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Caflisch, R. E.; Sammartino, M.

Existence and Singularities for the Prandtl Boundary Layer Equations

Prandtl's boundary layer equations, first formulated in 1904, resolve the differences between the viscous and inviscid description of fluid flows. This paper presents a review of mathematical results, both analytic and computational, on the unsteady boundary layer equations. This includes a review of the derivation and basic properties of the equations, singularity formation, well-posedness results, and infinite Reynolds number limits.

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1. Introduction

One of the fundamental problems of fluid mechanics is to resolve the differences between inviscid flow and viscous flow with small viscosity. The issues include drag, vorticity production, and boundary conditions:

- Inviscid flow does not correctly describe drag on an object. In irrotational flow ($\nabla \times \boldsymbol{u} = 0$), there is no drag resisting the motion of an object (d'Alembert's paradox) in the flow. For rotational flow, the pressure distribution produces form drag, but that does not account for the total drag.
- An inviscid flow does not produce vorticity (i.e. swirl).
- Along a boundary, an inviscid flow allows only vanishing normal velocity (i.e., flow cannot cross the boundary); whereas viscous flow requires vanishing velocity on the surface of a stationary object (i.e., fluid sticks to boundary).

Consider the initial value problem for incompressible flow over a plane $y = 0$ in 2D. The Euler equations for inviscid flow are

$$\begin{aligned} &\partial_t \boldsymbol{u}^E + \boldsymbol{u}^E \cdot \nabla \boldsymbol{u}^E + \nabla p^E = 0\,, \\ &\nabla \cdot \boldsymbol{u}^E = 0\,, \\ &v^E(y=0) = 0\,, \\ &\boldsymbol{u}^E(t=0) = \boldsymbol{u}_0^E\,. \end{aligned} \tag{1.1}$$

In these equations $\boldsymbol{x} = (x, y)$ is the space variable, $\boldsymbol{u} = (u, v)$ is velocity, and p is pressure.

Navier-Stokes equations for viscous flow are

$$\begin{aligned} &\partial_t \boldsymbol{u}^{NS} + \boldsymbol{u}^{NS} \cdot \nabla \boldsymbol{u}^{NS} + \nabla p^{NS} = \nu\, \Delta \boldsymbol{u}^{NS}\,, \\ &\nabla \cdot \boldsymbol{u}^{NS} = 0\,, \\ &\boldsymbol{u}^{NS}(y=0) = 0\,, \\ &\boldsymbol{u}^{NS}(t=0) = \boldsymbol{u}_0^{NS}\,. \end{aligned} \tag{1.2}$$

In these equations, the Reynolds number $\mathrm{Re} = UL/\nu$ is the relevant nondimensional parameter, in which U and L are characteristic values for the velocity and length scale. For typical flows, the viscosity ν is small, so that Re is large, and the flow should be nearly inviscid.

Ludwig Prandtl resolved the difference between viscous and inviscid flow, starting in 1904 [12]. This work contained the first development of boundary layer theory, which is now a standard part of singular perturbation theory. Prandtl found that the Euler equations are valid outside a thin "boundary layer" (BL) region. The BL thickness is $\varepsilon = \sqrt{\nu}$. Viscous drag, vorticity production and relaxation of no-slip boundary conditions all occur inside the BL.

Prandtl's boundary layer equations for flow inside the BL are

$$\begin{aligned} &\partial_t u^P + u^P\, \partial_x u^P + v^P\, \partial_Y u^P = (\partial_t + u^E\, \partial_x)\, u^E(y=0) + \partial_{YY} u^P\,, \\ &\partial_Y p^P = 0\,, \\ &\partial_x u^P + \partial_Y v^P = 0\,, \\ &u^P(Y=0) = 0\,, \\ &u^P(Y \to \infty) \to u^E(y=0)\,, \\ &u^P(t=0) = u_0^P \end{aligned} \tag{1.3}$$

in which Y is a scaled variable normal to the boundary, as discussed in the next section.

The focus of this paper is on mathematical results, both numerical and analytic, for time-dependent Prandtl equations. These include derivation of Prandtl's equations in Section 2, separation and singularities in Section 3, and existence theory in Section 4. Validity of boundary layer theory, including convergence of the Navier-Stokes solution to an Euler solution outside boundary layer and to a Prandtl's solution inside boundary layer, is discussed in Section 5. A summary is given in Section 6.

