Chapter 1

MONTE CARLO SIMULATION FOR AMERICAN OPTIONS

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Abstract

This paper reviews the basic properties of American options and the difficulties of applying Monte Carlo valuation to American options. Asymptotic results by Keller and co-workers are described for the singularity in the early exercise boundary for time t near the final time T. Recent progress on application of Monte Carlo to American options is described including the following: Branching processes have been constructed to obtain upper and lower bounds on the American option price. A Martingale optimization formulation for the American option price can be used to obtain an upper bound on the price, which is complementary to the trivial lower bound. The Least Squares Monte Carlo (LSM) provides a direct method for pricing American options. Quasi-random sequences have been used to improve performance of LSM; a brief introduction to quasi-random sequences is presented. Conclusions and prospects for future research are discussed. In particular, we expect that the asymptotic results of Keller and co-workers could be useful for improving Monte Carlo methods.

Keywords:

American options, Monte Carlo, martingale optimization, least squares Monte Carlo, American put, American call with dividends, quasi-Monte Carlo, Brownian bridge, Least Squares Monte Carlo

1. Introduction

American options are derivative securities for which the holder of the security can choose the time of exercise. In an American put, for example, the option holder has the right to sell an underlying security for a specified price K (the strike price) at any time between the initiation of the agreement ($t = 0$) and the expiration date ($t = T$). The exercise time τ can be represented be represented as a stopping time; so that American options are an example of optimal stopping time problems.

Valuation of American options presents at least two difficulties. First, there is a singularity in the option characteristics at the expiration time. For American puts and calls on equities with dividends, a thorough analysis of this singularity was performed by Evans, Kuske and Keller [10]. These results are briefly described in Section 3

A second difficulty occurs for Monte Carlo valuation of American options, the main subject of this paper. Monte Carlo methods are required for options that depend on multiple underlying securities or that involve path dependent features. Since determination of the optimal exercise time depends on an average over future events, Monte Carlo simulation for an American option has a "Monte Carlo on Monte Carlo" feature that makes it computationally complex.

In this paper, we review several methods for overcoming this difficulty with American options. The first, developed by Broadie and Glasserman [5] and presented in Section 4, involves two branching processes, the first of which provides an upper bound and the second a lower bound on option price. The second method, presented in Section 5, is a martingale optimization formula developed in [29] that provides a dual formulation of the Monte Carlo valuation formula and leads naturally to an upper bound on the option price. The third (Section 6) is the Least Squares Monte Carlo (LSM) method derived by Longstaff and Schwartz [19]. Finally we describe work by the authors on use of quasi-random sequences in LSM [8] in Section 7.

A brief introduction to the salient features of American options is given in Section 2 and a discussion of conclusions and prospects for future research is described in Section 8.

2. American options

In this section we describe some of the basic features of American options. These include the Black-Scholes PDE and the risk-neutral valuation formula for option price, the optimal exercise boundary, and the "Monte Carlo on Monte Carlo" difficulty.

Consider an equity price process $S(t)$ that follows an exponential Brownian motion process according to the following stochastic differential equation

$$dS = \mu S dt + \sigma S d\omega \tag{1.1}$$

in which μ and σ are the average growth rate and volatility (both assumed to be constant) and $\omega = \omega(t)$ is standard Brownian motion.

2.1 Option payout and early exercise

The option payout function is $u(S,t)$. A path dependent option is one for which $u(S,t)$ depends on the entire path $\{S(t') : 0 < t' < t\}$; whereas a simple (non-path dependent) option has $u(S,t) = u(S(t),t)$. For a simple European option the payout may only be collected at the final time so that it is $f(T) = u(S(T),T)$. For a simple American option, exercise may be at any time before T so that the payout is $f(\tau) = u(S(\tau),\tau)$ in which τ is an optimally chosen stopping time. The reason τ is a stopping time is that the decision of whether to exercise at time t can only depend on the values of S up to and including t.

