Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem

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Abstract

The Muskat, or Muskat-Leibenzon, problem describes the evolution of the interface between two immiscible fluids in a porous medium or Hele-Shaw cell under applied pressure gradients or fluid injection/extraction. In contrast to the Hele-Shaw problem (the one-phase version of the Muskat problem), there are few nontrivial exact solutions or analytic results for the Muskat problem. For the stable, forward Muskat problem, in which the higher-viscosity fluid expands into the lower-viscosity fluid, we show global-in-time existence for initial data that is a small perturbation of a flat interface. The initial data in this result may contain weak (e.g., curvature) singularities. For the unstable, backward problem, in which the higher-viscosity fluid contracts, we construct singular solutions that start off with smooth initial data but develop a point of infinite curvature at finite time. © 2004 Wiley Periodicals, Inc.

1 Introduction

The Muskat, or Muskat-Leibenzon, problem describes the evolution of the interface between two immiscible fluids in a porous medium or Hele-Shaw cell under applied pressure gradients or fluid injection/extraction. Originally proposed [11] as a simple model for displacement of oil by water in a porous medium, it has since emerged as a challenging free boundary problem in its own right. The one-phase version of the problem, in which one of the fluids has zero viscosity (or infinite mobility) so that it is purely passive, is commonly known as the Hele-Shaw problem (it is also the zero-specific heat version of the one-phase Stefan problem) and has been intensively studied for half a century. Significant progress has been made, largely exploiting the convenient fact that, when surface tension is neglected, the pressure, which is a potential for the flow, is harmonic and vanishes on the fluid

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interface. Many explicit solutions can be constructed using complex variable methods [7], and based on these and on more theoretical analyses, the following stylized (because subject to qualifications and exceptions) facts are known.

The problem is time-reversible if injection is replaced by the equivalent extraction, and following on from this, there is a diametric difference between "forward" problems, in which the "active" fluid region expands, and 'backward" ones, in which it contracts. The former are linearly stable with an exponential decay rate of small perturbations proportional to wave number, while the latter are, by time-reversibility, correspondingly unstable. Indeed, finite-time blowup of the interface via a cusp or other singularity is generic for backward problems; conversely, forward problems have interfaces that are eventually smooth even if they start out with singularities. We say "eventually" because, as shown in [10], if the initial interface has a finite-angle corner there may be a "waiting time" during which the corner persists before the interface eventually becomes smooth.

Like the Hele-Shaw problem, the Muskat problem, in which the second fluid has finite mobility, is time-reversible, and there is still a distinction on grounds of linear stability between stable "forward" problems, in which the fluid with the greater viscosity (lower mobility) expands, and unstable "backward" problems, in which it contracts; the growth rate is again proportional to the wave number. However, the crucial step from linear stability to nonlinear behavior is much more difficult to make in this case, largely because the interface pressure is unknown. For this reason, very little is known either about explicit solutions (see [8]) or on general issues such as existence/uniqueness of classical solutions. Weak solutions are defined in [9, 12], and a regularized model, in which the mobility is a smooth function of saturation is discussed in [15], but neither of these approaches has led to progress on the question of classical solutions to sharp-interface models.

In this paper, we prove a global existence theorem (Theorem 4.2) for the forward case with small initial data satisfying certain smoothness conditions, and we address the issues of whether a finite-time interface singularity can occur in the backward case. Specifically, we are able to show the following regarding singularity formation (a precise statement is given below, in Corollary 8.1): it is possible to construct solutions to the backward problems that start with a smooth (analytic) interface, evolve for a finite time, and then develop a curvature singularity in the interface. This therefore is a step in the direction of showing that the Muskat problem can exhibit the full range of singular behavior of its one-phase version, the Hele-Shaw problem. Using these singularities, one can show (Corollary 8.2) that the backward Muskat problem is ill-posed in the sense that singularities can form in an arbitrarily short time for arbitrarily small initial data, as measured in a Sobolev norm.

It should be noted that this result, of finite-time blowup, is not a foregone conclusion. Arguments for and against finite-time blowup by a cusp are reviewed in [8]; briefly, the main arguments in favor are the linear stability result and the detailed numerical studies of [5], which indicate that cusps can form. Against cusp formation, one can note that the traveling-wave "finger" solution of [14], which for the one-fluid case has infinite velocity as its width tends to 0, always has bounded velocity in the two-fluid case, and insofar as this solution is relevant to the local behavior near a cusp tip, it suggests that infinite cusp velocity is not possible with two fluids. Loosely speaking, one may say that the second fluid can transmit the pressure gradient, allowing the interface pressure to drop below 0 and thus weakening the "runaway" that leads to cusp formation. Finally, we may mention the results of [12, 13], in which a weak formulation of the fingering problem is used to show that the "mixing zone" can only grow at finite speed. We have only shown blowup via a curvature singularity, and indeed, in view of the waiting-time behavior for the one-phase problem referred to above, it is likely that different techniques will be required to show whether the Muskat problem can develop cusps, corners, or other singularities of higher order than ours.

The first result of our analysis, Theorem 4.2, is a global (in time) existence theorem for initial data that is a small perturbation of a flat interface, in which the size of the perturbation is measured in an L^1 Fourier norm. The initial data is allowed to have a curvature singularity, but the solution is shown to be smooth (analytic) for all subsequent times, and in the corollary, we appeal to time-reversibility of this solution to show existence of a solution that blows up in finite time. The problem is first reformulated as an integrodifferential equation for the interface (cf. [3]), and the core of the proof lies in showing that this has a solution with the required properties. The estimates derived in order to do this require restrictions on the singular behavior of the initial interface, specifically that its first derivative be continuous but its second derivative be singular, and hence confines us to the case of a curvature singularity.

This approach is similar to the analysis developed in [3] for constructing singular solutions to the Kelvin-Helmholtz problem. New challenges presented by the Muskat problem are that the nonlinear term is considerably more complicated and that there is no natural parametrization of the interface. The additional nonlinearity of the equation required considerably more care in the inequalities that are the essence of the existence proof, but this was aided considerably by use of a Fourier norm rather than the Hölder norms used in [3]. Lack of a natural parametrization results in the presence of a nonphysical "reparametrization" mode. This mode, which is neutrally stable, is in addition to the unstable physical mode of the backward Muskat problem. For the Kelvin-Helmholtz problem, in contrast, there is always a single stable and a single unstable mode. We are able to modify the analysis to accommodate this neutrally stable mode by prescribing its data at ∞; i.e., by requiring it to go to 0 as t goes to ∞. This results in an existence theorem, Lemma 4.1, for what appears to be a restricted set of data. Finally, introduction of a reparametrization allows this result to be converted to existence for any initial data, as in Theorem 4.2. To the best of our knowledge, this global existence result is the first that relies on a stable decay rate that is proportional to k in order to show that solutions become analytic immediately after the initial time.

After the basic formulation of the Muskat problem is detailed in Section 2, in Section 3 we briefly present the linear theory in a form that shall be convenient for the subsequent analysis. Statements of the main global existence results, Lemma 4.1 and Theorem 4.2, are given in Section 4. As a preliminary to presenting proofs for the existence results, Section 5 derives equations for the nonlinear corrections to the solution of linear perturbation theory. Proof of Lemma 4.1 through an iteration method is described in Section 6, with some inequalities deferred to the appendix. Using Lemma 4.1, Theorem 4.2 is proven in Section 7 and the singularity formation and ill-posedness results of Corollaries 8.1 and 8.2 are proven in Section 8. Conclusions are discussed in Section 9.

2 Governing Equations

Consider the flow of two immiscible, incompressible fluids in a Hele-Shaw cell or porous medium. The fluids are assumed to be separated by a sharp interface that is 2π-periodic in the x-direction. The fluid motion is driven by a prescribed far-field pressure gradient, leading to a constant fluid velocity $V\mathbf{j}$ as $y \to \pm\infty$, where $\mathbf{j}$ is a unit vector in the y-direction. We denote the domain of the upper fluid by D_1 and the lower fluid by D_2, while the interface is denoted by ∂D. Physical quantities associated with the upper or lower domain are indicated by a subscript 1 or 2, respectively.

The equations governing flow in the cell are Darcy's law

$$\mathbf{u}_i = V\mathbf{j} - k_i \nabla p_i \tag{2.1}$$

together with the incompressibility condition

$$\nabla \cdot \mathbf{u}_i = 0$$

for $i = 1, 2$. Here we have introduced the velocities $\mathbf{u}_i(x, y) = (u_i(x, y), v_i(x, y))$, pressures $p_i(x, y)$, and fluid mobilities k_i, which in a Hele-Shaw cell are equal to $= h^2/(12\mu_i)$, where h is the gap width and μ_i are the viscosities. The velocity at ∞ has been explicitly represented in (2.1), so that the far-field boundary condition is $\mathbf{u}_i \to 0$ as $y \to \pm\infty$. This is equivalent to performing a Galilean transformation to a frame moving with velocity $V\mathbf{j}$ with respect to the laboratory frame. In the following, all velocities (e.g., fluid and interface velocities) are measured with respect to the moving frame. The boundary conditions at the interface ∂D are

$$p_1 = p_2\,, \qquad \mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} = V_n\,, \tag{2.2}$$

where V_n is the normal velocity of ∂D. Note that in (2.2) we have assumed that there is no surface tension.

The interface between the fluids is a "vortex sheet" since the tangential velocity may be discontinuous there. An integrodifferential equation governing the evolution of the sheet is derived from the governing differential equations and boundary conditions in [4, 16]. We use here a form of the equation that employs complex variable notation, following the presentation of [4]. Let $z(\xi, t) = x(\xi, t) + iy(\xi, t)$

denote the location of the interface in the complex $x + iy$ plane as a function of the parameter ξ and time t. Define also the complex interface velocity $w(\xi, t) = u - iv$. The evolution equation takes the form

$$\frac{\partial z^*}{\partial t} = w^*(\xi, t)\,, \tag{2.3}$$

$$w^*(\xi, t) = \frac{A}{2\pi i} \,\mathrm{PV} \int_{-\infty}^{\infty} \frac{\langle w(\xi') z_\xi^*(\xi') - i z_\xi(\xi') \rangle}{z(\xi') - z(\xi)}\, d\xi'\,, \tag{2.4}$$

where the operators $*$ and $\langle f \rangle$ are defined as follows: For ξ real, the operator $*$ denotes the complex conjugate. However, as discussed in [4], it is useful to analytically extend the governing equations to complex values of ξ by extending the complex conjugate via Schwarz reflection. More precisely, we define

$$f^*(\xi, t) = \overline{f(\bar{\xi}, t)}$$

where the overbar denotes the usual complex conjugate. The operator $\langle f \rangle$ is then given by

$$\langle f \rangle = f + f^*\,.$$

The parameter A that appears in (2.4) is the *Atwood number* and is defined by

$$A = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} = \frac{k_1 - k_2}{k_1 + k_2}\,.$$

Note that A is positive when the displacing fluid 2 is more viscous (the stable case). The integral in (2.4) is in the Cauchy principal value sense. In deriving (2.4) we have chosen the interface velocity to be the average of the upper and lower fluid velocities adjacent to the interface, which is permissible since it provides the required normal velocity. The assumption $V = 1$ has also been made, which is equivalent to nondimensionalization of the velocity using the far-field value (the far-field velocity is assumed to be in the positive y-direction for t increasing). Equations (2.3) and (2.4) are the main results of this section.

3 Linearized Theory

The flat interface described by $z = \xi$, $w = 0$, is an exact steady solution to (2.3)–(2.4) that describes a planar interface propagating with velocity $\mathbf{j}$ in the laboratory frame. Consider a small perturbation to this solution; the perturbed sheet is denoted by $z = \xi + s(\xi, t)$, $w = w^s(\xi, t)$. Linearization of the governing equations about the flat interface gives

$$\frac{\partial s^*}{\partial t} = w^{s*}\,, \quad w^{s*} = A\mathcal{H}\big(\langle w^s - i s_\xi \rangle\big)\,, \tag{3.1}$$

where $\mathcal{H}$ is the Hilbert transform, defined by

$$\mathcal{H}(f) = \frac{1}{2\pi i} \,\mathrm{PV} \int_{-\infty}^{\infty} \frac{f(\xi')}{\xi' - \xi}\, d\xi' = \frac{f_+(\xi) - f_-(\xi)}{2}\,, \tag{3.2}$$

the last equality being one of the Plemelj formulae. Here we denote by $f_+ = \sum_{k>0} \hat{f}(k)e^{ik\xi}$ the projection onto positive wave number Fourier modes, i.e., the part of f that is analytic in the upper half-plane. Similarly, $f_- = \sum_{k<0} \hat{f}(k)e^{ik\xi}$ is the projection onto negative wave number modes, i.e., the part that is analytic in the lower half-plane. The zero wave number mode is denoted by f_0. Substituting the representation of the Hilbert transform in terms of $+$ and $-$ functions into (3.1) leads to the equivalent linear system

$$\frac{\partial s_+}{\partial t} = w^s_+ = \frac{iA}{2}\left(s_{+\xi} - s^*_{-\xi}\right), \tag{3.3}$$

$$\frac{\partial s^*_-}{\partial t} = w^{s\,*}_- = -\frac{iA}{2}\left(s_{+\xi} - s^*_{-\xi}\right), \tag{3.4}$$

where we employ the notation $f^*_- = (f_-)^*$ and $f^*_+ = (f_+)^*$. In deriving (3.3) and (3.4) we have used the identity $\mathcal{H}(f^*) = -\mathcal{H}(f)^*$. Also, for convenience the equations are presented in terms of upper analytic functions, which will be a convention used throughout this paper. Note that there is no $k = 0$ mode for s, which follows from the equality in flux magnitudes at $y \to \pm\infty$ together with the incompressibility assumption.

It is easily seen that the linearized equation has normal mode solutions that are constant multiples of $(s_+, s^*_-, w^s_+, w^{s\,*}_-) = (1, -1, -Ak, Ak)e^{-Akt+ik\xi}$ and $(1, 1, 0, 0)e^{ik\xi}$ for $k > 0$. The first set of modes are linearly stable (unstable) for $A > 0$ (< 0) and correspond to a purely imaginary perturbation of the interface, while the second set of modes are neutrally stable and represent a purely "real" deformation of the interface along itself. This stability result is in agreement with the analysis of Saffman and Taylor [14], and the switch in stability when A changes sign is equivalent to a switch in stability under time reversal.

