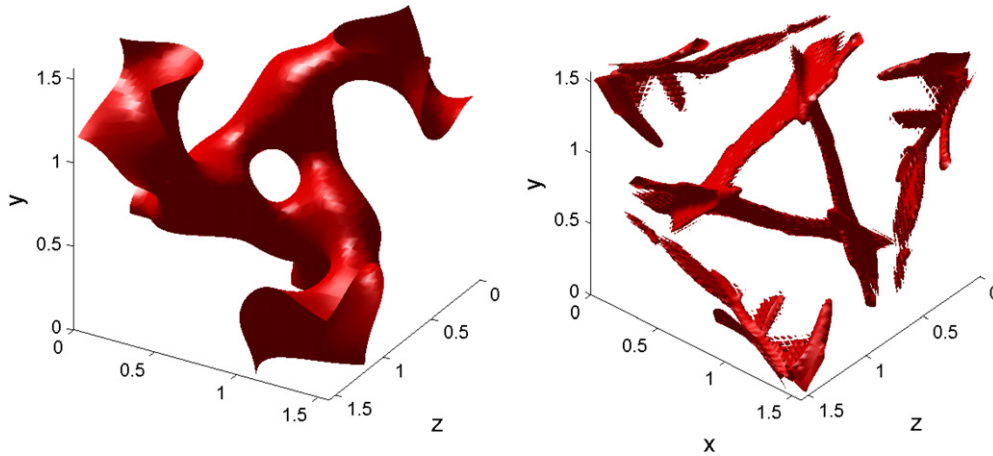


Fig. 11. Left: Vorticity magnitude isosurface for same parameter values as in Fig. 10, different view. Right: $t = 0.58$, $|\omega| = 37.5\omega_0$.

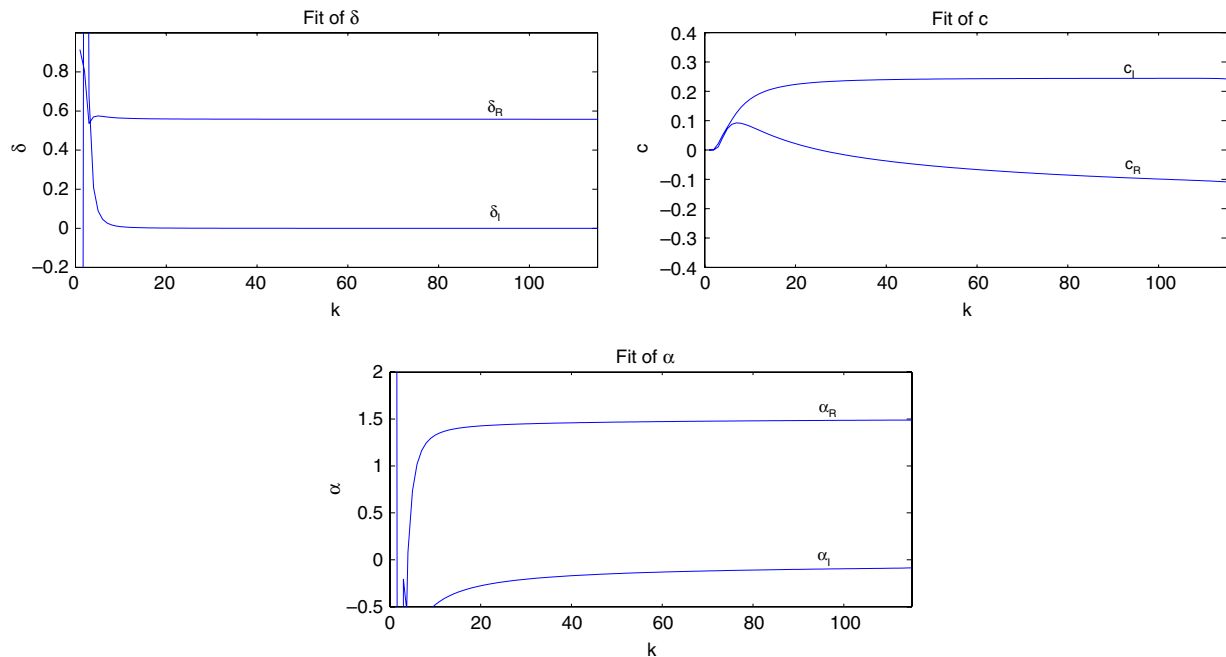


Fig. 12. Results from sliding 3 point fit of the Fourier coefficients at $(y, z) = (y_0, z_0)$ as a function of k . The graphs show δ_R , δ_I , c_R , c_I , α_R , and α_I .

with ξ given by (4.43). Similar fits are obtained for (y, z) near (y_0, z_0) and for the velocity components v and w .