Examples of simple payout function are a call, for which $u = \max(S - K, 0)$, and a put, for which $u = \max(K - S, 0)$. Examples of path dependent payouts are the Asian option u_A and the lookback u_L given by

$$u_A = U\left((t - t_0)^{-1} \int_{t_0}^{t} S(t')dt'\right) \tag{1.2}$$

$$u_L = U\left(\max_{t_0 < t' < t} S(t')\right) \tag{1.3}$$

in which U is some function such as the call or put payout. In u_A and u_L, the lower time limit t_0 could be 0 or it could be $t - \Delta$, in which the average is taken over a moving time window of length Δ. The latter case is particularly difficult because the resulting American price is not a Markov process.

The early exercise boundary is the set in time and state space on which exercise of the American option is optimal. For a simple option, this is just a curve $S = S^*(t)$ in the space (S,t). For a path dependent security, the exercise decision depends on more that $S(t)$ and t, so that the early exercise boundary is more complicated.

2.2 Black-Scholes PDE and risk-neutral valuation for American options

In their classic papers, Black and Scholes [3] and Merton [21] described two methods for valuation of derivative securities. The first is the Black-Scholes PDE. For an American option with value F, the Black-Scholes PDE is

$$F_t + rSF_S + \sigma^2 S^2 F_S S = rF \tag{1.4}$$

in which r is the risk-free rate of return. The "final condition" is

$$F(S,T) = u(S,T) \tag{1.5}$$

and the boundary conditions on the free boundary $S = S^*(t)$ are

$$F = u \tag{1.6}$$

$$F_S = u_S. \tag{1.7}$$

The second method, which is applicable to path-dependent options and other derivatives for which the PDE is either unavailable or intractable, is the risk-neutral valuation formula

$$F(S,t) = \max_{t<\tau<T} E'[e^{-r(\tau-t)}u(S(\tau),\tau) \mid S(t) = S] \tag{1.8}$$

in which E' is the risk-neutral expectation, for which the growth rate μ in (1.1) is replaced by r. This is the formula to which Monte Carlo quadrature can be applied.

This risk-neutral valuation approach provides a stochastic characterization of the early exercise boundary. Consider the exercise decision at a point (S,t). The value of early exercise is just the payoff $u(S,t)$. The expected value of deferred exercise is $\tilde{F}$ given by

$$\tilde{F} = \max_{t<\tau<T} E'[e^{-r(\tau-t)}u(S(\tau),\tau) \mid S(t) = S]. \tag{1.9}$$

The holder of the option will choose to exercise if $u \geq \tilde{F}$, so that

$$F = \max(u(S,t), \tilde{F}) \tag{1.10}$$

and $u(S^*(t),t) = \tilde{F}$ on the early exercise boundary.

A lower bound on the American option price follows from the formula (1.10). Let τ' be any stopping time and let F' be the price using this stopping time; i.e.

$$F' = E'[e^{-r(\tau'-t)}u(S(\tau'),\tau') \mid S(t) = S] \tag{1.11}$$

then

$$F \geq F'. \tag{1.12}$$

2.3 American options on trees: rolling-back on the tree

For a security whose price is modeled on a binary tree, evaluation of an American option with simple payout is straightforward by "rolling-back" on the tree. Suppose that the security price S_n follows the following process:

$$S_{n+1} = \begin{cases} uS_n & \text{with probability } p \\ dS_n & \text{with probability } 1-p \end{cases} \tag{1.13}$$

in which p is the risk-neutral probability and the discrete time variable runs over the values $n = 0, 1, \ldots N$. Assume that the discount factor over a single time period is $e^{r\delta t}$

At the final time N, exercise is determined by whether the payout is positive or not. Consider a time k before the final time and suppose that the price F_m has been found for all times m with $m > k$. The price F_k at a point S_k is determined as in (1.10). Set

$$\tilde{F}'_k = E'_{S_k}[e^{-r\delta t} F(S_{k+1})] \tag{1.14}$$

and then

$$F_k = \max(u(S_k), \tilde{F}_k). \tag{1.15}$$

In (1.14), the expectation is the empirical average over a chosen set of branches that continue from S_k.