4 Existence Theory

In order to specify the analytic properties of functions and quantify their magnitudes, we introduce the Fourier norm

$$\|f(\cdot, t)\|_\rho = \sum_{k=-\infty}^{\infty} e^{\rho|k|}|\hat{f}(k, t)| \tag{4.1}$$

where $\hat{f}(k, t)$ are the Fourier coefficients of f. If this norm is finite for $\rho > 0$, the Fourier inversion formula can be used to show that f is an analytic function in $|\mathrm{Im}\,\xi| < \rho$ and that $\sup_{|\mathrm{Im}\,\xi|<\rho} |f| \le \|f\|_\rho$. Other useful properties are (i) $\|fg\|_\rho \le \|f\|_\rho\|g\|_\rho$ for $\rho \ge 0$ and (ii) $\|f\|_\rho = \|f^*\|_\rho$. Although it is usual to restrict $\rho \ge 0$, we will use $\rho < 0$ in conditions on the initial data and for simplifying some derivations.

From now on, we assume the stable case $A > 0$ unless otherwise noted. For the existence theory we shall construct solutions for initial data of the form $z(\xi, 0) = \xi + S_0(\xi)$ where the function $S_0(\xi)$ is assumed to satisfy the following:

(1) $S_0(\xi)$ is small (of size ϵ) and purely imaginary, i.e., S_0 gives initial data only for the stable (linearized) problem.

(2) S_0 has at most a singularity in the $1+p$ derivative for $0 < p < 1$. A convenient (for the subsequent analysis) way of stating this is

$$\|S_0\|_{-\rho} + \|S_{0\xi}\|_{-\rho} < c\epsilon e^{-\rho}, \tag{4.2}$$

$$\|S_{0\xi\xi}\|_{-\rho} < c\epsilon e^{-\rho}(1+\rho^{p-1}), \tag{4.3}$$

for any $\rho \geq 0$. Note that we do not require analyticity of S_0, since the bounds hold for any function in a Sobolev space of high enough order.

Our general strategy to show existence for the stable problem $A > 0$ is to begin by deriving a preliminary existence result. This involves constructing a *particular* class of solutions to (2.3)–(2.4) of the form

$$z(\xi,t) = \xi + s(\xi,t) + r(\xi,t), \tag{4.4}$$

$$w(\xi,t) = w^s(\xi,t) + w^r(\xi,t), \tag{4.5}$$

where the dominant terms s and w^s constitute an exact decaying solution of the linearized system (3.3) and (3.4), and the remainder terms r and w^r are negligible in a sense that will be explained shortly. The part of the initial data given by $s_0 = s(\xi,0)$ is assumed to satisfy assumptions (1) and (2), but $r_0 = r(\xi,0)$ is a function of s_0 and in general is nonzero. The linearized solutions s and w^s satisfy

$$\|s\|_\rho + \|s_\xi\|_\rho + \|w^s\|_\rho < c\epsilon e^{\rho - At}, \tag{4.6}$$

$$\|s_{\xi\xi}\|_\rho + \|w^s_\xi\|_\rho < c\epsilon e^{\rho - At}(1+(At-\rho)^{p-1}), \tag{4.7}$$

for $\rho < At$ and (different) constants c. These inequalities follow from (4.2) and (4.3) (with S_0 replaced by s_0) upon noting that $\|\partial_\xi^i s\|_\rho(t) = \|\partial_\xi^i s_0\|_{\rho - At}$ for $i = 0, 1, 2$, and using $\|\partial_\xi^j w^s\|_\rho \leq \|\partial_\xi^j s_\xi\|_\rho$ for $j = 0, 1$. The terms s and w^s are therefore allowed to be singular at $t = 0$, are analytic in the time-dependent strip $|\mathrm{Im}\,\xi| < At$ for $t > 0$, and decay to zero as $t \to \infty$. The general existence theorem is proven from the preliminary existence result by showing, via a reparametrization, that there exists an s_0 such that $z(\xi,0) = \xi + s_0 + r_0$, where $z(\xi,0)$ is general initial data specified as above and r_0 depends on s_0.

An explicit example of functions s and w^s satisfying the requirements above can be given in terms of the *decaying* (linearized) normal mode solutions as

$$s(\xi,t) = c\epsilon \sum_{k=1}^{\infty} k^{-(p+2)} e^{-Atk}(e^{ik\xi} - e^{-ik\xi}),$$

$$w^s(\xi,t) = -cA\epsilon \sum_{k=1}^{\infty} k^{-(p+1)} e^{-Atk}(e^{ik\xi} - e^{-ik\xi}),$$

for which the perturbed interface is given by $y = 2c\epsilon \sum_{k=1}^{\infty} k^{-(p+2)} e^{-Atk} \sin kx$. The exponential decay with t in this solution guarantees analyticity in a strip of width $\rho < At$ for $t > 0$. The algebraic decay ensures that s and s_ξ are bounded at

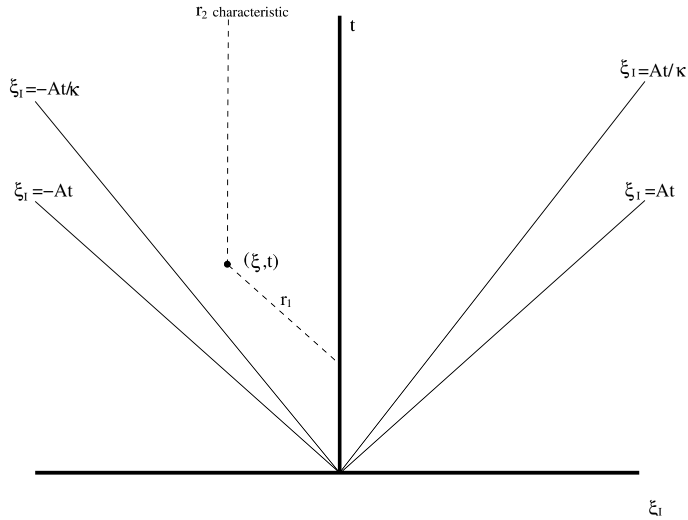

FIGURE 4.1. Sketch of the r_1 and r_2 characteristics emanating from the point (ξ, t), shown in the t vs. ξ_I plane (where $\xi = \xi_R + i\xi_I$ and ξ_R is fixed). The wide-angle wedge depicts the domain of analyticity of $s(\xi, t)$, $w^s(\xi, t)$, while the narrower wedge shows the domain of analyticity of $r(\xi, t)$, $w^r(\xi, t)$.

$t = 0$, but is not strong enough to give finiteness of $s_{\xi\xi}$. Indeed, it is easy to see that (4.6) and (4.7) are satisfied, and that $s_{\xi\xi} \sim O(\xi^{p-1})$ at $t = 0$ and for ξ near 0.

The aforementioned preliminary existence result, on which the main existence theorem of this paper is based, is the following:

LEMMA 4.1 *Let $A > 0$ and $0 < p < 1$, and let $\epsilon > 0$ be a sufficiently small real number. Let s and w^s solve the linearized equations* (3.3)–(3.4)*, with purely imaginary periodic initial data s_0 satisfying the bounds* (4.2) *and* (4.3)*. Then there are functions $r(\xi, t)$ and $w^r(\xi, t)$ and a constant $\kappa > 1$ such that* (4.4)–(4.5) *is an analytic solution of system* (2.3)–(2.4) *for $t > 0$ and $|\mathrm{Im}\,\xi| < At/\kappa$. The decaying mode $r_1 = r_+ - r_-^*$ can be initially chosen as* 0*, and the neutral mode $r_2 = r_+ + r_-^*$ satisfies $\lim_{t\to\infty} r_2 = 0$, although $r_2(t = 0)$ is generally nonzero. Moreover, there exists a constant c_0 (independent of ϵ) such that r and w^r satisfy*

$$\|r\|_0 + \|r_\xi\|_0 + \|w^r\|_0 \le \frac{c_0\epsilon^2}{p(1-p)(\kappa-1)} e^{-\frac{At}{2}} ,$$

$$\|r_{\xi\xi}\|_0 + \|w^r_\xi\|_0 \le \frac{c_0\epsilon^2}{p(1-p)(\kappa-1)} e^{-\frac{At}{2}} (1 + (At)^{p-1}) ; \tag{4.8}$$

i.e., r and w^r are negligible compared to s and w^s.

We note that the rate of exponential decay μ in inequalities (4.8) can be demonstrated to be any number $\mu > -A$; $-\frac{A}{2}$ is merely used for convenience. The characteristic directions and wedge of analyticity for the solution are depicted in Figure 4.1. Note that initial data for the decaying mode r_1 in the forward (stable) problem is chosen to be 0 at the singularity time, i.e., at $t = 0$. This means that in the backward problem, r_1 (which is now the growing component of r) is 0 at the singularity time, guaranteeing that growth in the "nonlinear" remainder term r does not overtake that due to the "linear" term s. The limited order of the singularity is also important for showing that the remainder term r is negligible compared to s.

The preliminary existence result is converted into a general existence theorem in Section 7. This requires an additional assumption on the initial data, namely, that

$$\|S_{0\xi}(\,\cdot\,)\|_{\mathrm{Lip}_{p+\nu}} < \infty \tag{4.9}$$

for some $0 < \nu$. Here, Lip_γ refers to the subspace of continuous 2π-periodic functions for which

$$\|f\|_{\mathrm{Lip}_\gamma} = \sup_{\substack{\xi \\ h\neq 0}} \frac{|f(\xi+h) - f(\xi)|}{|h|^\gamma} < \infty\,.$$

Inequality (4.9) implies that $\|\partial^{1+\gamma} S_0(\xi)/\partial\xi^{1+\gamma}\|_0$ is bounded for $\gamma = p$, but may be infinite for $\gamma > p$. (The fractional derivative is defined in Section 7.) The general existence theorem is the following:

THEOREM 4.2 *Let $A > 0$ and let $\epsilon > 0$ be a sufficiently small real number. Let $z(\xi, 0) = \xi + S_0(\xi)$ be initial data satisfying conditions* (1) *and* (2) *and assumption* (4.9). *Then there are functions $s(\xi, t)$, $w^s(\xi, t)$, $r(\xi, t)$, and $w^r(\xi, t)$ satisfying the conditions of Lemma* 4.1 *and a constant $\kappa > 1$ such that* (4.4)–(4.5) *is a solution of the system* (2.3)–(2.4) *with the given initial data*; *the solution is analytic in $|\mathrm{Im}\,\xi| < At/\kappa$ for $t > 0$. Moreover, r satisfies the bounds* (4.8); *i.e., r and w^r are negligible compared to s and w^s. This solution is unique.*

Additionally, time reversal of an initially singular solution leads to a solution of the Muskat problem that develops a finite time singularity from smooth initial data, as shown in Section 8.

In the next section we derive equations for the remainder terms r and w^r, and write these equations in a convenient form. The proof of Lemma 4.1 then follows in Section 6.

5 Equations for Remainder Terms r and w^r

5.1 Characteristic Form

We substitute the decomposition (4.4)–(4.5) in the governing equations (2.3)–(2.4) and use the fact that s and w^s solve the linear system (3.1) exactly to obtain

$$\frac{\partial r^*}{\partial t} = w^{r*}, \tag{5.1}$$

$$w^{r*} = \frac{A}{2\pi i}\,\mathrm{PV}\int_{-\infty}^{\infty}\left\{\frac{\langle w^{r\prime} - i r'_\xi\rangle}{\xi' - \xi} + \frac{\langle w^{*\prime}(s'_\xi + r'_\xi)\rangle}{\xi' - \xi} + \left(\frac{r + s - r' - s'}{\xi - \xi'}\right)\frac{\langle w^{*\prime} z^{*\prime}_\xi - i z'_\xi\rangle}{z - z'}\right\}d\xi' \tag{5.2}$$

$$= B_1^* + B_2^* \tag{5.3}$$

where B_1^* denotes the purely linear (first) integral term in (5.2) and B_2^* represents the remaining terms. Here the primes denote evaluation of a function at ξ'. The above expression is further simplified by noting that the linear term B_1^* can be evaluated using the Hilbert transform relation (3.2). Doing so yields

$$B_{1+} = -\frac{A}{2}\left[w^r_+ + w^{r\,*}_- - i(r_{+\xi} - r^*_{-\xi})\right] \tag{5.4}$$

$$= -B^*_{1-}\,. \tag{5.5}$$

The functions w^r_+ and $w^{r\,*}_-$ may be eliminated from (5.4) using $w^r_+ = B_{1+} + B_{2+}$ and $w^{r\,*}_- = B^*_{1-} + B^*_{2-}$ (see (5.3)) to give

$$B_{1+} = -\frac{A}{2}\left[B_{2+} + B^*_{2-} - i(r_{+\xi} - r^*_{-\xi})\right] = -B^*_{1-}\,, \tag{5.6}$$

where we have used (5.5) to simplify the resulting expression. Hence from (5.1), (5.3), and (5.6)

$$\frac{\partial r^*_-}{\partial t} = -\frac{iA}{2}\left(r_{+\xi} - r^*_{-\xi}\right) + \frac{A}{2}B_{2+} + \left(1 + \frac{A}{2}\right)B^*_{2-}\,, \tag{5.7}$$

$$\frac{\partial r_+}{\partial t} = \frac{iA}{2}\left(r_{+\xi} - r^*_{-\xi}\right) + \left(1 - \frac{A}{2}\right)B_{2+} - \frac{A}{2}B^*_{2-}\,. \tag{5.8}$$

The relation (5.6) may also be applied to replace the term B_1 in (5.3), yielding

$$w^r = \frac{iA}{2}\left(r_{+\xi} - r_{-\xi} + r^*_{+\xi} - r^*_{-\xi}\right) + \phi[r, w^r](\xi, t) \tag{5.9}$$

where

$$\phi[r, w^r](\xi, t) = \left(1 - \frac{A}{2}\right)B_{2+} + \frac{A}{2}B^*_{2+} + \left(1 + \frac{A}{2}\right)B_{2-} - \frac{A}{2}B^*_{2-}\,. \tag{5.10}$$

It is convenient to implement a change of variable so that (5.7) and (5.8) are in characteristic form. Define $r_1 = r_+ - r_-^*$ and $r_2 = r_+ + r_-^*$ as in Lemma 4.1. Then

$$\frac{\partial r_1}{\partial t} - iA\frac{\partial r_1}{\partial \xi} = (1-A)B_{2+} - (1+A)B_{2-}^* = \alpha(\xi, t)\,, \tag{5.11}$$

$$\frac{\partial r_2}{\partial t} = B_{2+} + B_{2-}^* = \beta(\xi, t)\,. \tag{5.12}$$

Note that r_1, r_2, α, and β are upper analytic; that is, their Fourier series contain only positive k wave numbers.

Equations (5.9)–(5.12) give the desired relations for the remainder terms r and w^r and are the main result of this section. We shall prove existence of analytic solutions for $t > 0$ by transforming this system into a set of integral equations and then solving by iteration. In the next section we first rewrite the differential equation for r_1 as an integral equation by employing a Green's function. This provides a representation of the solution for real ξ, and hence for complex ξ via analytic continuation. An integral equation representation of the equation for r_2 is obtained by integrating in t using data posed for complex ξ as $t \to \infty$. Equation (5.9) for w^r is already in the form of an integral equation. The decay of the Fourier coefficients in the solutions to these equations will be analyzed to show that r_1, r_2, and w^r are analytic in a time-dependent strip containing the real ξ-axis.