By way of comparison, Fig. 13 shows the Fourier fits for (c, α, x_0) corresponding to a travelling wave speed $\Sigma = (1, 0, 0)$ and data of the form (3.31) with

$$\begin{aligned}\hat{v}_{100} &= \hat{w}_{100} = a_0 i \\ \hat{u}_{101} &= -\hat{w}_{101} = 0.25b_0 \hat{w}_{100} \\ \hat{u}_{110} &= -\hat{v}_{110} = \hat{u}_{101}\end{aligned}\quad (4.61)$$

where (a_0, b_0) are $(-1.1, 0.7)$ (solid curves) and $(-0.95, 0.975)$ (dashed curves). The fits are constructed at $(y_0, z_0) = (0, 0)$, and show a slow decay toward constant values. In particular, the singularity exponent α_R at $k_{max} = 140$ is near 1.34 for the solid curve, 1.41 for the dashed curve, but in both cases the exponent is slowly increasing at k_{max} . Although the α_R profiles indicate an algebraic singularity, the value of the exponent is not determined reliably; fits for higher k are needed, but the computer costs become prohibitive. However, the fits are certainly consistent with $\alpha_R \approx 1.5$.

5. Discussion

The computations presented above reveal solutions to the 3D Euler equations that have singularities for complex values of the x -space variable, when y and z are taken as real. The singularities move with constant speed in $Im(x)$ and reach the real- x line in finite time. Thus, the computations show the development of singularities from smooth initial data for complex valued solutions of the Euler equations.

The significance of these results is that they provide simple examples of 3D solutions with singularities. In addition, the combination $\mathbf{u} = \mathbf{u}_+ + \bar{\mathbf{u}}_+$ is a real-valued velocity which is an approximate solution to the Euler equations. We conjecture that the complex singularities detected here are generic (i.e., stable with respect to perturbations) for real-valued flows [36]. Even if the width $\delta(t)$ of the analyticity strip in a real flow does not reach zero in finite time, the close proximity of complex singularities to the real-space domain is related to the growth of vorticity in the flow. Furthermore, the geometry of the singular surface is connected to the anisotropy of the region of maximum vorticity.