This straightforward method does not apply to path-dependent securities. Even at the first step, the value at the final time depends on the path history for each path separately, not just on the value S_N at the final time. These difficulties with path-dependent securities are similar to the difference between path-integrals and PDEs. Instead Monte Carlo methods are required for evaluation of path-dependent options. If the security is modeled as a process on a tree, then Monte Carlo can be performed on the tree as well.

Similarly for options that depend on a large number of underlying securities (or random factors), Monte Carlo is required because the corresponding Black-Scholes equation will have a large number of spatial dimensions, making it computationally intractable. For a tree in a large number of dimensions, a deterministic treatment on the tree would require branching for every dimension at every time step, which is also intractable.

3. Asymptotics for American puts and calls with dividends

The most common form of American options are American puts and calls. For an American call without dividends, early exercise of the option is never optimal, so that an American call (without dividends) is no different than a European call. On the other hand, early exercise may be optimal for an American put or an American call on a security that pays dividends.

Calls are not exercised early because of the risk premium that offers a better average rate of return for a risky asset than for a non-risky investment. For a put on the other hand, at some low value of the security the option holder can gain more by taking the payout and investing it at the risk-free rate of return than by holding the option in hopes of a small increase in payment later. This same argument works for a call with dividend.

When the early exercise boundary $S = S^*(t)$ hits the final time $t = T$, there is a singularity in the exercise boundary shape, which is characteristic of a many free boundary problems. In addition, $S^*(T)$ (the intersection of the early exercise boundary and the final time) may differ from K (the exercise boundary at the final time).

While these properties have long been recognized, the detailed asymptotics of the singularity in the early exercise boundary were not analyzed until recently. Evans, Kuske and Keller [10] derived the shape of the early exercise boundary for American put and call with dividends by two alternative methods: asymptotics for an integral equation formulation and matched asymptotics for the Black-Scholes PDE. The dividends are assumed to payout at a continuous rate D. The early exercise boundary $S_P^*(t)$ for the American put and $S_C^*(t)$ for the American call satisfy the following:

$$S_P^*(t) = \begin{cases} K + c_1\sqrt{(T-t)\log[1/(T-t)]} & \text{if } 0 \le D < r \\ K + c_2\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D = r \\ (r/D)(K + c_3\sqrt{T-t}) & \text{if } D > r \end{cases} \tag{1.16}$$

$$(1.17)$$

$$S_C^*(t) = \begin{cases} K + c_1\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D > r \\ K + c_2\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D = r \\ (r/D)(K + c_3\sqrt{T-t}) & \text{if } 0 \le D < r \end{cases} \tag{1.18}$$

in which c_1, c_2, c_3 are constants that depend on σ, D and r. Note that for $D > r$, $S_P^*(T) = (r/D)K < K$ and for $D < r$, $S_C^*(T) = (r/D)K > K$ which shows that the exercise boundary on the final boundary is not on the early exercise curve. Also as $D \to 0$, the early exercise boundary for the American call goes away to infinity.

Caflisch and Goldenfeld [7] have developed a cellular automata approach to evaluation of American calls and puts with dividends. In this representation, diffusing particles are emitted from the singularity and absorbed at the free boundary. In this method the principal effects are generated from the singularity point, so that the asymptotics of Evans, Kuske and Keller [10] could possibly be used to improve the convergence of the representation.

4. Branching processes

The "Monte Carlo on Monte Carlo" property can be seen in the decision formula (1.10). Consider a simulated path and a point $(S(T), t)$ on that path. In order to decide whether to exercise at that point, one must evaluate the expectation in (1.9). This in turns requires continuation from $(S(T), t)$ on many paths. Therefore this direct Monte Carlo simulation of the American option requires a set of continuously branching paths, which is computationally intractable.