5.2 Integral Equation Formulation

We first seek a Green's function solution for $r_1(\xi, t)$ for ξ real. The requirements are that the solution r_1 be 2π-periodic, have only positive wave number components, and vanish as $t \to \infty$. For convenience we also specify that $r_1(t = 0) = 0$. The solution is easily computed by taking the Fourier transform of (5.11) and solving the resulting ODE for the Fourier coefficients $\hat{r}_1(k, t)$ using a Green's function, which yields

$$\hat{r}_1(k, t) = \int_0^t e^{-Ak(t-t')}\hat{\alpha}(k, t')dt' \tag{5.13}$$

for $k = 1, 2, \dots$. Although this expression for $\hat{r}_1$ will prove to be of more use to us (in view of the choice of Fourier norm), we note in passing that a formula for $r_1(\xi, t)$ is easily found from (5.13) as

$$r_1(\xi, t) = \frac{1}{2\pi}\int_0^t \int_0^{2\pi} \frac{e^{-A(t-t')+i\xi'}}{1 - e^{-A(t-t')+i\xi'}}\alpha(\xi - \xi', t')d\xi'\, dt'\,, \tag{5.14}$$

which holds for ξ real but may be extended off the real line through analytic continuation. Equivalently, values of $r_1(\xi, t)$ for complex ξ can be found by direct integration of (5.12) along complex characteristics. An integral equation formulation for r_2 is obtained by assuming $r_2 \to 0$ as $t \to \infty$ and then integrating, i.e.,

$$r_2(\xi, t) = \int_\infty^t \beta(\xi, t')dt'\,, \tag{5.15}$$

which holds for ξ complex. The Fourier coefficients of r_2 are

$$\hat{r}_2(k,t) = \int_\infty^t \hat{\beta}(k,t')dt' \tag{5.16}$$

for $k \geq 1$. The function $r = r_+ + r_-$ is recovered from r_1 and r_2 via the relation

$$r = \frac{1}{2}\big(r_1 + r_2 - r_1^* + r_2^*\big)\,. \tag{5.17}$$

Let $I[r, w^r]$ denote the combination of the right-hand sides of (5.14) and (5.15) corresponding to the right-hand side of (5.17), and $J[r, \phi[r, w^r]]$ the right-hand side of (5.9). Then the original governing equations (2.3)–(2.4) for $z = \xi + s + r$, $w = w^s + w^r$, can be rewritten as

$$r = I[r, w^r]\,, \tag{5.18}$$

$$w^r = J[r, \phi[r, w^r]]\,, \tag{5.19}$$

which hold for complex ξ via analytic continuation. In the next section we demonstrate the convergence of an iteration method for solving this system, thus providing a proof of Lemma 4.1.

6 Proof of Lemma 4.1

6.1 Iteration Method

We solve system (5.18)–(5.19) by iteration. Define $r^0 = 0$ and $w^{r,0} = 0$. For $n \geq 0$ we let r^{n+1} and $w^{r,n+1}$ satisfy

$$r^{n+1} = I[r^n, w^{r,n}]\,, \tag{6.1}$$

$$w^{r,n+1} = J[r^{n+1}, \phi[r^n, w^{r,n}]]\,. \tag{6.2}$$

For convenience the local term in (6.2) is evaluated at iterate $n+1$, whereas the nonlocal term ϕ is taken at iterate n. In terms of equations (5.14) and (5.15), the iteration scheme takes the form

$$r_1^{n+1}(\xi,t) = \frac{1}{2\pi}\int_0^t \int_0^{2\pi} \frac{e^{-A(t-t')+i\xi'}}{1 - e^{-A(t-t')+i\xi'}} \alpha^n(\xi - \xi', t')d\xi'\,dt' \tag{6.3}$$

for ξ real (and hence complex ξ through analytic continuation) and

$$r_2^{n+1}(\xi,t) = \int_\infty^t \beta^n(\xi,t')dt' \tag{6.4}$$

for complex ξ, where α^n and β^n are defined as in (5.11) and (5.12) but with r^n and $w^{r,n}$ replacing r and w^r. The Fourier coefficients satisfy

$$\widehat{r_1^{n+1}}(k,t) = \int_0^t e^{-Ak(t-t')}\widehat{\alpha^n}(k,t')dt'\,, \tag{6.5}$$

$$\widehat{r_2^{n+1}}(k,t) = \int_\infty^t \widehat{\beta^n}(k,t')dt'\,, \tag{6.6}$$

for $k \geq 1$. The iteration scheme for equation (6.2) takes the form

$$w^{r,n+1} = \frac{iA}{2}\left(r_{1\xi}^{n+1} + r_{1\xi}^{*n+1}\right) + \phi^n \tag{6.7}$$

where ϕ^n is defined as in (5.10) but with r^n and $w^{r,n}$ replacing r and w^r.

To show convergence of the iterates we obtain estimates on the differences

$$R_1^{n+1} = r_1^{n+1} - r_1^n\,, \quad R_2^{n+1} = r_2^{n+1} - r_2^n\,, \quad W^{n+1} = w^{r,n+1} - w^{r,n}\,.$$

We shall also use the following differentiated equations for (6.3) and (6.4):

$$\partial_\xi^i R_1^{n+1}(\xi, t) = \frac{1}{2\pi}\int_0^t \int_0^{2\pi} \frac{e^{-A(t-t')+i\xi'}}{1-e^{-A(t-t')+i\xi'}}\partial_\xi^i[\alpha^n - \alpha^{n-1}]d\xi'\,dt'\,,$$
$$\partial_\xi^i R_2^{n+1}(\xi, t) = \int_\infty^t \partial_\xi^i[\beta^n - \beta^{n-1}]dt'\,,$$

for $i = 1, 2$ and $n \geq 0$ (with $\alpha^{-1} \equiv \beta^{-1} \equiv 0$), or equivalently in terms of the Fourier coefficients

$$\begin{aligned} \widehat{\partial_\xi^i R_1^{n+1}}(k, t) &= \int_0^t e^{-Ak(t-t')}\left[\widehat{\partial_\xi^i \alpha^n} - \widehat{\partial_\xi^i \alpha^{n-1}}\right]dt'\,, \\ \widehat{\partial_\xi^i R_2^{n+1}}(k, t) &= \int_\infty^t \left[\widehat{\partial_\xi^i \beta^n} - \widehat{\partial_\xi^i \beta^{n-1}}\right]dt'\,, \end{aligned} \tag{6.8}$$

for $i = 1, 2$ and $k \geq 1$. We shall repeatedly use the fact that, for the Fourier norm defined in (4.1), the Cauchy estimate for the derivative of a function f is

$$\left\|\frac{\partial^\gamma f}{\partial \xi^\gamma}(\,\cdot\,, t)\right\|_\rho \leq \frac{\|f(\,\cdot\,, t)\|_{\rho'}}{(\rho' - \rho)^\gamma} \tag{6.9}$$

where $\rho < \rho'$ and $0 < \gamma \leq 1$. Although this estimate is mainly applied for $\gamma = 1$, we will also use it for $\gamma = p$ in Section 7. Note that analyticity of f is not needed for $\rho < \rho' < 0$.

Crucial estimates on the nonlocal term B_2 are derived in the appendix. These estimates are repeatedly used in the subsequent sections. The estimates are summarized below, where we introduce the notation $\tilde{B}_2 = B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]$, $\|r, w^r\|_\rho = \|r\|_\rho + \|w^r\|_\rho$, and $\|r, w^r\|_\rho' = \|r\|_\rho + \|w^r\|_\rho + \|\tilde{r}\|_\rho + \|\tilde{w}^r\|_\rho$, with the obvious extension for more functions:

$$\|B_2[s, w^s, r, w^r]\|_\rho \leq c_1|A|\|s_\xi, w^s, r_\xi, w^r\|_\rho \|s_\xi, r_\xi\|_\rho\,, \tag{6.10}$$

$$\|B_{2\xi}[s, w^s, r, w^r]\|_\rho \leq c_1|A|\left\{\|s_\xi, r_\xi\|_\rho \|w_\xi^s, w_\xi^r\|_\rho + \|s_\xi, w^s, r_\xi, w^r\|_\rho \|s_{\xi\xi}, r_{\xi\xi}\|_\rho\right\}, \tag{6.11}$$

$$\|B_2 - \tilde{B}_2\|_\rho \leq c_1|A|\|s_\xi, w^s, r_\xi, w^r\|_\rho' \left\{\|s_\xi - \tilde{s}_\xi\|_\rho + \|r_\xi - \tilde{r}_\xi\|_\rho + \|w^r - \tilde{w}^r\|_\rho\right\}, \tag{6.12}$$

$$
\begin{aligned}
(6.13)\quad & \|B_{2\xi} - \tilde{B}_{2\xi}\|_\rho \le \\
& c_1|A|\big\{\|s_\xi, w^s, r_\xi, w^r\|'_\rho\big[\|s_{\xi\xi} - \tilde{s}_{\xi\xi}\|_\rho + \|r_{\xi\xi} - \tilde{r}_{\xi\xi}\|_\rho + \|w^r_\xi - \tilde{w}^r_\xi\|_\rho\big] \\
& \quad + \|s_{\xi\xi}, w^s_\xi, r_{\xi\xi}, w^r_\xi\|'_\rho\big[\|s_\xi - \tilde{s}_\xi\|_\rho + \|r_\xi - \tilde{r}_\xi\|_\rho + \|w^r - \tilde{w}^r\|_\rho\big]\big\}\,.
\end{aligned}
$$

In deriving these estimates we have assumed that ϵ is small enough so that

$$\|r_\xi\|_\rho < \|s_\xi\|_\rho \le C \le \frac{1}{2}\,. \tag{6.14}$$

This condition will be checked at every stage of the iteration. The constant c_1 is independent of ϵ, ρ, and t, although it may tend to infinity as $\|s_\xi\|_\rho$ and $\|r_\xi\|_\rho$ (and hence C) tend to $\frac{1}{2}$. Note that we have used (3.3) and (3.4) to eliminate $\|w^s - \tilde{w}^s\|_\rho$ and $\|w^s_\xi - \tilde{w}^s_\xi\|_\rho$ in favor of $\|s - \tilde{s}\|_\rho$ and $\|s_\xi - \tilde{s}_\xi\|_\rho$. In Sections 6.4 and 6.5 these estimates are applied for $s = \tilde{s}$ and $w^s = \tilde{w}^s$, in which case several terms are eliminated. The full estimates are utilized in Section 7.

6.2 First Approximation

Set $r^0 = 0$ and $w^{r,0} = 0$. The bounds (4.6), (4.7), and (6.11) and the definition of α and β imply that

$$\|\alpha^0_\xi\|_\rho + \|\beta^0_\xi\|_\rho \le 3\|B^0_{2\xi}\|_\rho \le dA\epsilon^2 e^{2(\rho - At)}[1 + (At - \rho)^{p-1}]\,, \tag{6.15}$$

where $d = 3c_1c^2$ and it is assumed that $0 \le \rho < At$. It then follows from (6.5) that for $\rho < At$

$$
\begin{aligned}
\|r^1_{1\xi}\|_\rho &\le \sum_{k=1}^{\infty} e^{\rho k} \int_0^t e^{-Ak(t-t')}\big|\widehat{\alpha^0_\xi}(k, t')\big|dt' \\
&= \left(\int_0^{t-\frac{\rho}{A}} + \int_{t-\frac{\rho}{A}}^{t}\right)\|\alpha^0_\xi\|_{\rho_1}\,dt' \\
&\equiv I_1(t) + I_2(t)\,,
\end{aligned}
$$

where we have introduced the quantity

$$\rho_1 = \rho - A(t - t') \le \rho < At \quad \text{for } t' \le t. \tag{6.16}$$

Note that $\rho_1 \le 0$ over the interval $[0, t - \frac{\rho}{A}]$. Hence we estimate I_1 as

$$
\begin{aligned}
I_1(t) &\le \int_0^{t-\frac{\rho}{A}} e^{\rho_1}\|\alpha^0_\xi\|_0\,dt' \\
&\le dA\epsilon^2 e^{\rho - At}\int_0^{t-\frac{\rho}{A}} e^{-At'}[1 + (At')^{p-1}]dt' \quad \text{by (6.15), (6.16)} \\
&\le \frac{4d\epsilon^2}{p}e^{\rho - At}\,, \qquad (6.17)
\end{aligned}
$$

where the integral is bounded using (A.21) (after setting $\lambda = 1$ and $At - \kappa\rho = 0$ in (A.19)). We use (6.15) to estimate the integral I_2, which is allowed in view of the fact that $0 \le \rho_1 \le At'$ on the integration interval. Then

$$\begin{aligned} I_2(t) &\le dA\epsilon^2 \int_{t-\frac{\rho}{A}}^{t} e^{2(\rho_1 - At')}[1 + (At' - \rho_1)^{p-1}]dt' \\ &= d\epsilon^2 e^{2(\rho - At)}\rho[1 + (At - \rho)^{p-1}] \quad \text{using } At' - \rho_1 = At - \rho . \end{aligned} \tag{6.18}$$

In order to bound I_2 for large ρ, it is necessary to shrink the wedge of existence, thereby obtaining stricter control over $\|r^1_{1\xi}\|_\rho$. This is effectively achieved by replacing the requirement $\rho < At$ with

$$\kappa\rho < At, \tag{6.19}$$

where $\kappa = 1 + \delta$ with $0 < \delta < 1$. This reduction in size of the domain of existence forces the boundaries of the wedge to be transverse to the characteristic directions of the PDE (5.11). Thus integration along characteristics effectively reduces the order of the singularity. The reduction in wedge size only need be performed once; i.e., the domain of existence does not need to shrink at each step of the iteration as in a Nash-Moser type of proof.