2. Derivation and basic properties of Prandtl's equations

Within the flow, the only parameter is the Reynolds number $\mathrm{Re} = LU/\nu$. Near a boundary, however, the relative distance to the boundary is a second parameter. This suggests that away from boundary, only simple scaling is correct, yielding the Euler equations, but that near a boundary, a different scaling may apply. Prandtl's boundary layer scaling (in dimensional form) is the following:

$$\begin{aligned} Y &= y/\varepsilon, \\ \boldsymbol{u} &= (u, \varepsilon V) \end{aligned} \tag{2.1}$$

so that $\partial_y = \varepsilon^{-1}\partial_Y$. This allows rapid variation normal to boundary and requires the normal velocity to be small near the boundary.

Under this scaling, Navier-Stokes equations become

$$\begin{aligned} &\partial_t u + u\,\partial_x u + V\,\partial_Y u + \partial_x p = \nu\,\partial_x^2 u + (\nu/\varepsilon^2)\,\partial_Y^2 u\,, \\ &\partial_t V + u\,\partial_x V + V\,\partial_Y V + \varepsilon^{-2}\,\partial_Y p = \nu\,\partial_x^2 V + (\nu/\varepsilon^2)\,\partial_Y^2 V\,, \\ &\partial_x u + \partial_Y V = 0\,, \\ &u = V = 0 \quad \text{on} \quad Y = 0\,. \end{aligned} \tag{2.2}$$

Set $\varepsilon = \sqrt{\nu}$ and take $\varepsilon \to 0$ to obtain Prandtl's equations

$$\begin{aligned} &\partial_t u + u\,\partial_x u + V\,\partial_Y u + \partial_x p = \partial_Y^2 u\,, \\ &\partial_Y p = 0\,, \\ &\partial_x u + \partial_Y V = 0\,, \\ &u = V = 0 \quad \text{on} \quad Y = 0\,. \end{aligned} \tag{2.3}$$

Since $p = p^P$ is independent of Y, set it to limiting Euler value $p^P(x,t) = p^E(x,0,t)$ so that

$$\begin{aligned} \partial_x p^P(x,t) &= \partial_x p^E(x,0,t) \\ &= -\left(\partial_t u^E + u^E \partial_x u^E\right)(x,0,t) \end{aligned} \tag{2.4}$$

which implies

$$\lim_{y\to\infty} u^P(x,y,t) = u^E(x,0,t)\,. \tag{2.5}$$

This results in the Prandtl equations (1.3), after a change in notation.

Here is a summary of the properties of a Prandtl solution, showing that it accounts for the differences between inviscid and viscous flow that were mentioned in the Introduction. The vorticity for Navier-Stokes, written in the Prandtl scaling, is

$$\omega = \partial_x v - \partial_y u = \varepsilon\,\partial_x V - \varepsilon^{-1}\partial_Y u\,. \tag{2.6}$$

It follows that the vorticity in Prandtl equations is

$$\omega^P = -\partial_Y u\,. \tag{2.7}$$

Since the flow is incompressible, the normal velocity is

$$v^P(x,Y,t) = -\int_0^Y \partial_x u^P(x,Y',t)\,dY'\,. \tag{2.8}$$

The drag is

$$\int \nu\,|\nabla \boldsymbol{u}|^2\,d\boldsymbol{x} \;\to \varepsilon \int |\partial_Y V|^2\,dx\,dY\,. \tag{2.9}$$

Boundary conditions for the Prandtl equations are

- no slip at $Y = 0$, as in Navier-Stokes,
- zero normal velocity to leading order (i.e. $v = O(\varepsilon)$) at $Y = \infty$, corresponding to $y = 0$, as required in Euler.

An important complication in this picture is the occurrence of boundary layer separation. Prandtl equations describe vorticity as produced and confined to the BL. In the flow past objects, however, vorticity is observed out in flow; e.g. lift on an airfoil is proportional to the circulation, which is equal to the total vorticity in the flow. Transport of vorticity by the Navier-Stokes equations is described by

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = \nu \Delta \omega\,. \tag{2.10}$$

For small viscosity ν, the diffusion of vorticity is too small to effectively get vorticity into flow. So convection of vorticity must be mechanism for vorticity shedding.