Broadie and Glasserman [5] consider a Bermudan option; i.e., an option in which exercise can occur at any one of a discrete number $d+1$ times $t_0, \ldots, t_d$. They constructed two branching processes, each with b branches at each exercise time. The first process provides an upper estimate F_u and the second a lower estimate F_ℓ, on average; i.e.

$$E[F_\ell] \leq F \leq E[F_u]. \tag{1.19}$$

In addition, both processes converge to the correct price as the branching number b and the number of paths N increase; i.e.

$$\lim_{b\to\infty,\ N\to\infty} F_\ell = \lim_{b\to\infty,\ N\to\infty} F_u = F. \tag{1.20}$$

On the other hand, this construction is computationally complex with CPU time that scales like $O(Nb^d)$.

In both processes the price is determined by "rolling-back" on the branched paths. At the final time, exercise is determined by whether the payout is positive or not. Consider a time t_k before the final time and suppose that the price has been found for all times t_m with $m > k$. The price F_k at a point (S_k, t_k) is determined as in (1.10). Set

$$\tilde{F}_k' = E'_{S_k,t_k}[e^{-r(t_{k+1}-t_k)}u(S_{k+1}, t_{k+1})] \tag{1.21}$$

and then

$$F_k = \max(u(S_k, t_k), \tilde{F}_k). \tag{1.22}$$

In (1.21), the expectation is the empirical average over a chosen set of branches that continue from (S_k, t_k).

The difference between the upper and lower processes is in which paths are used in the expectation of (1.21). In the upper process all of the branches are used. Since the early exercise decision uses knowledge of the future for the finite set of branching paths, then the price estimate F_u is biased high. This gives the upper estimate in (1.19).

For the lower process, at each decision points, one of the branches is designated to be the continuation branch. The average in (1.21) is determined using the other $b-1$ branches. The value of this empirical average is independent of the continuation branch, but since the average is approximate, the resulting exercise decision is suboptimal. Therefore the resulting price estimate F_ℓ is biased low. This gives the lower estimate in (1.19).

As stated in [5], it seems quite likely that there is no unbiased, convergent Monte Carlo estimator of the American option price. Their construction shows that this should be true because correlations with the future lead to upward bias and independence of the future leads to non-optimal early exercise causing downward bias.

5. Martingale optimization

Rogers [29] derived a formula for the American option price that is dual to the formula in (1.8):

$$F(0) = \min_M E'[\max_{0<t'<T} \left(e^{-rt'}u(t') - M(t')\right)] \tag{1.23}$$

in which the minimum is taken over all martingales for which $M(0) = 0$. Similar formulas were derived by Anderson and Broadie [2] and Kogan and Haugh [17].

By insertion of a (non-optimal) martingale M into (1.23), one gets a upper bound on F. This has been carried out for various choices of M in [2, 17, 18, 29]. Chaudhary [9] has used this to form an approximate method for hedging the American option.

In this method, it is difficult to determine the accuracy of the upper bound, because it is difficult to quantify the degree of non-optimality for the martingale that is used in (1.23) in order to generate the upper bound. The following characterization of the optimal Martingale may be a starting point for determining the degree of optimality. The optimal martingale $M^*(t)$ is the one coming from the American itself, for which (using the Martingale decomposition of $F(t)$)

$$M^*(t) = e^{-rt}F(t) - F(0) + A(t) \tag{1.24}$$

in which $A(t)$ is a nondecreasing process with $A(0) = 0$. As shown by Rogers [29],

$$\begin{aligned} F(0) &\leq E[sup_{0<t<T}(e^{-rt}u(t) - M^*(t))] \\ &= F(0) + E[sup_{0<t<T}\left(e^{-rt}(u(t) - F(t)) - A(t)\right)] \end{aligned} \tag{1.25}$$

Since both $-u(t) + F(t)$ and $A(t)$ are nonnegative, they must both be 0; i.e.

$$sup_{0<t<T}(e^{-rt}u(t) - M^*(t)) = F(0). \tag{1.26}$$

One can also show that (1.26) uniquely characterizes the correct price $F(0)$.

6. Least squares Monte Carlo (LSM)

Longstaff and Schwartz [19] introduced a new approach to Monte Carlo evaluation of American options by replacing the future expectation by a least squares interpolation, based on earlier work of Tsitsiklis and Van Roy [33]. The method starts with N random paths (S_n^k, t_n) for $1 \leq k \leq N$ and $t_n = ndt$. Valuation is performed by rolling-back on these paths.