With the aforementioned reduction, (6.18) is bounded as

$$I_2 \le d\epsilon^2 e^{\rho - At}\rho e^{-\delta\rho}(1 + (\delta\rho)^{p-1}) \le \frac{2d\epsilon^2}{\delta} e^{\rho - At} , \tag{6.20}$$

using the the fact that $\sup_{x>0} e^{-x}x^q \le 1$ for $q = 1$ or $q = p$. Combining estimates (6.17) and (6.20) leads to

$$\|r^1_{1\xi}\|_\rho \le \frac{6d\epsilon^2}{p\delta} e^{\rho - At} . \tag{6.21}$$

Next we estimate $\|r^1_{1\xi\xi}\|_\rho$. We have, from (6.5),

$$\begin{aligned} \|r^1_{1\xi\xi}\|_\rho &\le \sum_{k=1}^{\infty} e^{\rho k} \int_0^t e^{-Ak(t-t')} \left|\widehat{\alpha^0_{\xi\xi}}(k, t')\right| dt' \\ &= \int_0^t \|\alpha^0_{\xi\xi}\|_{\rho_1}\, dt' \equiv J(t) , \end{aligned} \tag{6.22}$$

where ρ_1 is defined in (6.16). The integral J is approximated using the Cauchy estimate (6.9). Let

$$\rho_2 = \rho_1 + \frac{At' - \rho_1}{2} = \frac{\rho_1 + At'}{2} . \tag{6.23}$$

Then

$$
\begin{aligned}
J(t) &\le \int_0^t \frac{\|\alpha_\xi^0\|_{\rho_2}}{\rho_2-\rho_1}\,dt' \\
&= 2\Big(\int_0^{\frac{1}{2}(t-\frac{\rho}{A})} + \int_{\frac{1}{2}(t-\frac{\rho}{A})}^t\Big)\frac{\|\alpha_\xi^0\|_{\rho_2}}{At'-\rho_1}\,dt' \\
&\equiv J_1(t)+J_2(t)\,.
\end{aligned}
$$

Note that $\rho_2 \le 0$ for t' in the interval $[0, \frac{1}{2}(t-\frac{\rho}{A})]$. Hence we estimate J_1 as

$$
\begin{aligned}
J_1(t) &\le 2\int_0^{\frac{1}{2}(t-\frac{\rho}{A})} e^{\rho_2}\frac{\|\alpha_\xi^0\|_0}{At'-\rho_1}\,dt' \\
&\le 2\,dA\epsilon^2\frac{e^{(\rho-At)/2}}{At-\rho}\int_0^{\frac{1}{2}(t-\frac{\rho}{A})} e^{-At'}[1+(At')^{p-1}]dt' \quad \text{using (6.15)} \\
&\le \frac{2\,d\epsilon^2}{p}e^{\frac{\rho-At}{2}}[1+(At-\rho)^{p-1}]\,, \qquad (6.24)
\end{aligned}
$$

after estimating the integral by neglecting the factor $e^{-At'}$. J_2 is estimated by applying (6.15), which is allowed in view of the fact that $0<\rho_2<At'$ for $\frac{1}{2}(t-\frac{\rho}{A})<t'<t$. This gives

$$
\begin{aligned}
J_2(t) &\le 2\,dA\epsilon^2\int_{\frac{1}{2}(t-\frac{\rho}{A})}^t e^{2(\rho_2-At')}\frac{1+(At'-\rho_2)^{p-1}}{At'-\rho_1}\,dt' \\
&= 2\,dA\epsilon^2\int_{\frac{1}{2}(t-\frac{\rho}{A})}^t e^{\rho-At}\frac{1+2^{1-p}(At-\rho)^{p-1}}{At-\rho}\,dt' \\
&\equiv d\epsilon^2(J_2'(t)+J_2''(t)) \qquad (6.25)
\end{aligned}
$$

where

$$
\begin{aligned}
J_2'(t) &= e^{\rho-At}[1+2^{1-p}(At-\rho)^{p-1}] \\
&\le 2e^{\rho-At}[1+(At-\rho)^{p-1}]\,, \qquad (6.26)
\end{aligned}
$$

and

$$
J_2''(t) = 2e^{\rho-At}\rho\frac{1+2^{1-p}(At-\rho)^{p-1}}{At-\rho}\,.
$$

Now apply the reduced domain to $J_2''(t)$, which leads to the bound $(At-\rho)^{-1} < 1/(\delta\rho)$. It easily follows that

$$
J_2''(t) \le \frac{4}{\delta}e^{\rho-At}[1+(At-\rho)^{p-1}] \qquad (6.27)
$$

for $\rho < At$. Combining estimates (6.24)–(6.27) leads to

$$\|r^1_{1\xi\xi}\|_\rho \le \frac{8\,d\epsilon^2}{\delta p} e^{\frac{\rho - At}{2}}[1 + (At - \kappa\rho)^{p-1}] \tag{6.28}$$

for $\kappa\rho < At$.

Next we consider estimates on $\|r^1_{2\xi}\|_\rho$ and $\|r^1_{2\xi\xi}\|_\rho$ for $\rho < At$. We have from (6.6) and (6.15)

$$\begin{aligned}\|r^1_{2\xi}\|_\rho &\le \int_t^\infty \|\beta^0_\xi\|_\rho \, dt' \\ &\le dA\epsilon^2 \int_t^\infty e^{2(\rho - At')}[1 + (At' - \rho)^{p-1}]dt' \\ &\le \frac{3\,d\epsilon^2}{p} e^{2(\rho - At)} \end{aligned} \tag{6.29}$$

using (A.21) (set $\lambda = 2$ and $\kappa = 1$ in the formula there). The norm $\|r^1_{2\xi\xi}\|_\rho$ is bounded using the Cauchy estimate (6.9). From (6.6) we have

$$\|r^1_{2\xi\xi}\|_\rho \le \int_t^\infty \|\beta^0_{\xi\xi}\|_\rho \, dt' \le \int_t^\infty \frac{\|\beta^0_\xi\|_{\rho_3}}{\rho_3 - \rho} \, dt' \,, \tag{6.30}$$

where we have defined

$$\rho_3 = \rho + \frac{At' - \rho}{2} \,.$$

Note that $\rho_3 < At'$ for $\rho/A < t' < \infty$, so that (6.15) may be applied to (6.30), with the result

$$\begin{aligned}\|r^1_{2\xi\xi}\|_\rho &\le dA\epsilon^2 \int_t^\infty e^{2(\rho_3 - At')} \frac{[1 + (At' - \rho_3)^{p-1}]}{\rho_3 - \rho} \, dt' \\ &\le 4\,dA\epsilon^2 \int_t^\infty e^{\rho - At'} \frac{[1 + (At' - \rho)^{p-1}]}{At' - \rho} \, dt' \\ &\le \frac{16\,d\epsilon^2}{1 - p} e^{\rho - At}[1 + (At - \rho)^{p-1}] \,. \end{aligned} \tag{6.31}$$

using the estimate (A.24) (with $\lambda = \kappa = 1$).

Finally, from (5.17) it follows that $\|\partial^i_\xi r^1\|_\rho \le \|\partial^i_\xi r^1_1\|_\rho + \|\partial^i_\xi r^1_2\|_\rho$ for $i = 1, 2$ so that combining the estimates (6.21) and (6.29) and estimates (6.28) and (6.31) leads to

$$\|r^1_\xi\|_\rho \le \frac{9\,d\epsilon^2}{\delta p} e^{\kappa\rho - At} \,, \tag{6.32}$$

$$\|r^1_{\xi\xi}\|_\rho \le \frac{24\,d\epsilon^2}{\delta p(1 - p)} e^{\frac{\kappa\rho - At}{2}}[1 + (At - \kappa\rho)^{p-1}] \,, \tag{6.33}$$

where for convenience we have replaced $At - \rho$ with $At - \kappa\rho$. This change anticipates the form of the singularity in the subsequent iterates due to the use of the Cauchy estimate in the induction step.

We turn now to the first approximation $w^{r,1}$, which is easily bounded. From equation (6.7)

$$\|w^{r,1}\|_\rho \le \|r_\xi^1\|_\rho + 2\|B_2[w^{r,0}, r^0]\|_\rho \,,$$

so that by (4.6), (4.7), (6.10), and (6.32),

$$\|w^{r,1}\|_\rho \le \left(\frac{9d}{\delta p} + 2c_1c^2\right)\epsilon^2 e^{\kappa\rho - At} \,, \tag{6.34}$$

where we have also used $A \le 1$. Similarly,

$$\begin{aligned} \|w_\xi^{r,1}\|_\rho &\le \|r_{\xi\xi}^1\|_\rho + 2\|B_{2\xi}[w^{r,0}, r^0]\|_\rho \\ &\le \left(\frac{24\,d}{\delta p(1-p)} + 2c_1c^2\right)\epsilon^2 e^{\frac{\kappa\rho - At}{2}}[1 + (At - \kappa\rho)^{p-1}] \,, \end{aligned} \tag{6.35}$$

where we have used (6.11) and (6.33).

A compact representation of these estimates may be obtained by introducing the norm $\|\!|\cdot|\!\|$, defined by

$$\|\!|u|\!\| = \sup_{\substack{0\le\rho<\infty \\ t>0,\ \kappa\rho<At}} \left[\left(\|u\|_\rho + \frac{\|u_\xi\|_\rho}{1 + (At - \kappa\rho)^{p-1}}\right) e^{\frac{At - \kappa\rho}{2}}\right]. \tag{6.36}$$

In terms of the above norm, (6.32) and (6.33) become

$$\|\!|R_\xi^1|\!\| = \|\!|r_\xi^1|\!\| \le \frac{33\,d\epsilon^2}{\delta p(1-p)} \,, \tag{6.37}$$

whereas (6.34) and (6.35) take the form

$$\|\!|W^1|\!\| = \|\!|w^{r,1}|\!\| \le \frac{33\,d\epsilon^2}{\delta p(1-p)} + 4c_1c^2\epsilon^2 \,. \tag{6.38}$$

Estimates (6.37) and (6.38) are the main result of this section.

6.3 Induction Hypothesis

The induction argument is related to that used in the proof of the abstract Cauchy-Kowalewski theorem given in [2], but with changes necessitated by the particular application here. To begin, define

$$a = 2\left[\frac{33\,d}{\delta p(1-p)} + 4c_1c^2\right]\epsilon^2 \equiv a_0\epsilon^2 \tag{6.39}$$

so that from (6.37) and (6.38)

$$\|\!|R_\xi^1|\!\| = \|\!|r_\xi^1|\!\| \le \frac{a}{2}\,, \quad \|\!|W^1|\!\| = \|\!|w^{r,1}|\!\| \le \frac{a}{2}\,. \tag{6.40}$$

By way of induction, assume that

$$\|\!|r_\xi^k|\!\| \le a\,, \quad \|\!|w^{r,k}|\!\| \le a\,, \quad \text{for } 2 \le k \le n, \tag{6.41}$$

and estimate $|||R_\xi^{n+1}|||$ and $|||W^{n+1}|||$. It will frequently be necessary to use the bound

$$
\begin{aligned}
&\|\alpha_\xi^n - \alpha_\xi^{n-1}\|_\rho + \|\beta_\xi^n - \beta_\xi^{n-1}\|_\rho \\
&\quad \le 3\|B_{2\xi}^n - B_{2\xi}^{n-1}\|_\rho \\
&\quad \le bA\epsilon e^{\frac{\kappa\rho - At}{2}} \Big\{ \|R_{\xi\xi}^n\|_\rho + \|W_\xi^n\|_\rho \\
&\qquad\qquad + [1 + (At - \kappa\rho)^{p-1}](\|R_\xi^n\|_\rho + \|W^n\|_\rho) \Big\},
\end{aligned} \tag{6.42}
$$

where $b = 3c_1(2c + 4a_0\epsilon)$ with a_0 defined in (6.39). This estimate readily follows from (4.6), (4.7), (6.13) (with $s = \tilde{s}$), and the induction hypothesis. Note in particular that the expression $(2c + 4a_0\epsilon)\epsilon e^{(\kappa\rho - At)/2}$ arises in bounding the first primed norm on the right-hand side of (6.13); this same expression when multiplied by $[1 + (At - \kappa\rho)^{p-1}]$ bounds the second primed norm there.

6.4 Estimate of $|||R_\xi^{n+1}|||$

First we bound $\|R_{1\xi}^{n+1}\|_\rho$ for $\rho < At/\kappa$. We have, from (6.8),

$$
\begin{aligned}
\|R_{1\xi}^{n+1}\|_\rho &\le \sum_{k=1}^{\infty} e^{\rho k} \int_0^t e^{-Ak(t-t')} \left|\widehat{\alpha_\xi^n}(k,t') - \widehat{\alpha_\xi^{n-1}}(k,t')\right| dt' \\
&= \left(\int_0^{t-\frac{\rho}{A}} + \int_{t-\frac{\rho}{A}}^t \right) \|\alpha_\xi^n - \alpha_\xi^{n-1}\|_{\rho_1}\, dt' \\
&\equiv K_1(t) + K_2(t),
\end{aligned} \tag{6.43}
$$

where we have used the definition of ρ_1 given in (6.16). Introduce the notation $|||u, v||| = |||u||| + |||v|||$. Since $\rho_1 \le 0$ for $t' \in [0, t - \frac{\rho}{A}]$, we estimate K_1 as

$$
\begin{aligned}
K_1 &\le \int_0^{t-\frac{\rho}{A}} e^{\rho_1} \|\alpha_\xi^n - \alpha_\xi^{n-1}\|_0\, dt' \\
&\le bA\epsilon e^{\rho - At} \int_0^{t-\frac{\rho}{A}} e^{\frac{At'}{2}} \Big\{ \|R_{\xi\xi}^n\|_0 + \|W_\xi^n\|_0 \\
&\qquad\qquad + [1 + (At')^{p-1}](\|R_\xi^n\|_0 + \|W^n\|_0) \Big\} dt'
\end{aligned}
$$

using (6.16), (6.42)

$$
\begin{aligned}
&\le bA\epsilon e^{\rho - At} |||R_\xi^n, W^n||| \int_0^{t-\frac{\rho}{A}} [1 + (At')^{p-1}] dt' \\
&= b\epsilon e^{\rho - At} |||R_\xi^n, W^n||| \left[At - \rho + \frac{(At - \rho)^p}{p} \right] \\
&\le \frac{2b\epsilon}{p} e^{\frac{\rho - At}{2}} |||R_\xi^n, W^n|||,
\end{aligned} \tag{6.44}
$$

where we have used the comment following equation (6.20) to obtain the latter bound.

The term K_2 is estimated using (6.42), which is allowed in view of the fact that $0 \leq \rho_1 < At'$ for $t' \in [t - \frac{\rho}{A}, t]$. Then

$$\begin{aligned} K_2 &\leq bA\epsilon \int_{t-\frac{\rho}{A}}^{t} e^{\frac{\kappa\rho_1 - At'}{2}} \Big\{ \| R^n_{\xi\xi} \|_{\rho_1} + \| W^n_{\xi} \|_{\rho_1} \\ &\qquad\qquad + [1 + (At' - \kappa\rho_1)^{p-1}] (\| R^n_{\xi} \|_{\rho_1} + \| W^n \|_{\rho_1}) \Big\} dt' \\ &\leq bA\epsilon \vert\vert\vert R^n_{\xi}, W^n \vert\vert\vert \int_0^t e^{(\kappa\rho_1 - At')} [1 + (At' - \kappa\rho_1)^{p-1}] dt' \\ &\leq \frac{4b\epsilon}{\delta p} e^{(\kappa\rho - At)} \vert\vert\vert R^n_{\xi}, W^n \vert\vert\vert \,, \end{aligned} \tag{6.45}$$

where an estimate for the integral term is provided in the appendix (see (A.23) with $\lambda = 1$). It immediately follows from (6.44) and (6.45) that

$$\| R^{n+1}_{1\xi} \|_{\rho} \leq \frac{6b\epsilon}{\delta p} e^{(\kappa\rho - At)} \vert\vert\vert R^n_{\xi}, W^n \vert\vert\vert \tag{6.46}$$

for $\rho < \frac{At}{\kappa}$.