For an object with a corner, vorticity shedding occurs smoothly at the corner (Kutta condition). Shedding of vorticity for smooth objects, on the other hand, occurs through boundary layer separation. Can Prandtl's equations describe separation? For steady separation, GOLDSTEIN [6] found that the Prandtl flow becomes singular. For unsteady flow, singularity at separation was conjectured by SEARS and TELIONIS [16], but the nature of separation was unclear until VAN DOMMELEN and SHEN [20] numerically showed singularity formation at separation. This is described in detail in the next section.

3. Singularities

VAN DOMMELEN and SHEN [20] considered an impulsively started circular cylinder, for which the equations are

$$\begin{aligned} &\partial_t u + u\, \partial_x u + V\, \partial_Y u = UU'(x) + \partial_Y^2 u\,,\\ &\partial_x u + \partial_Y V = 0 \end{aligned} \tag{3.1}$$

with boundary and initial conditions

$$\begin{aligned} &u(x,\infty,t) \;=\; U(x) \;=\; \sin x\,,\\ &u(x,0,t) \;=\; V(x,0,t) \;=\; u(0,Y,t) \;=\; u(\pi,Y,t) \;=\; 0\,. \end{aligned} \tag{3.2}$$

The domain is $0 \le Y < \infty$, $0 \le x \le \pi$, in which $Y = r$ is the variable normal to cylinder and $x = \theta$ is the tangential variable along cylinder.

They applied a transformation to Lagrangian variables (ξ, η), with $(\xi, \eta) = (x, Y)$ at $t = 0$, to obtain

$$\begin{aligned} &u_t = .5 \sin 2x + x_\xi^2 u_{\eta\eta} - 2x_\xi x_\eta u_{\xi\eta} + x_\eta^2 u_{\xi\xi} - x_\xi u_\xi x_{\eta\eta} + (x_\xi u_\eta + x_\eta u_\xi) x_{\xi\eta} - x_\eta u_\eta x_{\xi\xi}\,,\\ &x_t = u\,. \end{aligned} \tag{3.3}$$

The variable Y is found from

$$x_\xi Y_\eta - x_\eta Y_\xi = 1\,. \tag{3.4}$$

A singularity occurs when x has a stationary point, at which

$$\begin{aligned} &|\nabla x| = 0\,,\\ &|\nabla Y| = \infty\,,\\ &|u_x| = |Y_\eta u_\xi - Y_\xi u_\eta| \;=\; \infty\,. \end{aligned} \tag{3.5}$$

At this singularity, which we call the "Van Dommelen singularity", the normal velocity blows up, sending flow out of the BL to $Y = \infty$. The singularity location is off of the boundary and away from the trailing edge ($x = \pi$). This implies the surprising result that the initial separation is away from trailing edge. This singularity formation and its properties were confirmed by COWLEY [3], using a Pade expansion in t with numerical computation of the coefficients. SHEN [17] conjectured that singularity generation is due to "wave steepening" as in compressible flow, which will be confirmed in the analytic result described next.

E and ENGQUIST [4] performed a construction of singular solutions for unsteady Prandtl. Their method is analogous to Lax's proof of shock formation from smooth data. They simplified the flow by assuming that $p_{xx} = u_0 = 0$ on $x = 0$. The solution then has form $u(x,y,t) = x\, b(x,y,t)$, and the equation for $a(y,t) = b(0,y,t)$ is

$$a_t = a_{YY} - a^2 + a_Y \int_0^Y a(Y',t)\, dY'\,. \tag{3.6}$$

They showed the following result:

Theorem 3.1: *Assume that* $\int_0^\infty \left(2a_{0Y}^2 + a_0^3\right) dY < 0$, *in which* $a_0(Y) = a(Y,0)$. *Then* $a(Y,t)$ *becomes singular at finite time* T*; i.e.*

$$\lim_{t\to T} \max_Y |a| = \infty \quad \textit{or} \quad \lim_{t\to T} |a_Y(0,t)| = \infty\,. \tag{3.7}$$

4. Well-posedness results

Singularity occurrence in the numerical results of Van Dommelen and Shen [20] and analytic results of E and Engquist [4] is at a finite time T. An important mathematical question is whether T is bounded away from 0, and whether Prandtl's equations are a well-posed system. For example, inviscid Burgers equation $u_t + uu_x = 0$ is a well-posed equation, since singularity formation occurs at a finite time that is controlled by total variation of the initial data. On the other hand, the Cauchy-Riemann equations in space-time

$$\begin{aligned} &u_t + v_x = 0\,, \\ &v_t - u_x = 0 \end{aligned} \tag{4.1}$$

are an ill-posed system; singularities can occur in arbitrarily short time for initial data that is bounded in any Sobolev norm. Well-posedness for Prandtl is still an open problem.