Suppose that $F_{n+1}^k = F(S_{n+1}^k, t_{n+1})$ is known. For points (S_n^k, t_n) set $X = S_n^k$ the current equity value and $Y^k = e^{-rdt}F(S_{n+1}^k, t_{n+1})$ the value of deferred exercise. Then perform regression of Y as a function of the polynomials $X, X^2, \ldots, X^m$ for some small value of m; i.e. approximate Y by a least squares fit of these polynomials in X. Use this regressed value as an approximation to $\tilde{F}$ in (1.9) and apply it in deciding whether to exercise early.

In this method the least squares fit provides coupling between the prices on different Monte Carlo paths. This coupling replaces the Monte Carlo-on-Monte Carlo feature of direct Monte Carlo evaluation of American options, without the computational intractability of direct method. On the other hand, the efficiency of the LSM method depends on only using a small number m of polynomials in the least squares fit. The strong accuracy that is attained with such small degree for the polynomial fit is remarkable.

Longstaff and Schwartz have applied this method to puts, Asian options, swaps, swaptions and other options with excellent results for small m. A convergence proof for the LSM method has been constructed in [32].

7. Quasi-Monte Carlo for LSM

In their LSM paper, Longstaff and Schwartz [19] suggested that their method might be improved by the use of quasi-random points. There are two potential difficulties with this extension of the method: the problem is high dimensional and the prices along different paths in the LSM method are correlated, both of which can be problematic for quasi-Monte Carlo.

7.1 Quasi-Monte Carlo

Quasi-random sequences [26, 14] are a deterministic alternative to random or pseudo-random points. The distribution of quasi-random points is much more uniform that than of random points, because of correlations between the points that are designed to keep them from clumping. As a result, Monte Carlo quadrature in d dimensions using N quasi-random points (i.e. quasi-Monte Carlo) can converge at a rate $N^{-1}(\log N)^d$, as opposed to convergence at rate $N^{-1/2}$ for random points. The uniformity of a quasi-random sequence is measured in terms of its discrepancy, which is defined as the maximum error in the Monte Carlo estimate of the volume of a rectangular set.

The exponent d for the log indicates that the advantages of the method can breakdown for large dimension [23]. In addition, this convergence rate is not attained for quasi-Monte Carlo quadrature on functions that are not smooth [24]. Making effective use of quasi-random sequences requires specially adapted techniques in order to avoid these limitations on the method [6, 31], including smoothing of nonsmooth integrands and reduction of effective dimension. Note, however, that when the effectiveness of quasi-Monte Carlo is lost, its typical performance is the same as that of standard Monte Carlo.

Examples of quasi-random sequences include the sequences of Haselgrove [13], Halton [12], Faure [11], Sobol [30] and Niederreiter [34]. Software for generating quasi-random sequences can be found, for example, in [4, 28].

A promising alternative acceleration of quasi-Monte Carlo is through randomization of the quasi-random sequence [27] which gives convergence rates of almost $N^{-3/2}$ for quadrature of smooth functions.

When using quasi-random sequences in Gaussian random variables [22], one must use the inverse of the normal CDF (see [16] for an algorithm) rather than the Box-Muller method (e.g. [20]).

7.2 The Brownian bridge method

The standard discretization of a random walk is to represent the position $b(t+\Delta t)$ in terms of the previous position $b(t)$ by the formula

$$b(t+\Delta t) = b(t) + \sqrt{\Delta t}\ \nu \tag{1.27}$$

in which ν is an $N(0,1)$ random variable. Using a sequence of independent samples of ν we can generate the random walk sequentially by

$$y_0 = 0,\ y_1 = b(\Delta t),\ y_2 = b(2\Delta t),\ \ldots \tag{1.28}$$

Evaluation of path integrals, such as the risk-neutral expectation for the price of a path-dependent security, using this representation for Brownian motion results in a high dimensional integral, for which the effectiveness of quasi-Monte Carlo is lost.