Next we estimate $\| R^{n+1}_{1\xi\xi} \|_{\rho}$ for $\rho < \frac{At}{\kappa}$. Employing the first equation of (6.8), we have

$$\begin{aligned} \| R^{n+1}_{1\xi\xi} \|_{\rho} &\leq \sum_{k=1}^{\infty} e^{\rho k} \int_0^t e^{-Ak(t-t')} \left| \widehat{\alpha^n_{\xi\xi}}(k, t') - \widehat{\alpha^{n-1}_{\xi\xi}}(k, t') \right| dt' \\ &= \int_0^t \| \alpha^n_{\xi\xi} - \alpha^{n-1}_{\xi\xi} \|_{\rho_1} \, dt' \,. \end{aligned} \tag{6.47}$$

Define

$$\rho_4 = \rho_1 + \frac{At' - \kappa\rho_1}{2\kappa} \,. \tag{6.48}$$

Then from the Cauchy estimate

$$\begin{aligned} \| R^{n+1}_{1\xi\xi} \|_{\rho} &\leq \int_0^t \frac{\| \alpha^n_{\xi} - \alpha^{n-1}_{\xi} \|_{\rho_4}}{\rho_4 - \rho_1} \, dt' \\ &= \left(\int_0^{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})} + \int_{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})}^{t} \right) \frac{\| \alpha^n_{\xi} - \alpha^{n-1}_{\xi} \|_{\rho_4}}{\rho_4 - \rho_1} \, dt' \\ &\equiv L_1(t) + L_2(t) \,. \end{aligned}$$

Since $\rho_4 \le 0$ for t' in the interval $[0, \frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})]$, we estimate $L_1(t)$ by

$$
\begin{aligned}
L_1 &\le \int_0^{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})} e^{\rho_4} \frac{\|\alpha_\xi^n - \alpha_\xi^{n-1}\|_0}{\rho_4 - \rho_1} \\
&\le 2\kappa b A\epsilon e^{\frac{\rho - At}{2}} \int_0^{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})} e^{\frac{At'}{2\kappa}} (At' - \kappa\rho_1)^{-1} \big\{ \|R_{\xi\xi}^n\|_0 + \|W_\xi^n\|_0 \\
&\qquad + [1 + (At')^{p-1}](\|R_\xi^n\|_0 + \|W^n\|_0) \big\} dt' \\
&\qquad \text{using (6.16), (6.42)} \\
&\le \frac{2\kappa b A\epsilon}{At - \rho} e^{\frac{\rho - At}{2}} \|\!| R_\xi^n, W^n |\!\|\int_0^{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})} e^{\frac{-\delta At'}{2\kappa}} [1 + (At')^{p-1}] dt' \\
&\qquad \text{since } At' - \kappa\rho_1 > At - \rho \text{ for } t' \in \left[0, \frac{\kappa}{1+\kappa}\left(t - \frac{\rho}{A}\right)\right]
\end{aligned}
$$

$$
\le \frac{2\kappa b\epsilon}{p} e^{\frac{\rho - At}{2}} \|\!| R_\xi^n, W^n |\!\|[1 + (At - \rho)^{p-1}], \tag{6.49}
$$

where the integral is estimated by neglecting the exponentially decaying factor.

To bound L_2 note that $0 \le \rho_4 < \frac{At'}{\kappa}$ for t' in the interval $[\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A}), t]$ so that (6.42) may be applied. Then

$$
\begin{aligned}
L_2 &\le bA\epsilon \int_{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})}^t e^{\frac{\kappa\rho_4 - At'}{2}} (\rho_4 - \rho_1)^{-1} \big\{ \|R_{\xi\xi}^n\|_{\rho_4} + \|W_\xi^n\|_{\rho_4} \\
&\qquad + [1 + (At' - \kappa\rho_4)^{p-1}](\|R_\xi^n\|_{\rho_4} + \|W^n\|_{\rho_4}) \big\} dt' \\
&\le bA\epsilon \|\!| R_\xi^n, W^n |\!\| \int_{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})}^t e^{(\kappa\rho_4 - At')} \frac{[1 + (At' - \kappa\rho_4)^{p-1}]}{\rho_4 - \rho_1} dt' \\
&= 2bA\kappa\epsilon \|\!| R_\xi^n, W^n |\!\| \int_{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})}^t e^{\frac{\kappa\rho_1 - At'}{2}} \frac{[1 + 2^{1-p}(At' - \kappa\rho_1)^{p-1}]}{At' - \kappa\rho_1} dt' \\
&\qquad \text{using } At' - \kappa\rho_4 = \frac{At' - \kappa\rho_1}{2} \text{ and } \rho_4 - \rho_1 = \frac{At' - \kappa\rho_1}{2\kappa} \\
&\le 4bA\kappa\epsilon \|\!| R_\xi^n, W^n |\!\| \int_{\frac{\kappa}{1+\kappa}(t-\frac{\rho}{A})}^t e^{\frac{\kappa\rho_1 - At}{2}} \frac{[1 + (At' - \kappa\rho_1)^{p-1}]}{At' - \kappa\rho_1} dt' \\
&\le \frac{24b\kappa\epsilon}{\delta(1-p)} \|\!| R_\xi^n, W^n |\!\| e^{\frac{\kappa\rho - At}{2}} [1 + (At - \kappa\rho)^{p-1}].
\end{aligned}
$$

where the integral is estimated in the appendix (see (A.25) with $\lambda = \frac{1}{2}$). It follows that

$$
\|R_{1\xi\xi}^{n+1}\|_\rho \le \frac{26b\kappa\epsilon}{\delta p(1-p)} e^{\frac{\kappa\rho - At}{2}} \|\!| R_\xi^n, W^n |\!\|[1 + (At - \kappa\rho)^{p-1}] \tag{6.50}
$$

for $\rho < \frac{At}{\kappa}$.

We next estimate $\|R^{n+1}_{2\xi}\|_\rho$ for $\rho < \frac{At}{\kappa}$. From (6.8) it follows that

$$\begin{aligned}
\|R^{n+1}_{2\xi}\|_\rho &\le \int_t^\infty \|\beta^n_\xi - \beta^{n-1}_\xi\|_\rho\, dt' \\
&\le bA\epsilon \int_t^\infty e^{\frac{\kappa\rho - At'}{2}} \big\{ \|R^n_{\xi\xi}\|_\rho + \|W^n_\xi\|_\rho \\
&\qquad + [1 + (At' - \kappa\rho)^{p-1}](\|R^n_\xi\|_\rho + \|W^n\|_\rho)\big\} dt' \\
&\le bA\epsilon |\!|\!| R^n_\xi, W^n |\!|\!| \int_t^\infty e^{\kappa\rho - At'}[1 + (At' - \kappa\rho)^{p-1}] dt' \\
&\le \frac{4b\epsilon}{p} e^{\kappa\rho - At} |\!|\!| R^n_\xi, W^n |\!|\!| \qquad (6.51)
\end{aligned}$$

using estimate (A.21) in the appendix with $\lambda = 1$.

The bound on $\|R^{n+1}_{2\xi\xi}\|_\rho$ for $\rho < \frac{At}{\kappa}$ is obtained using the Cauchy estimate (6.9). We have

$$\|R^{n+1}_{2\xi\xi}\|_\rho \le \int_t^\infty \|\beta^n_{\xi\xi} - \beta^{n-1}_{\xi\xi}\|_\rho\, dt' \le \int_t^\infty \frac{\|\beta^n_\xi - \beta^{n-1}_\xi\|_{\rho_5}}{\rho_5 - \rho}\, dt',$$

where

$$\rho_5 = \rho + \frac{At' - \kappa\rho}{2\kappa}. \qquad (6.52)$$

Note that $\kappa\rho_5 < At'$ for $\frac{\kappa\rho}{A} < t < t' < \infty$. Thus using (6.42)

$$\begin{aligned}
\|R^{n+1}_{2\xi\xi}\|_\rho &\le bA\epsilon \int_t^\infty e^{\frac{\kappa\rho_5 - At'}{2}} (\rho_5 - \rho)^{-1} \\
&\quad \big\{ \|R^n_{\xi\xi}\|_{\rho_5} + \|W^n_\xi\|_{\rho_5} + [1 + (At' - \kappa\rho_5)^{p-1}](\|R^n_\xi\|_{\rho_5} + \|W^n\|_{\rho_5})\big\} dt' \\
&\le bA\epsilon |\!|\!| R^n_\xi, W^n |\!|\!| \int_t^\infty e^{\kappa\rho_5 - At'} \frac{[1 + (At' - \kappa\rho_5)^{p-1}]}{\rho_5 - \rho}\, dt' \\
&\le 4bA\kappa\epsilon |\!|\!| R^n_\xi, W^n |\!|\!| \int_t^\infty e^{\frac{\kappa\rho - At'}{2}} \frac{[1 + (At' - \kappa\rho)^{p-1}]}{At' - \kappa\rho}\, dt' \\
&\qquad \text{using } At' - \kappa\rho_5 = \frac{At' - \kappa\rho}{2} \text{ and } \rho_5 - \rho = \frac{At' - \kappa\rho}{2\kappa} \\
&\le \frac{24b\kappa\epsilon}{1 - p} |\!|\!| R^n_\xi, W^n |\!|\!| e^{\frac{\kappa\rho - At}{2}} [1 + (At - \kappa\rho)^{p-1}] \qquad (6.53)
\end{aligned}$$

using estimate (A.24) in the appendix with $\lambda = \frac{1}{2}$.

In summary, estimates (6.45) and (6.50) imply that

$$|\!|\!| R^{n+1}_{1\xi} |\!|\!| \le \frac{32b\kappa\epsilon}{\delta p(1 - p)} |\!|\!| R^n_\xi, W^n |\!|\!|,$$

whereas (6.51) and (6.53) show that

$$\| R_{2\xi}^{n+1} \| \leq \frac{28 b\kappa\epsilon}{p(1-p)} \| R_\xi^n, W^n \| .$$

Therefore, from (5.17)

$$\| R_\xi^{n+1} \| \leq \frac{60 b\kappa\epsilon}{\delta p(1-p)} \| R_\xi^n, W^n \| \tag{6.54}$$

for $n \geq 1$, which is the main result of this section. The following estimates, which result from (6.45, 6.51) and (6.50, 6.53), will also be useful in the next section:

$$\| R_\xi^{n+1} \|_\rho \leq \frac{10 b\epsilon}{\delta p} e^{\frac{\kappa\rho - At}{2}} \| R_\xi^n, W^n \| , \tag{6.55}$$

$$\| R_{\xi\xi}^{n+1} \|_\rho \leq \frac{50 b\kappa\epsilon}{\delta p(1-p)} e^{\frac{\kappa\rho - At}{2}} \| R_\xi^n, W^n \| [1 + (At - \kappa\rho)^{p-1}] , \tag{6.56}$$

for $n \geq 1$.

6.5 Estimate of $\| W^{n+1} \|$

It is easily seen from the integral equation for w^r given by (5.9) and (5.10) and the iteration scheme specified by (6.7) that

$$\| W^{n+1} \|_\rho \leq \| R_\xi^{n+1} \|_\rho + 2 \| B_2^n - B_2^{n-1} \|_\rho .$$

Hence from (6.12)

$$\begin{aligned} \| W^{n+1} \|_\rho &\leq \| R_\xi^{n+1} \|_\rho + b\epsilon e^{\frac{\kappa\rho - At}{2}} \left(\| R_\xi^n \|_\rho + \| W^n \|_\rho \right) \\ &\leq \| R_\xi^{n+1} \|_\rho + b\epsilon e^{\frac{\kappa\rho - At}{2}} \| R_\xi^n, W^n \| , \end{aligned}$$

where the constant b arises in the bound on $\| B_2^n - B_2^{n-1} \|_\rho$ as noted in the discussion following equation (6.42). Note that b incorporates the induction hypothesis through the presence of the term a_0, defined in (6.39). Substitution of (6.55) then leads to

$$\| W^{n+1} \|_\rho \leq \left(\frac{10 b}{\delta p} + b \right) \epsilon e^{\frac{\kappa\rho - At}{2}} \| R_\xi^n, W^n \| . \tag{6.57}$$

Similarly, we have

$$\| W_\xi^{n+1} \|_\rho \leq \| R_{\xi\xi}^{n+1} \|_\rho + 2 \| B_{2\xi}^n - B_{2\xi}^{n-1} \|_\rho ,$$

so that from (6.13) and (6.56)

$$\begin{aligned} \| W_\xi^{n+1} \|_\rho &\leq \| R_{\xi\xi}^{n+1} \|_\rho \\ &\quad + b\epsilon e^{\frac{\kappa\rho - At}{2}} \Big\{ \| R_{\xi\xi}^n \|_\rho + \| W_\xi^n \|_\rho \\ &\qquad + [1 + (At - \kappa\rho)^{p-1}] (\| R_\xi^n \|_\rho + \| W^n \|_\rho) \Big\} \\ &\leq \left(\frac{50 b\kappa}{\delta p(1-p)} + b \right) \epsilon e^{\frac{\kappa\rho - At}{2}} [1 + (At - \kappa\rho)^{p-1}] \| R_\xi^n, W^n \| . \end{aligned} \tag{6.58}$$

Therefore, we can write

$$\|\!| W^{n+1} |\!\| \leq a_1 \epsilon \|\!| R_\xi^n, W^n |\!\| \,, \tag{6.59}$$

where $a_1 = 60b\kappa/(\delta p(1-p)) + 2b$. Equation (6.59) is the main result in this section.