In this section we shall review some of the most relevant results about well posedness of Prandtl equations. First we give a brief review of the work of Oleinik and her coworkers. In a series of papers, most of which date back to the sixties, she gave a complete theory of the Prandtl equations, the main limitation being an assumption of some kind of monotonicity. We shall make no attempt to provide a complete account of her work, and refer to the recent book of Oleinik and Samokhin [11] for an exhaustive review and for a complete list of references. Next we describe an existence theorem recently proven by Caflisch and Sammartino in [14]. In their theorem no assumption is made on the monotonicity of the initial data or of the Euler flow, but they had to restrict the initial data to be analytic.

4.1 The Oleinik results

Consider a domain $\mathbf{\Omega} = \{(x, y, t) : 0 < x < M,\ 0 < y < \infty,\ 0 < t < T\}$. Write Prandtl equations in the following form:

$$\partial_t u + u\,\partial_x u + v\,\partial_Y u = \partial_t U + U\,\partial_x U + \partial_{YY} u\,, \tag{4.2}$$

$$\partial_x u + \partial_Y v = 0\,, \tag{4.3}$$

$$u|_{t=0} = u_0\,, \tag{4.4}$$

$$u|_{Y=0} = 0\,, \tag{4.5}$$

$$u|_{x=0} = 0\,, \tag{4.6}$$

$$u \to U \quad \text{when} \quad Y \to \infty\,. \tag{4.7}$$

4.1.1 The hypotheses

We now introduce the hypotheses that will be necessary to prove the main Theorem:

$$U,\ \partial_x U,\ \partial_t U/U\,, \qquad \text{bounded with bounded derivatives}\,, \tag{h.1}$$

$$\begin{aligned} &u_0|_{Y=0} = 0\,, \\ &u_0(x, Y \to \infty) \to U(x, t = 0)\,, \end{aligned} \tag{h.2}$$

$$u_0/U,\ \partial_Y u_0/U \qquad \text{continuous}\,, \tag{h.3}$$

$$\partial_Y u_0 > 0\,, \tag{h.4}$$

$$K_1(U(x, t = 0) - u_0(x, y)) \le \partial_Y u_0(x, y) \le K_2(U(x, t = 0) - u_0(x, y)) \quad \text{with} \quad K_1, K_2 > 0\,, \tag{h.5}$$

$$\partial_Y u_0, \partial_{YY} u_0, \partial_{YYY} u_0 \quad \text{bounded}\,, \tag{h.6}$$

$$\frac{\partial_{YY} u_0}{\partial_Y u_0}, \qquad \frac{\partial_{YYY} u_0\,\partial_Y u_0 - (\partial_{YY} u_0)^2}{(\partial_Y u_0)^2} \quad \text{bounded}\,, \tag{h.7}$$

$$v_0(x, t = 0)\,\partial_Y u_0(x, Y = 0) + \partial_x p\,(x, t = 0) = \partial_{YY} u_0\,(x, t = 0)\,, \tag{h.8}$$

$$\left| \frac{\partial_{xY} u_0\,\partial_Y u_0 - \partial_x u_0\,\partial_{YY} u_0}{\partial_Y u_0} + \partial_x U \frac{u_0\,\partial_{YY} u_0 - (\partial_Y u_0)^2}{U\,\partial_Y u_0} \right| \le K_3(U - u_0)\,. \tag{h.9}$$

4.1.2 The main result

Theorem 4.1: *Suppose the hypotheses* (h.1)–(h.9) *are verified. Then eqs.* (4.2)–(4.7) *admit a unique solution* (u, v). *Moreover* (u, v) *have the following properties:*

$$\begin{aligned} &u/U,\ \partial_Y u/U \quad \textit{continuous and bounded}\,, \\ &v,\ \partial_t u,\ \partial_x u,\ \partial_{xY} u,\ \partial_{YY} u,\ \partial_Y v, \quad \textit{bounded measurable and continuous in } y\,, \\ &\exp(-C_1 Y) \le 1 - u/U \le \exp(-C_2 Y)\,, \qquad C_1, C_2 > 0\,, \\ &\partial_Y u/U > 0\,, \quad \textit{for} \quad Y \ge 0\,, \\ &\partial_Y u/U \to 0 \quad \textit{when} \quad Y \to \infty\,. \end{aligned}$$