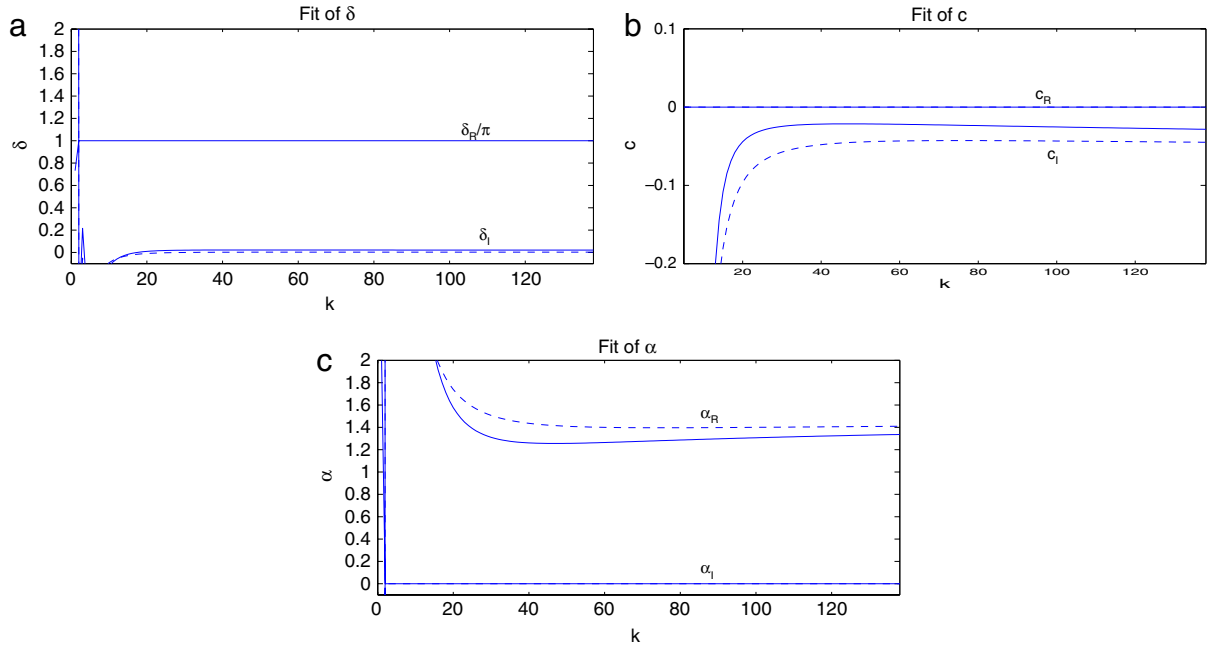


Fig. 13. Results from sliding fit for the Fourier coefficients at $(y_0, z_0) = (0, 0)$ as a function of k . In (a) the graphs are of δ_I and δ_R/π , (b) c_R and c_I , (c) α_R and α_I .

The connection between the complex singular surface and growth of vorticity in a real-valued flow is made concrete by considering γ , the vortex stretching rate projected onto the unit vorticity vector, i.e.,

$$\gamma = \mathbf{v} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega} \quad \text{where } \mathbf{v} = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}. \quad (5.62)$$

For real velocity $\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-$ and vorticity $\boldsymbol{\omega} = \boldsymbol{\omega}_+ + \boldsymbol{\omega}_-$, the stretching rate γ contains terms with products of like wavenumber such as $\boldsymbol{\omega}_+ \cdot \nabla \mathbf{u}_+ \cdot \boldsymbol{\omega}_+$ and cross terms like $\boldsymbol{\omega}_- \cdot \nabla \mathbf{u}_+ \cdot \boldsymbol{\omega}_-$. We estimate γ by setting $\mathbf{u} = \mathbf{u}_+$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_+$, i.e., neglecting products of cross terms. (Equivalently, we neglect the interaction with the complex conjugate singular surface.) This is expected to give the dominant contribution to γ in the neighbourhood of the singular surface.

We compute γ corresponding to a log singularity in the velocity. Assume that the leading order singular behaviour of $\mathbf{u}$ and $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \mathbf{W} \xi^{-1} \quad \mathbf{u} = \mathbf{U} \log \xi \quad (5.63)$$

where $\mathbf{U} = \mathbf{U}_0 + O(\ln^{-1} \xi)$, $\mathbf{W} = \mathbf{W}_0 + O(\xi \ln \xi)$ with ξ given by (4.43), and where $\mathbf{W}_0$ and $\mathbf{U}_0$ are analytic in a neighbourhood of $\xi = 0$. Note that

$$\nabla \cdot \mathbf{u} = (\nabla \cdot \mathbf{U}) \log \xi + \nabla \xi \cdot \mathbf{U} \xi^{-1} = 0 \quad (5.64)$$

which implies

$$\nabla \xi \cdot \mathbf{U} = -\xi \log \xi \nabla \cdot \mathbf{U}. \quad (5.65)$$

Similarly, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ implies that

$$\mathbf{W} = \nabla \xi \times \mathbf{U} + (\nabla \times \mathbf{U}) \xi \log \xi. \quad (5.66)$$

Eqs. (5.65) and (5.66) show that $\nabla \xi$, $\mathbf{U}$, and $\mathbf{W}$ form an orthogonal triad on the singular surface $\xi = 0$. Calculate

$$\boldsymbol{\omega} \cdot \nabla \mathbf{u} = (\mathbf{W} \cdot \nabla \mathbf{U}) \xi^{-1} \log \xi + (\mathbf{W} \cdot \nabla \xi) \mathbf{U} \xi^{-2}. \quad (5.67)$$

Taking the dot product of (5.66) with $\nabla \xi$ and substituting into (5.67) gives

$$\boldsymbol{\omega} \cdot \nabla \mathbf{u} = [\mathbf{W} \cdot \nabla \mathbf{U} + (\nabla \times \mathbf{U} \cdot \nabla \xi) \mathbf{U}] \xi^{-1} \log \xi. \quad (5.68)$$

Here, the orthogonality of $\nabla \xi$ and $\mathbf{W}$ is critical since it reduces the leading order singularity from ξ^{-2} to $\xi^{-1} \log \xi$. The main result follows from taking the dot product with $\mathbf{v}$ and applying the sup norm

$$\|\boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \mathbf{v}\|_\infty = \left\| \mathbf{W} \cdot \nabla \mathbf{U} \cdot \frac{\mathbf{W}}{|\mathbf{W}|} \xi^{-1} \log \xi \right\|_\infty, \quad (5.69)$$

where we have used the orthogonality of $\mathbf{U}$ and $\mathbf{W}$.