The effectiveness of quasi-Monte Carlo for path integrals will be regained using an alternative representation of the random walk, the Brownian Bridge Discretization, which was first introduced as a quasi-Monte Carlo technique by [25]. This representation relies on the following Brownian bridge formula [15] for $b(t+\Delta t_1)$ knowing $b(t)$ and $b(T = t+\Delta t_1+\Delta t_2)$:

$$b(t+\Delta t_1) = ab(t) + (1-a)b(T) + c\nu \tag{1.29}$$

in which

$$\begin{aligned} a &= \Delta t_2/(\Delta t_1+\Delta t_2) \\ c &= \sqrt{a\Delta t_1}. \end{aligned} \tag{1.30}$$

Using this representation, the random walk can be generated by successive subdivision. Suppose for simplicity that M is a power of 2. Then generate the random walk in the following order:

$$y_0 = 0,\ y_M,\ y_{M/2},\ y_{M/4},\ y_{3M/4},\ \ldots. \tag{1.31}$$

The significance of this representation is that it first chooses the large time steps over which the changes in $b(t)$ are large. Then it fills in the small time steps in between, in which the changes in $b(t)$ are quite small. The advantage of this representation is that it concentrates the variance into the early, large time steps. A similar, more general but slower, approach was developed in [1] using principal component analysis.

Although the actual dimension of the problem is not changed, in some sense the effective dimension of the problem is lowered, so that quasi-Monte Carlo retains its effectiveness. To make this statement quantitative, suppose that at some given value of N, the discrepancy is of size

N^{-1} for dimension d (omitting logarithmic terms for simplicity), but is of size $N^{-1/2}$ for the remaining dimensions. We expect that the integration error is roughly of the size of the variance times the discrepancy. Using the Brownian bridge discretization, the variance over the first d dimensions is σ_0, which is about the same size as the original value of σ; whereas the variance over the remaining $M - d$ dimensions is σ_1, which is much smaller; i.e.,

$$\sigma_1 << \sigma_0 \approx \sigma \tag{1.32}$$

Denote ε_s and ε_{bb} to be the errors for quasi-Monte Carlo using the standard discretization and the Brownian bridge discretization, respectively. Then approximately

$$\begin{aligned} \varepsilon_s &= \sigma N^{-1/2} \\ \varepsilon_{bb} &= \sigma_0 N^{-1} + \sigma_1 N^{-1/2}. \end{aligned} \tag{1.33}$$

The ordering (1.32) then implies that

$$\varepsilon_{bb} << \varepsilon_s. \tag{1.34}$$

This shows that the Brownian bridge discretization can provide a significant improvement in quasi-Monte Carlo integration for path integral problems.

7.3 Application to the LSM method

Chaudhary [8] implemented a Brownian bridge construction for the paths in the LSM method. As described above, this can reduce or remove the high dimensionality difficulty for quasi-Monte Carlo quadrature of path dependent securities. In addition, the Brownian bridge method shows that the memory requirements of the LSM method can be significantly reduced. The potential difficulty with correlations between the paths did not turn out to be much of a problem, perhaps because the true correlations are via the early exercise boundary which is deterministic.

8. Conclusions

Our intention in writing this paper is to describe the difficulties involved in applying Monte Carlo evaluation to American options, as well as several recent methods that are quite promising for overcoming these difficulties. Here are some directions that we believe to be promising for future research.

The singularity in the early exercise boundary at the final time has been well characterized by Keller and co-workers [10], at least for call and put options. The information in these asymptotic results could be valuable in improving Monte Carlo simulations.

For Martingale optimization, there is not yet a good method for choosing the Martingale in order to get a good approximation. In particular, one might hope to find an iterative method, in which an approximate Martingale would be modified at each step in order to improve the approximation over that of the previous step.

The LSM method with random or quasi-random sequences has been shown to work well on a good selection of examples, but it still needs to be validated for more complicated examples, such as American Asian options with the average taken over a moving window.

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