4.1.3 Sketch of the proof: The Crocco variables

We introduce the Crocco variables (τ,ξ,η). They are defined as

$$\tau = t, \qquad \xi = x, \qquad \eta = \frac{u(x,y,t)}{U(x,t)}, \qquad w(\tau,\xi,\eta) = \frac{\partial_Y u(x,y,t)}{U(x,t)}\,. \tag{4.8}$$

Then in the domain $\Omega' = \{(\xi,\eta,\tau):\ 0<\xi<M,\ 0<\eta<1,\ 0<\tau<T\}$, w satisfies the following equation:

$$\partial_\tau w + \eta U\,\partial_\xi w - A\partial_\eta w - Bw = w^2\,\partial_{\eta\eta} w\,, \tag{4.9}$$

with the following initial and boundary conditions:

$$w|_{\tau=0} \equiv w_0 = \partial_Y u_0/U\,, \tag{4.10}$$

$$w|_{\eta=1} = 0\,, \tag{4.11}$$

$$(w\,\partial_\eta w - v_0 w + C)|_{\eta=0} = 0\,, \tag{4.12}$$

where A, B, C in (4.9) and (4.12) are defined by

$$A = (\eta^2-1)\,\partial_x U + (\eta-1)\,\frac{\partial_t U}{U}\,, \tag{4.13}$$

$$B = -\,\eta\,\partial_x U - \frac{\partial_t U}{U}\,, \tag{4.14}$$

$$C = \partial_x U + \frac{\partial_t U}{U}\,. \tag{4.15}$$

4.1.4 Sketch of the proof: The discretization procedure

We now construct the solution of eqs. (4.9)–(4.12). Discretize the variables ξ and τ, and get an ODE in the variable η. For a function $f(\xi,\eta,\tau)$ on Ω', define

$$f^{k,m}(\eta) = f(kh,\eta,mh), \qquad k=0,1,\dots,[M/h], \qquad m=1,\dots,[T/h], \qquad h>0\,.$$

Consider the following discretized (in the variables ξ and τ) version of eq. (4.9):

$$\left(w^{k,m-1}+h\right)^2\partial_{\eta\eta}w^{k,m} + A^{k,m}\,\partial_\eta w^{k,m} - \frac{w^{k,m}-w^{k,m-1}}{h} - \eta U^{k,m}\,\frac{w^{k,m}-w^{k-1,m}}{h} + B^{k,m}w^{k,m} = 0\,. \tag{4.16}$$

The boundary conditions for eq. (4.16) are

$$w^{k,m}|_{\eta=1} = 0\,, \tag{4.17}$$

$$(w^{k,m-1}\,\partial_\eta w^{k,m} - v_0^{k,m}w^{k,m-1} + C^{k,m})|_{\eta=0} = 0\,. \tag{4.18}$$

Therefore, keeping the step size h fixed, for each m and k one has a linear second order ordinary differential equation. Initialize the system using the initial condition (4.10), i.e.

$$w^{k,0} = w_0(\xi = kh,\eta) \equiv w_0^h\,. \tag{4.19}$$

Then pass to the time steps $m=1$, and solve (4.16)–(4.18) for each $k=1,2\dots$. One can therefore proceed to the other time steps $m=2$, $k=1,2\dots$, and so on. The following Proposition ensures the existence of a unique solution of (4.16)–(4.18) up to the time T_0 independent of h.

Proposition 4.2: *Suppose A, B, C, v_0 w_0 are bounded with their derivatives, and $K_1(1-\eta) \le w_0 \le K_2(1-\eta)$, $|\partial_\xi w_0| \le K_3(1-\eta)$. Then (4.16)–(4.18) admit a unique solution $w^{k,m}$ for $mh<T_0$, where T_0 depends only on the data. Moreover the solution $w^{k,m}$ is such that*

$$\partial_\eta w^{k,m}\,, \qquad \frac{w^{k,m}-w^{k,m-1}}{h}\,, \qquad (1-\eta+h)\,\partial_{\eta\eta}w^{k,m} \tag{4.20}$$

are uniformly bounded with respect to h, and satisfies the following estimate:

$$V_1(mh,\eta) \le w^{k,m} \le V_2(mh,\eta)\,. \tag{4.21}$$

In these inequalities V_1 and V_2 are two positive continuous functions, satisfying $V_i \sim (1-\eta)$ in a neighborhood of $\eta=1$.