Eq. (5.69) provides the following estimate on the maximum vortex stretching projected onto the direction of vorticity

$$\|\boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \mathbf{v}\|_\infty \sim C \|\boldsymbol{\omega}\|_\infty \log \|\boldsymbol{\omega}\|_\infty, \quad (5.70)$$

where C is a constant. Here we have employed $\|\xi^{-1}\|_\infty \sim \|\boldsymbol{\omega}\|_\infty$ and $\|\log \xi\|_\infty \sim \log \|\boldsymbol{\omega}\|_\infty$. Using the equation that governs the magnitude of vorticity [6],

$$\frac{\partial |\boldsymbol{\omega}|}{\partial t} + \mathbf{u} \cdot \nabla |\boldsymbol{\omega}| = \boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \mathbf{v},$$

one can easily show that in a real-valued flow (5.70) implies double exponential growth in maximum vorticity $\|\boldsymbol{\omega}\|_\infty \sim e^{e^t}$. Thus, the logarithmic singularity (5.63) with a smooth surface given by (4.43) cannot reach the real domain in finite time (i.e., singularities of this type cannot occur in a real-valued Euler flow), although we do not rule out nonsmooth singular surfaces or those with stronger singularities than log from reaching the real domain. The relation (5.70) and double exponential growth in maximum vorticity are consistent with the maximum vortex stretching and growth in vorticity observed by Hou and Li [22] in their numerical calculations for antiparallel vortex tube initial data, although additional study is necessary to connect the behaviour there to the approach of a particular complex singular surface.

In addition, the paraboloidal geometry of the complex singular surface is connected to the anisotropy of the region of maximum vorticity. If we replace x by $x' = x - \mathbf{A} \cdot \mathbf{Y}$ in (4.43) and then introduce variables y', z' so that (x', y', z') is an orthogonal coordinate system, Eq. (4.43) becomes

$$\xi = x' - i\sigma t + iZ(x', y', z') = 0 \quad (5.71)$$

where Z is a quadratic function of x', y', z' . For a singular surface near $\text{Im } x' = 0$, relations (5.71) and (5.66) show the region of large

vorticity in the neighbourhood of the maximum at $\text{Re } x' = y' = z' = 0$ is a thin sheet-like domain. The reason is that vorticity decays rapidly (linearly) as x' ranges away from 0, but slowly (quadratically) as y' and z' veer from 0. This is also consistent with the simulations of Hou and Li, who find that a sheet-like region of maximum vorticity develops during evolution.

6. Conclusion

The computations presented above describe complex singular solutions of the fully 3D incompressible Euler equations. The numerical instability affecting the computations in [25,26] is avoided by utilising a highly accurate, fully spectral numerical method which is devoid of aliasing or truncation error in the restriction to finite $\mathbf{k}$, and by generating the unstable travelling wave in a simple way. The computations show that singularities in the velocity field $\mathbf{u}_+$ have a simple form, given by (1.8) and (1.9), with the singular surface given as the zero level set of (4.43). The singularities can be made to have a small amplitude, thereby satisfying a condition for the perturbation construction of a real singular solution proposed in [26]. While such a construction is beyond the scope of this paper, it is noted that complex singular solutions of the type found here are not expected to yield real (finite time) singular solutions. The reason, as discussed in Section 5, is the vortex stretching rate γ (cf. (5.62)) does not grow fast enough to be consistent with a real Euler singularity. Instead, the growth of γ for the logarithmic singularity (1.9) is shown to imply a double exponential growth in the maximum vorticity, $\|\omega\|_\infty = e^{e^t}$, consistent with the numerical calculations in [22].

In future work, we hope to extend the search for different singularity types and to find a way to predict or control the singularity exponent, with the goal of finding complex singularities in which the growth in γ is consistent with a real finite time singularity.