6.6 Completion of Induction Proof

Choose ϵ_0 small enough so that $a_1\epsilon_0 \leq \frac{1}{4}$, which also implies that

$$\left(\frac{60b\kappa}{\delta p(1-p)}\right)\epsilon_0 \leq \frac{1}{4}\,.$$

Then (6.54) and (6.59) imply that

$$\|\!| R_\xi^{n+1} |\!\| \leq \frac{1}{4} \|\!| R_\xi^n, W^n |\!\| \,,$$

$$\|\!| W^{n+1} |\!\| \leq \frac{1}{4} \|\!| R_\xi^n, W^n |\!\| \,,$$

for $0 < \epsilon < \epsilon_0$. The above inequalities combined with (6.40) therefore show that $\|\!| R_\xi^n |\!\| \leq a/2^n$ and $\|\!| W^n |\!\| \leq a/2^n$ for $n \geq 1$, which in turn implies that

$$\|\!| r_\xi^{n+1} |\!\| \leq \|\!| R_\xi^1 |\!\| + \cdots + \|\!| R_\xi^{n+1} |\!\| \leq \frac{a}{2} + \frac{a}{4} + \cdots + \frac{a}{2^n} \leq a$$

and similarly

$$\|\!| w^{r,n+1} |\!\| \leq a\,.$$

This completes the induction step. Since R_ξ^n and W^n are geometrically decreasing in size, it follows that $r^n \to r$ and $w^{r,n} \to w^r$ in the norm $\|\!| \cdot |\!\|$, with the pair (r, w^r) solving (2.3)–(2.4) and with $\|\!| r |\!\| \leq \|\!| r_\xi |\!\| \leq a$, $\|\!| w^r |\!\| \leq a$. Here we recall that $\|\!| \cdot |\!\|$ is defined in (6.36) and $a = a_0\epsilon^2$ is given by (6.39), with $\delta = \kappa - 1$. This completes the proof of Lemma 4.1.

7 Existence for General Initial Data

The analysis above produces a solution with initial data from a special class. In particular at $t = 0$, $s_0 \equiv s(\xi, 0)$ is purely imaginary and $r_1(\xi, 0) = 0$, which in turn implies (via (5.17)) that $r_0 \equiv r(\xi, 0)$ is real. In order to produce a general solution from the special initial data, a reparametrization is required. The reparametrization is introduced for technical reasons and does not have a physical significance.

Consider initial data S_0 that satisfies conditions (1) and (2) of Section 4. We find functions s_0 which is of size ϵ, ζ which is real and $O(1)$, and $r_0[s_0]$ of size ϵ^2 so that

$$\zeta(\xi) + S_0(\zeta(\xi)) = \xi + s_0(\xi) + r_0[s_0](\xi)\,, \tag{7.1}$$

where we use the notation $r_0[s_0]$ to signify the initial value (i.e., at $t = 0$) produced by the above iteration. Since ξ, ζ, and $r_0[s_0]$ are real, while S_0 and s_0 are imaginary, then

$$s_0(\xi) = S_0(\zeta(\xi))\,, \tag{7.2}$$

$$\zeta(\xi) = \xi + r_0[S_0(\zeta(\,\cdot\,))](\xi)\,. \tag{7.3}$$

Equations (7.2) and (7.3) are solved by an iteration method of the form

$$s_0^{n+1}(\xi) = S_0(\zeta^n(\xi))\,, \tag{7.4}$$

$$\zeta^{n+1}(\xi) = \xi + r_0\big[S_0(\zeta^n(\,\cdot\,))\big](\xi)\,, \tag{7.5}$$

with $\zeta^0(\xi) = \xi$. To show convergence of the iteration scheme, introduce the norm

$$|||u|||_0 = \sup_{\substack{0\le\rho<\infty\\ t>0,\ \kappa\rho<At}} \left[\left(\|u\|_\rho + \|u_p\|_\rho + \frac{\|u_\xi\|}{1+(At-\kappa\rho)^{p-1}}\right)e^{\frac{At-\kappa\rho}{2}}\right], \tag{7.6}$$

where $u_p \equiv \partial^p u/\partial\xi^p$ $(0 < p < 1)$ refers to the fractional (p^{th}) derivative of u, defined by

$$\widehat{u}_p(k) = (ik)^p \widehat{u}(k)\,.$$

The fractional derivative term in (7.6) balances the third term within brackets, and allows a bound on $|||s|||_0$ in terms of $\|s_0\|_0$ and $\|s_{0p}\|_0$. Note that we need to go to $t > 0$ to show convergence of (7.4) and (7.5), since part of r_0 is determined by integrating in time (see (5.15)), and this is reflected in the norm (7.6). This norm will be applied to functions even with a bounded ρ-norm as a way of controlling singular terms that are generated through use of the Cauchy estimate.

The main result used to show convergence of the iteration scheme (7.4)–(7.5) is the following:

LEMMA 7.1 *Let s, w^s and $\tilde{s}, \tilde{w}^s$ be any two sets of functions satisfying the conditions of Lemma* 4.1, *and let r, w^r and $\tilde{r}, \tilde{w}^r$ be the corresponding solutions to* (5.1)–(5.2). *Also, let r_0 and $\tilde{r}_0$ denote the value of r and $\tilde{r}$ at $t = 0$ for prescribed data s_0 and $\tilde{s}_0$. Then under the assumptions of Lemma* 4.1

$$|||r - \tilde{r}|||_0 \le \frac{c_2\epsilon}{p(1-p)(\kappa-1)}|||s - \tilde{s}|||_0\,. \tag{7.7}$$

PROOF: Introduce the notation

$$R = r - \tilde{r}, \quad R_j = r_j - \tilde{r}_j \text{ for } j = 1, 2, \qquad W = w^r - \tilde{w}^r, \qquad S = s - \tilde{s}\,.$$

We make frequent use of the following inequality, which is easily derived from the definitions of $\alpha, \tilde{\alpha} = \alpha[\tilde{s}, \tilde{r}]$, β, and $\tilde{\beta} = \beta[\tilde{s}, \tilde{r}]$ in (5.11) and (5.12) as well as the inequality (6.12):

$$\|\alpha - \tilde{\alpha}\|_\rho + \|\beta - \tilde{\beta}\|_\rho \le d_1 A\epsilon e^{\frac{\kappa\rho - At}{2}}\{\|S_\xi\|_\rho + \|R_\xi\|_\rho\}\,, \tag{7.8}$$

where d_1 is a constant independent of ϵ. Note that the term $\|W\|_\rho$ that would normally appear in (7.8) has been eliminated in favor of $\|S_\xi\|_\rho + \|R_\xi\|_\rho$. This is done by following the analysis of Section 6.5 to derive the bound

$$\|W\|_\rho \le \|R_\xi\|_\rho + d_2 \epsilon e^{\frac{\kappa\rho - At}{2}} (\|S_\xi\|_\rho + \|R_\xi\|_\rho + \|W\|_\rho),$$

from which a bound on $\|W\|_\rho$ in terms of $\|S_\xi\|_\rho + \|R_\xi\|_\rho$ is easily obtained.

We provide details for the bound on $\|R_p\|_\rho$ (the p^{th} derivative of R); bounds on $\|R\|_\rho$ and $\|R_\xi\|_\rho$ follow similarly. Following the analysis just below equation (6.48), we have

$$\begin{aligned}
\|R_{1p}\|_\rho &\le \int_0^t \|\alpha_p - \tilde{\alpha}_p\|_{\rho_1}\, dt' \quad \text{where } \rho_1 \text{ is defined in (6.16)} \\
&\le \int_0^t \frac{\|\alpha - \tilde{\alpha}\|_{\rho_4}}{(\rho_4 - \rho_1)^p}\, dt' \quad \text{using (6.9) with } \rho_4 \text{ defined in (6.48)} \\
&\le \int_0^{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})} e^{\rho_4} \frac{\|\alpha - \tilde{\alpha}\|_0}{(\rho_4 - \rho_1)^p}\, dt' + \int_{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})}^t \frac{\|\alpha - \tilde{\alpha}\|_{\rho_4}}{(\rho_4 - \rho_1)^p}\, dt' \\
&\equiv M_1(t) + M_2(t).
\end{aligned}$$

Next, introduce the notation $|||S, R|||_0 = |||S|||_0 + |||R|||_0$ and use (6.48) and (7.8) to estimate

$$\begin{aligned}
M_1 &\le (2\kappa)^p\, d_1 A \epsilon e^{\frac{\rho - At}{2}} \int_0^{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})} e^{\frac{At'}{2\kappa}} \frac{\{\|S_\xi\|_0 + \|R_\xi\|_0\}}{(At' - \kappa\rho_1)^p}\, dt' \\
&\le (2\kappa)^p\, d_1 A \epsilon |||S, R|||_0 e^{\frac{\rho - At}{2}} \int_0^{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})} \frac{e^{-\delta At'/(2\kappa)}}{(At')^p}\, dt' \\
&\qquad \text{since } \rho_1 \le 0 \text{ on the integration interval} \\
&\le \frac{6\kappa^2 d_1 \epsilon}{\delta(1-p)} e^{\frac{\rho - At}{2}} |||S, R|||_0, \qquad (7.9)
\end{aligned}$$

after bounding the integral. Following the analysis just below equation (6.49), the integral term M_2 is estimated using (7.8) as

$$\begin{aligned}
M_2 &\le \int_{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})}^t e^{\frac{\kappa\rho_4 - At'}{2}} \frac{\{\|S_\xi\|_{\rho_4} + \|R_\xi\|_{\rho_4}\}}{(At' - \kappa\rho_1)^p}\, dt' \\
&\le (2\kappa)^p\, d_1 A \epsilon |||S, R|||_0 \int_{\frac{\kappa}{1+\kappa}(t - \frac{\rho}{A})}^t \frac{e^{(\kappa\rho_1 - At')/2}}{(At' - \kappa\rho_1)^p}\, dt' \\
&\le \frac{6\kappa d_1 \epsilon}{\delta(1-p)} e^{\frac{\kappa\rho - At}{2}} |||S, R|||_0. \qquad (7.10)
\end{aligned}$$

Combining estimates (7.9) and (7.10) gives

$$\|R_{1p}\|_\rho \le \frac{12\kappa^2 d_1 \epsilon}{\delta(1-p)} e^{\frac{\kappa\rho - At}{2}} |||S, R|||_0. \qquad (7.11)$$

Similarly, following the arguments just below equation (6.51), we have

$$\begin{aligned}
\|R_{2p}\|_\rho &\leq \int_t^\infty \|\beta_p - \tilde{\beta}_p\|_\rho \, dt' \\
&\leq \int_t^\infty \frac{\|\beta - \tilde{\beta}\|_{\rho_5}}{(\rho_5 - \rho)^p} \, dt' \quad \text{using (6.9), where } \rho_5 \text{ is defined in (6.52)} \\
&\leq (2\kappa)^p \, d_1 A\epsilon \int_t^\infty e^{\frac{\kappa\rho_5 - At'}{2}} \frac{\{\|S_\xi\|_{\rho_5} + \|R_\xi\|_{\rho_5}\}}{(At' - \kappa\rho)^p} \, dt' \\
&\leq (2\kappa)^p \, d_1 A\epsilon \|\!|S, R|\!\|_0 \int_t^\infty \frac{e^{(\kappa\rho - At')/2}}{(At' - \kappa\rho)^p} \, dt' \\
&\leq \frac{6\kappa d_1 \epsilon}{1-p} e^{\frac{\kappa\rho - At}{2}} \|\!|S, R|\!\|_0
\end{aligned} \tag{7.12}$$

after bounding the integral.

Combining (7.11) amd (7.12) gives the estimate

$$\|R_p\|_\rho \leq \frac{c_2 \epsilon}{\delta(1-p)} e^{\frac{\kappa\rho - At}{2}} \|\!|S, R|\!\|_0 \tag{7.13}$$

for constant c_2. The following estimates are similarly derived:

$$\|R\|_\rho \leq \frac{c_2\epsilon}{\delta p} e^{\frac{\kappa\rho - At}{2}} \|\!|S, R|\!\|_0 \,, \tag{7.14}$$

$$\|R_\xi\|_\rho \leq \frac{c_2\epsilon}{\delta p(1-p)} e^{\frac{\kappa\rho - At}{2}} \|\!|S, R|\!\|_0 [1 + (At - \kappa\rho)^{p-1}] \,, \tag{7.15}$$

where c_2 is taken large enough so that (7.13)–(7.15) hold. Equation (7.14) follows from arguments similar to those used to derive equation (6.55) (the Cauchy estimate is not used), while the derivation of (7.15) is similar to that of (6.56). Dividing (7.15) by $[1 + (At - \kappa\rho)^{p-1}]$, adding to (7.13) and (7.14), and taking the sup leads, after a redefinition of the constant, to equation (7.7). This completes the proof of the lemma. □

An additional estimate is needed to show convergence of the iteration scheme (7.4)–(7.5). As discussed, this estimate requires the further assumption on S_0 given in (4.9), which in the notation of this section takes the form $\|S_{0\zeta}(\,\cdot\,)\|_{\mathrm{Lip}_{p+\nu}} < \infty$ for some $\nu > 0$. The desired estimate is given by the following:

LEMMA 7.2 *Let $\epsilon > 0$ and let $S_0(\zeta)$ satisfy conditions* (1) *and* (2) *of Section* 4, *as well as assumption* (4.9). *Furthermore, let $\zeta(\xi) = \xi + r_0(\xi)$ and $\tilde{\zeta}(\xi) = \xi + \tilde{r}_0(\xi)$, with r_0 and $\tilde{r}_0$ defined as in Lemma* 7.1. *Then there exists a constant c independent of ϵ such that*

$$\|S_0(\zeta(\,\cdot\,)) - S_0(\tilde{\zeta}(\,\cdot\,))\|_0 + \|S_{0p}(\zeta(\,\cdot\,)) - S_{0p}(\tilde{\zeta}(\,\cdot\,))\|_0 \leq c\epsilon(\|\zeta - \tilde{\zeta}\|_0 + \|\zeta_p - \tilde{\zeta}_p\|_0) \,, \tag{7.16}$$

where the subscript p denotes the fractional derivative with respect to ξ.