Acknowledgements

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Appendix

Consider an upper analytic function $f(z)$ with singularities at $z = z_j, j = 1, \dots, n$, with local behaviour

$$f(z) \approx c_j(z - z_j)^\nu \log^p(z - z_j),$$

for z near z_j , where $\nu > -1$. Following [33], we calculate the Fourier integral by contour deformation. Define the Fourier integral

$$I(k) = \int_C e^{-ikz} f(z) dz = \sum_{j=1}^n I_j(k) \quad (\text{A.72})$$

where the original contour C and deformed contour $\bigcup_{j=1}^n C_j$ is illustrated in Fig. 14, and $I_j(k)$ is the contribution to $I(k)$ from the integral over C_j . The leading order contribution to the integral over C_j is given by

$$I_j(k) \sim c_j \int_{C_j} e^{-ikz} (z - z_j)^\nu \log^p(z - z_j) dz.$$

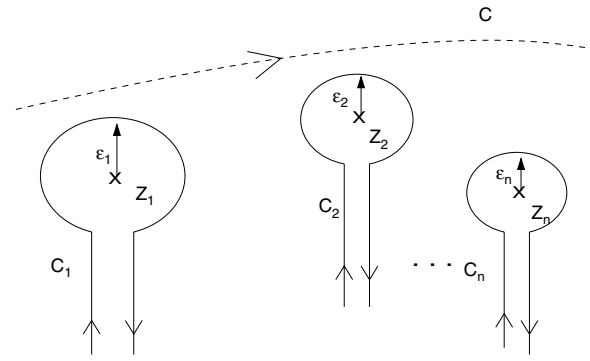


Fig. 14. Illustration of original integration contour C and deformed contour $\bigcup_{j=1}^n C_j$.

Set $z - z_j = re^{i\theta}$ where $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Then

$$\begin{aligned} I_j(k) &\sim c_j \left\{ \int_0^\infty e^{-ik(z_j + re^{\frac{3\pi i}{2}})} r^\nu e^{\frac{3\pi i}{2}\nu} \log^p(re^{\frac{3\pi i}{2}}) e^{\frac{3\pi i}{2}} dr \right. \\ &\quad \left. + \int_0^\infty e^{-ik(z_j + re^{-\frac{\pi i}{2}})} r^\nu e^{-\frac{\pi i}{2}\nu} \log^p(re^{-\frac{\pi i}{2}}) e^{-\frac{\pi i}{2}} dr \right\} \\ &\sim ic_j e^{-ikz_j} \int_0^\infty e^{-kr} r^\nu \left\{ e^{\frac{3\pi i}{2}\nu} \log^p(re^{\frac{3\pi i}{2}}) \right. \\ &\quad \left. - e^{-\frac{\pi i}{2}\nu} \log^p(re^{-\frac{\pi i}{2}}) \right\} dr \end{aligned} \quad (\text{A.73})$$

where we have used the fact that the integral around the circular part of C_j tends to zero as $\epsilon_j \rightarrow 0$. Now, let $r' = kr$. Then (A.73) becomes

$$\begin{aligned} I_j(k) &\sim ic_j e^{-ikz_j} k^{-(1+\nu)} \int_0^\infty e^{-kr'} r'^\nu \\ &\quad \times \left\{ e^{\frac{3\pi i}{2}\nu} \left(\log r' - \log k + \frac{3\pi i}{2} \right)^p \right. \\ &\quad \left. - e^{-\frac{\pi i}{2}\nu} \left(\log r' - \log k - \frac{i\pi}{2} \right)^p \right\} dr'. \end{aligned} \quad (\text{A.74})$$

After factoring $(-\log k)$ out of the terms in parentheses and expanding in $\frac{1}{\log k}$ for large k , we obtain (after some algebra)

$$\begin{aligned} I_j(k) &\sim c_j e^{-ikz_j} k^{-(1+\nu)} (-\log k)^p \left\{ -2(\sin \pi \nu) e^{\frac{\pi i}{2}\nu} \Gamma(1+\nu) \right. \\ &\quad \left. - \frac{p\pi}{2} e^{-\frac{\pi i}{2}\nu} (1 + 3e^{\pi i \nu}) \Gamma(1+\nu) (-\log k)^{-1} \right. \\ &\quad \left. - 2p(\sin \pi \nu) e^{\frac{\pi i}{2}\nu} \left(\int_0^\infty e^{-r'} r'^\nu \log r' dr' \right) (-\log k)^{-1} \right\}, \end{aligned} \quad (\text{A.75})$$

at leading order for $k \gg 1$, where $\Gamma(z)$ is the Gamma function. The leading order contribution to $f(z)$ comes from the singularity z_j which maximises $\text{Im } z_j$ over $j = 1, \dots, n$. Setting $x_0 = z_j$, $c = c_j$, $\beta = p - 1$ and $\alpha = \nu + 1$, we obtain (3.35)–(3.37).

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