PROOF: Define

$$h(\xi) = \int_0^1 S_{0\zeta}\big(\tilde{\zeta}(\xi) + x(\zeta(\xi) - \tilde{\zeta}(\xi))\big)dx\,. \tag{7.17}$$

Then

$$\begin{aligned}
&\|S_{0p}(\zeta(\,\cdot\,)) - S_{0p}(\tilde{\zeta}(\,\cdot\,))\|_0 \\
&\quad = \left\| \frac{\partial^p}{\partial \xi^p} \int_{\tilde{\zeta}(\,\cdot\,)}^{\zeta(\,\cdot\,)} S_{0\zeta}(\overline{\zeta})d\overline{\zeta} \right\|_0 \\
&\quad = \left\| \frac{\partial^p}{\partial \xi^p}\big[h(\,\cdot\,)(\zeta(\,\cdot\,) - \tilde{\zeta}(\,\cdot\,))\big] \right\|_0 \quad \text{after a change of variable} \\
&\quad \le \|h_p(\,\cdot\,)\|_0\, \|\zeta(\,\cdot\,) - \tilde{\zeta}(\,\cdot\,)\|_0 + \|h(\,\cdot\,)\|_0\, \|\zeta_p(\,\cdot\,) - \tilde{\zeta}_p(\,\cdot\,)\|_0 \\
&\quad \le \|h(\,\cdot\,)\|_{\mathrm{Lip}_{p+\nu}}\, \|\zeta(\,\cdot\,) - \tilde{\zeta}(\,\cdot\,)\|_0 + \|h(\,\cdot\,)\|_{Lip_\nu}\, \|\zeta_p(\,\cdot\,) - \tilde{\zeta}_p(\,\cdot\,)\|_0\,,
\end{aligned}$$

where in the last line above we have used the inequalities $\|f(\,\cdot\,)\|_0 \le c_\nu \|f(\,\cdot\,)\|_{\mathrm{Lip}_\nu}$ (see [17, p. 136]) and $\|f_p(\,\cdot\,)\|_0 \le c_\nu \|f_p(\,\cdot\,)\|_{\mathrm{Lip}_\nu} \le c\|f(\,\cdot\,)\|_{\mathrm{Lip}_{p+\nu}}$ [17, p 225], which holds for functions with f and f_p of bounded variation and satisfying condition (4.9) for some $\nu > 0$. The constants c_ν and c depend only on the Lipschitz exponents. The function h clearly satisfies the bounded variation requirements. Furthermore, it is easy to show that the finiteness of $\|h(\,\cdot\,)\|_{\mathrm{Lip}_{p+\nu}}$ (which is of size ϵ) follows from the boundedness assumption (4.9). The result (7.16) immediately follows. □

We use (7.7) and (7.16) to show that the iteration (7.4)–(7.5) converges to a solution s_0, ζ solving the original equations. Introduce the notation $r^n = r[s_0^n]$. We estimate

$$\begin{aligned}
&\|s_0^{n+1} - s_0^n\|_0 + \|s_{0p}^{n+1} - s_{0p}^n\|_0 \\
&\quad = \|S_0(\zeta^n) - S_0(\zeta^{n-1})\|_0 + \|S_{0p}(\zeta^n) - S_{0p}(\zeta^{n-1})\|_0 \\
&\quad \le c\epsilon\big(\|\zeta^n - \zeta^{n-1}\|_0 + \|\zeta_p^n - \zeta_p^{n-1}\|_0\big) \quad \text{using (7.16)} \\
&\quad \le c\epsilon |||r^n - r^{n-1}|||_0 \quad \text{by (7.5) and (7.6)} \\
&\quad \le c\epsilon |||s^n - s^{n-1}|||_0 \quad \text{by (7.7)} \\
&\quad \le c\epsilon\big(\|s_0^n - s_0^{n-1}\|_0 + \|s_{0p}^n - s_{0p}^{n-1}\|_0\big)
\end{aligned} \tag{7.18}$$

where the constant c depends only on p. The last inequality in (7.18) easily follows from the definition of $|||\cdot|||_0$, along with the known properties of s (in particular, $\hat{s}(k,t) = \hat{s}(k,0)e^{-A|k|t}$). Inequality (7.18) shows that for sufficiently small ϵ the iteration is contracting and therefore converges.

The uniqueness of the solution easily follows from the uniqueness of the fixed point (which implies that representation (7.1) is unique), combined with inequality (7.7). This finishes the proof of Theorem 4.2.

8 Demonstration of Ill-Posedness

Theorem 4.2 is used to derive solutions to the Hele-Shaw equations (2.3)–(2.4) that develop singularities in finite time during unstable evolution (i.e., with $A < 0$) starting from analytic initial data. This is then used to show that the initial value problem for these equations is ill-posed in the Sobolev space H^k for $k > \frac{3}{2}$.

Following the related analysis for the Birkhoff-Rott equation [3] we use three symmetry properties to obtain the desired results. Let $z(\xi, t)$, $w(\xi, t, A)$ be a solution of (2.3)–(2.4). Then it is easily seen that the following are also solutions:

(i) $z_1(\xi, t) = z^*(\xi, -t)$, $w_1(\xi, t) = -w^*(\xi, -t, -A)$,

(ii) $z_2(\xi, t) = z(\xi, t - t_0)$, $w_2(\xi, t) = w(\xi, t - t_0, A)$,

(iii) $z_3(\xi, t) = N^{-1}z(N\xi, Nt)$, $w_3(\xi, t) = w(N\xi, Nt, A)$.

Properties (i) and (ii) imply that $z_b = z^*(\xi, t_0 - t)$, $w_b = -w^*(\xi, t_0 - t, -A)$, is a solution to (2.3)–(2.4) which is analytic at time 0 but which develops a (curvature) singularity at time t_0. Thus we have the following:

COROLLARY 8.1 *There exists initial data $z_b(\xi, 0) = z^*(\xi, t_0)$ which is analytic in a neighborhood of ξ real such that the solution $z_b(\xi, t)$, $w_b(\xi, t)$ of system* (2.3)–(2.4) *in the unstable case* ($A < 0$) *develops an infinite* $(1 + p)^{th}$ *derivative at a finite time t_0.*

Note that setting $t_0 = 0$ in Corollary 8.1 gives a solution $z_b(\xi, t)$ that is defined for $t < 0$, decays to 0 as $t \to -\infty$, and has a singularity in the $(1 + p)^{\text{th}}$ derivative at $t = 0$. This fact is combined with the rescaling of z to z_N to show ill-posedness of the initial value problem in the unstable case. Specifically, let $z_N(\xi, t) = N^{-2}z_b(N^2\xi, N^2t - 2N)$ so that $S_N = z_N - \zeta = N^{-2}S_b(N^2\xi, N^2t - 2N)$. Then at $t = 0$ the H^k norm of S_N satisfies the bound

$$\begin{aligned} \|S_N(\,\cdot\,, t = 0)\|_{H^k} &\le N^{2k-3}\|S(\,\cdot\,, -2N)\|_{H^k} \le KN^{2k-3}e^{2At} \\ &\to 0 \quad \text{as } N \to \infty\,, \end{aligned} \tag{8.1}$$

where K is a constant independent of N and $A < 0$. However, the time T_N of singularity formation satisfies $T_N = 2/N \to 0$ as $N \to \infty$. This proves the following:

COROLLARY 8.2 *Let $A < 0$. For any positive ϵ there is initial data $z = \zeta + S_0$ with $\|S_0\|_{H^k} < \epsilon$ such that $\|S\|_{H^k} \to \infty$ for $t = t_0$, where $t_0 > 0$ and $k > \frac{3}{2}$. In other words, the initial value problem for* (2.3)–(2.4) *is ill-posed in the Sobolev spaces H^k for $k > \frac{3}{2}$.*

9 Conclusion

The analysis presented above establishes global existence for the stable Muskat problem with small initial data that may contain singularities, showing that the solutions are analytic immediately after the initial time. It also shows existence of singular solutions for the unstable case of the Muskat problem. The singular solutions start with smooth initial data and develop singularities of order $1 + p$ with $p < 1$ at a finite time. Since the singularity time can be made arbitrarily small by adjusting the choice of initial data, this shows that the unstable case of the Muskat problem is ill-posed. The construction of singular solutions for the unstable problem is effected by applying time reversal to solutions of the stable Muskat problem with singular initial data. This construction uses analyticity and a version of the abstract Cauchy-Kowalewski theorem, but it does not require analyticity of the initial data to show global existence for the stable Muskat problem. As (one of) the first analytic results on the Muskat problem, this construction delineates some of the boundaries for possible further existence results.

The construction of singular solutions presented here is made possible by an unstable growth rate that is proportional to k (the wave number), as in [3]. The global existence result is (to the best of our knowledge) the first result that relies on a stable decay rate that is proportional to k in order to show that solutions become analytic immediately after the initial time.

The singularities found here are important for applications because they indicate the onset of complex geometry and evolution for two-phase fronts in Hele-Shaw systems. The present analysis does not include corners or cusps, but it does not rule them out either. Further work is required to assess the possibility of these stronger singularities and to determine the typical, or generic, singularity types.

Appendix

A.1 Lemma on the Fourier Norm

In the appendix we derive important estimates on the nonlocal term B_2. We begin with a lemma that proves useful in constructing the estimates.

LEMMA A.1 *Let $f_1, \dots, f_n$ and g be any functions satisfying $\|f_i\|_{\rho_0} < \infty$ and $\|g\|_{\rho_0} < \infty$ for some $\rho_0 > 0$. Define*

$$F^{(n)}(\xi) = \mathrm{PV} \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \frac{f_i(\xi + \gamma') - f_i(\xi)}{\gamma'} \right) \frac{g(\xi + \gamma')}{\gamma'} \, d\gamma' .$$

Then

$$|\hat{F}^{(n)}(k)| \leq \pi \sum_{\substack{k_1, \dots, k_{n+1} \\ k_1 + \dots + k_{n+1} = k}} \left| \hat{g}(k_{n+1}) \right| \prod_{i=1}^{n} |k_i \hat{f}_i(k_i)| \tag{A.1}$$

and

$$\|F^{(n)}\|_\rho \le \pi \|g\|_\rho \prod_{i=1}^{n} \|f_{i\xi}\|_\rho \tag{A.2}$$

for $0 \le \rho < \rho_0$*, where* $\hat{F}^{(n)}(k)$ *denotes the* k^{th} *Fourier coefficient of* $F^{(n)}$*.*

PROOF: The proof is a straightforward extension of a result in [1]. Define

$$h(\xi,\gamma') = \left(\prod_{i=1}^{n} \frac{f_i(\xi+\gamma') - f_i(\xi)}{\gamma'}\right)\frac{g(\xi+\gamma')}{\gamma'}\,.$$

Taking a Fourier transform in ξ gives

$$\widehat{h}(k,\gamma') = \sum_{\substack{k_1,\dots,k_{n+1}\\ k_1+\cdots+k_{n+1}=k}} \left(\prod_{i=1}^{n} \widehat{f_i}(k_i)\frac{e^{ik_i\gamma'}-1}{\gamma'}\right)\left(\frac{\widehat{g}(k_{n+1})e^{ik_{n+1}\gamma'}}{\gamma'}\right).$$

Therefore,

$$\hat{F}^{(n)}(k) = \sum_{\substack{k_1,\dots,k_{n+1}\\ k_1+\cdots+k_{n+1}=k}} \left(\prod_{i=1}^{n} \widehat{f_i}(k_i)\right)\widehat{g}(k_{n+1}) J(k_1,\dots,k_{n+1})\,, \tag{A.3}$$

where

$$J(k_1,\dots,k_{n+1}) = \mathrm{PV}\int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \frac{e^{ik_i\gamma'}-1}{\gamma'}\right)\frac{e^{ik_{n+1}\gamma'}}{\gamma'}\,d\gamma'\,,$$

with the interchange of sum and integral allowed in view of the analyticity of f_i and g. Now,

$$\begin{aligned} J(k_1,\dots,k_{n+1}) &= (-i)^n\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{ik\gamma'/2}e^{ik_{n+1}\gamma'/2}}{\gamma'}\left(\prod_{i=1}^{n}\frac{\sin(k_i\gamma'/2)}{\gamma'/2}\right)d\gamma' \\ &= -(-i)^{n+1}\int_0^{\infty} \frac{\sin((k+k_{n+1})\gamma'/2)}{\gamma'/2}\left(\prod_{i=1}^{n}\frac{\sin(k_i\gamma'/2)}{\gamma'/2}\right)d\gamma' \\ &= -(-i)^{n+1} I_p\,. \end{aligned} \tag{A.4}$$

An exact formula [6, formula 3.746] for the integral I_p in the case $k + k_{n+1} > \sum_{i=1}^{n} |k_i|$ is $I_p = \pi\prod_{i=1}^{n} k_i$ (i.e., the result is independent of $k + k_{n+1}$) so that

$$|J| \le \pi \prod_{i=1}^{n} |k_i|\,, \tag{A.5}$$

although estimate (A.5) holds in the general case (see [1]). Equations (A.3) and (A.5) then imply (A.1), while (A.2) readily follows from (A.1). □

A.2 Estimates on $B_2[s, w^s, r, w^r]$

Employing the change of variable $\gamma' = \xi' - \xi$, we write B_2 as (see (5.2))

$$B_2^* = \frac{A}{2\pi i}\,\mathrm{PV}\int_{-\infty}^{\infty}\Bigg\{\frac{\langle w^{*\prime}(s'_\xi + r'_\xi)\rangle}{\gamma'} - \left(\frac{s'+r'-s-r}{\gamma'}\right)\frac{\langle w^{*\prime} z'_\xi - i z'_\xi\rangle}{(\gamma'+s'+r'-s-r)}\Bigg\}d\gamma'\,, \tag{A.6}$$

where we use the notation $f' = f(\xi + \gamma')$. Assume

$$\|r_\xi\|_\rho < \|s_\xi\|_\rho \le C < \frac{1}{2}. \tag{A.7}$$

Then we can expand B_2 as

$$B_2 = \sum_{n=0}^{\infty} B_{2n}\,,$$

where B_{20}^* is the first term in (A.6) and

$$B_{2n}^* = \frac{A}{2\pi i}\,\mathrm{PV}\int_{-\infty}^{\infty}(-1)^{1+n}\left(\frac{s'+r'-s-r}{\gamma'}\right)^n \frac{g'}{\gamma'}\,d\gamma'$$

for $n \ge 1$. Here we have defined

$$g' = g(\xi+\gamma') = \big\langle w^*(\xi+\gamma') z_\xi(\xi+\gamma') - i z_\xi(\xi+\gamma')\big\rangle\,. \tag{A.8}$$

Now, from Lemma A.1,

$$\|B_{2n}\|_\rho \le \frac{|A|}{2}\|g\|_\rho \|s_\xi + r_\xi\|_\rho^n \tag{A.9}$$

for $n \ge 1$. Furthermore, $B_{20}^* = (A/2)(h_+ - h_-)$ where $h = \langle w^*(s_\xi + r_\xi)\rangle$. This in turn implies that

$$\begin{aligned}\|B_{20}\|_\rho = \frac{|A|}{2}\sum_{k=-\infty}^{\infty} e^{\rho|k|}|\hat{h}(k,t)| &= \frac{|A|}{2}\|h\|_\rho \\ &\le |A|\,\|w\|_\rho \|s_\xi + r_\xi\|_\rho\,,\end{aligned} \tag{A.10}$$

where in the last inequality we have used the definition of h and the fact that $\|f^*\|_\rho = \|f\|_\rho$. Summing (A.9) over n and adding the result to (A.10) yields

$$\|B_2\|_\rho \le |A|\left[\|w\|_\rho + \frac{\|g\|_\rho}{2(1-\|s_\xi\|_\rho - \|r_\xi\|_\rho)}\right](\|s_\xi\|_\rho + \|r_\xi\|_\rho)\,. \tag{A.11}$$

Finally, substituting the inequality

$$\|g\|_\rho \le 2\big[\|w\|_\rho(1 + \|s_\xi\|_\rho + \|r_\xi\|_\rho) + \|s_\xi\|_\rho + \|r_\xi\|_\rho\big] \tag{A.12}$$

and using (A.7), we may write (A.11) in simplified form as

$$\|B_2\|_\rho \le c_1|A|(\|w^s\|_\rho + \|w^r\|_\rho + \|s_\xi\|_\rho + \|r_\xi\|_\rho)(\|s_\xi\|_\rho + \|r_\xi\|_\rho)\,,$$

which is the desired estimate. The constant c_1 is independent of ϵ, ρ, and t. A similar calculation leads to estimate (6.11). (The constant c_1 is chosen large enough so that each of the estimates in this and the next subsection apply.)

A.3 Estimates on $\|B_2[s, w^s, r, w^r] - B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]\|_\rho$

We also need to estimate $\|B_2 - \tilde{B}_2\|_\rho$ where $\tilde{B}_2 = B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]$. We write

$$\begin{aligned}\|B_2 - \tilde{B}_2\|_\rho \le \|B_2[s, w^s, r, w^r] - B_2[s, w_s, \tilde{r}, \tilde{w}^r]\|_\rho \\ + \|B_2[s, w^s, \tilde{r}, \tilde{w}^r] - B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]\|_\rho\end{aligned} \tag{A.13}$$

and first estimate $\|B_2[s, w^s, r, w^r] - B_2[s, w_s, \tilde{r}, \tilde{w}^r]\|_\rho$. (To simplify the notation, we temporarily suppress writing the s and w^s in the argument list of B_2.) It is a simple matter to bound

$$\begin{aligned}&\|B_{20}[r, w^r] - B_{20}[\tilde{r}, \tilde{w}^r]\|_\rho \le \\ &\qquad |A|\left\{\|s_\xi\|_\rho\|w^r - \tilde{w}^r\|_\rho + \|w\|_\rho\|r_\xi - \tilde{r}_\xi\|_\rho + \|\tilde{r}_\xi\|_\rho\|w^r - \tilde{w}^r\|_\rho\right\},\end{aligned} \tag{A.14}$$

where we have used the identity $r_\xi w^* - \tilde{r}_\xi \tilde{w}^* = (r_\xi - \tilde{r}_\xi)w^* + \tilde{r}_\xi(w^* - \tilde{w}^*)$. Note that estimate (A.14) is not symmetric in r and $\tilde{r}$ or w^r and $\tilde{w}^r$ in view of this choice of identity. Nevertheless, we shall later add terms to make the final relation symmetric.

More work is necessary to estimate $\|B_{2n}[r, w^r] - B_{2n}[\tilde{r}, \tilde{w}^r]\|_\rho$ for $n \ge 1$. Denote by $\tilde{g}'$ the quantity in (A.8), but with $\tilde{w} = w^s + \tilde{w}^r$ and $\tilde{z}$ replacing w and z, respectively. Also introduce

$$\begin{aligned}p &= s + r\,, \qquad q = \frac{s' + r' - s - r}{\gamma'}\,,\\ \tilde{p} &= s + \tilde{r}\,, \qquad \tilde{q} = \frac{s' + \tilde{r}' - (s + \tilde{r})}{\gamma'}\,.\end{aligned}$$

Then

$$\begin{aligned}&\|B_{2n}[r, w^r] - B_{2n}[\tilde{r}, \tilde{w}^r]\|_\rho \\ &\quad\le \frac{|A|}{2\pi}\left\|\mathrm{PV}\int_{-\infty}^{\infty}\left(q^n g' - \tilde{q}^n \tilde{g}'\right)\frac{d\gamma'}{\gamma'}\right\|_\rho \\ &\quad= \frac{|A|}{2\pi}\left\|\mathrm{PV}\int_{-\infty}^{\infty}\left[(q^n - \tilde{q}^n)g' + \tilde{q}^n(g' - \tilde{g}')\right]\frac{d\gamma'}{\gamma'}\right\|_\rho \\ &\quad= \frac{|A|}{2\pi}\left\|\mathrm{PV}\int_{-\infty}^{\infty}\left[(q - \tilde{q})(q^{n-1} + q^{n-2}\tilde{q} + \cdots + q\tilde{q}^{n-2} + \tilde{q}^{n-1})g'\right.\right. \\ &\qquad\qquad\left.\left. + \tilde{q}^n(g' - \tilde{g}')\right]\frac{d\gamma'}{\gamma'}\right\|_\rho,\end{aligned}$$

so that upon applying Lemma A.1,

$$\begin{aligned}&\|B_{2n}[r, w^r] - B_{2n}[\tilde{r}, \tilde{w}^r]\|_\rho \\ &\quad \leq \frac{|A|}{2}\left\{\|r_\xi - \tilde{r}_\xi\|_\rho \left(\|p_\xi\|_\rho^{n-1} + \cdots + \|\tilde{p}_\xi\|_\rho^{n-1}\right)\|g\|_\rho + \|\tilde{p}_\xi\|_\rho^n \|g - \tilde{g}\|_\rho\right\} \\ &\quad \leq \frac{|A|}{2}\left\{\|r_\xi - \tilde{r}_\xi\|_\rho n \left(\|p_\xi\|_\rho^{n-1} + \|\tilde{p}_\xi\|_\rho^{n-1}\right)\|g\|_\rho + \|\tilde{p}_\xi\|_\rho^n \|g - \tilde{g}\|_\rho\right\}.\end{aligned}$$

Summing over n, substituting for p and $\tilde{p}$, and using the triangle inequality then gives

$$\begin{aligned}&\|B_2[r, w^r] - B_2[\tilde{r}, \tilde{w}^r]\|_\rho \leq \\ &|A|\Bigg\{(\|s_\xi\|_\rho + \|\tilde{r}_\xi\|_\rho)\|w^r - \tilde{w}^r\|_\rho + \|w\|_\rho \|r_\xi - \tilde{r}_\xi\|_\rho \\ &\quad + \|g\|_\rho \left[\frac{1}{(1 - \|s_\xi\|_\rho - \|r_\xi\|_\rho)^2} + \frac{1}{(1 - \|s_\xi\|_\rho - \|\tilde{r}_\xi\|_\rho)^2}\right]\|r_\xi - \tilde{r}_\xi\|_\rho \\ &\quad + \frac{(\|s_\xi\|_\rho + \|\tilde{r}_\xi\|_\rho)}{1 - \|s_\xi\|_\rho - \|\tilde{r}_\xi\|_\rho}\|g - \tilde{g}\|_\rho\Bigg\},\end{aligned} \tag{A.15}$$

where the first line above comes from the estimate for B_{20} in (A.14). We next substitute (A.12) and the easily derived inequality

$$\|g - \tilde{g}\|_\rho \leq 2\left\{(1 + \|s_\xi\|_\rho + \|r_\xi\|_\rho)\|w^r - \tilde{w}^r\|_\rho + \|\tilde{w}\|_\rho \|r_\xi - \tilde{r}_\xi\|_\rho\right\}$$

into (A.15). The result may be written in simplified form as

$$\begin{aligned}&\|B_2[r, w^r] - B_2[\tilde{r}, \tilde{w}^r]\|_\rho \leq \\ &\qquad c_1|A|\left\{\|s_\xi, r_\xi\|'_\rho \|w^r - \tilde{w}^r\|_\rho + \|s_\xi, w^s, r_\xi, w^r\|'_\rho \|r_\xi - \tilde{r}_\xi\|_\rho\right\},\end{aligned} \tag{A.16}$$

where we have used the notation of Section 6.1. In going from (A.15) to (A.16) we have added terms so that the estimate is symmetric in $r, \tilde{r}$ etc., and used (6.14) to simplify the resulting estimate.

We next estimate $\|B_2[s, w^s, \tilde{r}, \tilde{w}^r] - B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]\|_\rho$. Note that B_2 is invariant under the interchange $s \leftrightarrow r$ and $w^s \leftrightarrow w^r$, so it immediately follows from (A.16) that

$$\begin{aligned}&\|B_2[s, w^s, \tilde{r}, \tilde{w}^r] - B_2[\tilde{s}, \tilde{w}^s, \tilde{r}, \tilde{w}^r]\|_\rho \leq \\ &\qquad c_1|A|\left\{\|s_\xi, r_\xi\|'_\rho \|w^s - \tilde{w}^s\|_\rho + \|s_\xi, w^s, r_\xi, w^r\|'_\rho \|s_\xi - \tilde{s}_\xi\|_\rho\right\}.\end{aligned} \tag{A.17}$$

The term $\|w^s - \tilde{w}^s\|_\rho$ may be replaced using the identity

$$\|w^s - \tilde{w}^s\|_\rho \leq \|s - \tilde{s}\|_\rho, \tag{A.18}$$

which is easily derived from (3.3), (3.4). Together, (A.13) and (A.16)-(A.18) imply the final estimate (6.12). Note that some terms have been added to the final inequality in order to give the estimate a compact form. A similar calculation is used to derive (6.13).

A.4 Bounds on Time Integrals

We derive estimates on the time integrals that arise in the proof of Lemma 4.1.

1. (a) We first estimate

$$\mathcal{F}(t) = \int_t^\infty e^{\lambda(\kappa\rho - At')}(1 + (At' - \kappa\rho)^{p-1})dt'$$

where $\lambda > 0$. After the substitution $u = At' - \kappa\rho$, the integral becomes

$$\mathcal{F}(t) = \frac{1}{A}\int_{At-\kappa\rho}^\infty e^{-\lambda u}[1 + u^{p-1}]du\,. \tag{A.19}$$

The integrand is bounded above by $e^{-\lambda(At-\kappa\rho)}(1 + u^{p-1})$, and this estimate is used to simplify the integrand when $At - \kappa\rho < 1$ and $u \in [At - \kappa\rho, 1]$. For $u > 1$ the integrand is simplified by using $1 + u^{p-1} < 2$. These remarks justify the inequality

$$\begin{aligned}\mathcal{F}(t) &\le \frac{H[1-(At-\kappa\rho)]}{A}e^{-\lambda(At-\kappa\rho)}\int_{At-\kappa\rho}^1 (1+u^{p-1})du \\ &\quad + \frac{2}{A}\int_{At-\kappa\rho}^\infty e^{-\lambda u}\,du \\ &\equiv \mathcal{F}_1(t) + \mathcal{F}_2(t)\end{aligned}$$

where $H[x]$ is the Heaviside function, and $\mathcal{F}_1$ and $\mathcal{F}_2$ refer to the two integral terms above. Now, by direct calculation

$$\begin{aligned}\mathcal{F}_1(t) &= \frac{H[1-(At-\kappa\rho)]}{A}e^{-\lambda(At-\kappa\rho)}\left[1-(At-\kappa\rho) + \frac{1-(At-\kappa\rho)^p}{p}\right] \\ &\le \frac{2}{Ap}e^{-\lambda(At-\kappa\rho)}\end{aligned} \tag{A.20}$$

and

$$\mathcal{F}_2(t) = \frac{2e^{-\lambda(At-\kappa\rho)}}{\lambda A}\,.$$

It follows that

$$\mathcal{F}(t) \le \frac{2+2\lambda^{-1}}{Ap}e^{-\lambda(At-\kappa\rho)}\,, \tag{A.21}$$

which is the desired result.

(b) A related integral that we estimate is

$$\mathcal{G}(t) = \int_0^t e^{\lambda(\kappa\rho_1 - At')}\big[1 + (At' - \kappa\rho_1)^{p-1}\big]dt'\,,$$

where ρ_1 is defined in (6.16). After the substitution $u = At' - \kappa\rho_1$, the integral is written as

$$\begin{aligned} \mathcal{G} &= \frac{1}{A\delta}\int_{At-\kappa\rho}^{\kappa(At-\rho)} e^{-\lambda u}(1+u^{p-1})du \\ &\le \frac{1}{A\delta}\int_{At-\kappa\rho}^{\infty} e^{-\lambda u}(1+u^{p-1})du\,. \end{aligned} \tag{A.22}$$

Noting the similarity with (A.19), we can immediately write

$$\mathcal{G} \le \frac{2+2\lambda^{-1}}{A\delta p} e^{-\lambda(At-\kappa\rho)}\,. \tag{A.23}$$

2. (a) We similarly estimate

$$\mathcal{I}(t) = \int_t^{\infty} e^{\lambda(\kappa\rho - At')} \frac{t[1+(At'-\kappa\rho)^{p-1}]}{At'-\kappa\rho}\,dt'\,.$$

Changing variables, we have

$$\mathcal{I}(t) = \frac{1}{A}\int_{At-\kappa\rho}^{\infty} e^{-\lambda u}\frac{(1+u^{p-1})}{u}\,du\,.$$

Following the arguments in 1, we have

$$\begin{aligned} \mathcal{I}(t) &\le \frac{H[1-(At-\kappa\rho)]}{A} e^{-\lambda(At-\kappa\rho)} \int_{At-\kappa\rho}^{1} \frac{1+u^{p-1}}{u}\,du + \frac{2}{A}\int_{At-\kappa\rho}^{\infty} e^{-\lambda u}\,du \\ &\equiv \mathcal{I}_1(t) + \mathcal{I}_2(t)\,. \end{aligned}$$

By direct calculation

$$\begin{aligned} \mathcal{I}_1(t) &= \frac{H[1-(At-\kappa\rho)]}{A} e^{-\lambda(At-\kappa\rho)} \left[-\|(At-\kappa\rho) + \frac{1-(At-\kappa\rho)^{p-1}}{p-1}\right] \\ &\le \frac{2}{A(1-p)} e^{-\lambda(At-\kappa\rho)}[1+(At-\kappa\rho)^{p-1}]\,, \end{aligned}$$

since $|\ln x| < \frac{x^{p-1}}{1-p}$ for $0 < x < 1$ and $0 < p < 1$. Also,

$$\mathcal{I}_2(t) = \frac{2e^{-\lambda(At-\kappa\rho)}}{\lambda A}.$$

It follows that

$$\mathcal{I}(t) \le \frac{2+2\lambda^{-1}}{A(1-p)} e^{-\lambda(At-\kappa\rho)}[1+(At-\kappa\rho)^{p-1}]\,, \tag{A.24}$$

which is the desired estimate.

(b) We also give an estimate for

$$\mathcal{J}(t) = \int_0^t e^{\lambda(\kappa\rho_1 - At')} \frac{[1+(At'-\kappa\rho_1)^{p-1}]}{At'-\kappa\rho_1}\,dt'\,.$$

The estimate is obtained by following the steps leading to (A.23), with the result

$$\mathcal{J}(t) \leq \frac{2 + 2\lambda^{-1}}{A\delta(1-p)} e^{-\lambda(At-\kappa\rho)}[1 + (At - \kappa\rho)^{p-1}]. \tag{A.25}$$

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