The next issue is to analyze the limit $h\to 0$.

4.1.5 Sketch of the proof: Compactness

So far we have constructed a family of functions $w^{k,m}(\eta)$ defined on the *grid* $k=0,1,\dots,[M/h]$, $\quad m=1,\dots,[T/h]$. Next extend them to the whole Ω' by linear interpolation between the points of the grid. Denote by $w_h(\xi,\eta,\tau)$ these linear interpolations.

Due to the Proposition 4.2, these functions on Ω' are uniformly bounded (see (4.21)) and equicontinuous (see (4.20)). By the Ascoli-Arzelà Theorem, there is a subsequence converging, when $h \to 0$, to a function $w(\xi, \eta, \tau)$. This leads to the following Proposition:

Proposition 4.3: *Suppose the hypotheses of Proposition 4.2 are verified. Then eqs.* (4.9)–(4.12) *admit, up to a time* T_1 *that is dependent on the data, a unique continuous solution* $w(\xi, \eta, \tau)$ *with bounded weak derivatives* $\partial_\xi w$, $\partial_\eta w$, $\partial_\tau w$. *Moreover,* w *has the following properties:*

$$D_1(1-\eta) \le w \le D_2(1-\eta) \tag{4.22}$$

and

$$|\partial_\xi w| \le D_3(1-\eta)\,, \tag{4.23}$$

$$|\partial_\tau w| \le D_4(1-\eta) \tag{4.24}$$

in which $D_i > 0$ *are constant.*

4.1.6 Sketch of the proof: Conclusion

Under the assumptions (h.1)–(h.9) and using the transformation (4.8), the Prandtl equations are solved by finding w satisfying (4.9)–(4.12). In addition, the assumptions (h.1)–(h.9) imply validity of the hypotheses of Proposition 4.3. This establishes existence of a solution for (4.9)–(4.12). Inverting the Crocco transformation, one obtains the solution u of (4.2)–(4.7) with the properties claimed in Theorem 4.1.

4.2 The analyticity result

Next is a brief account of the results of Sammartino and Caflisch [14]. Their work, inspired by an unpublished analysis of Asano, [1], is based on the hypothesis of analytic initial data; this hypothesis allowed them to analyze Prandtl equations in the framework of the abstract Cauchy-Kowalewski Theorem.

Introduce the variable $\tilde{u} \equiv u - U$. Eqs. (4.2)–(4.7) therefore can be written as

$$\partial_t \tilde{u} + U\,\partial_x \tilde{u} - \tilde{u}\,\partial_x U + \tilde{u}\,\partial_x \tilde{u} - \left(\int_0^Y \partial_x \tilde{u}\, dY' + Y\,\partial_x U\right)\partial_Y \tilde{u} = \partial_{YY}\tilde{u}\,, \tag{4.25}$$

$$v = -\int_0^Y \partial_x \tilde{u}\, dY' - Y\,\partial_x U\,, \tag{4.26}$$

$$\tilde{u}|_{t=0} = u_0 - U|_{t=0} \equiv \tilde{u}_0\,, \tag{4.27}$$

$$\tilde{u}|_{Y=0} = -U\,, \tag{4.28}$$

$$\tilde{u} \to 0 \quad \text{when} \quad Y \to \infty\,. \tag{4.29}$$

If $\rho > 0$ and $0 < \theta < \pi/4$, define

$$D(\rho) \equiv \{x \in \mathbb{C} \mid \quad |\Im x| \le \rho\}\,,$$

$$\Sigma(\theta) \equiv \{y \in \mathbb{C} \mid \quad |\Im y| \le \tan\theta\,\Re y,\ \Re y > 0\}\,.$$

Then define the following function spaces:

$$\begin{aligned} K'^{l,\rho}_{\beta,T} \;&\equiv\; \{f : D(\rho)\times[0,T] \to \mathbb{R} \mid \partial_x^i f,\ \partial_t\partial_x^j f \quad \text{are analytic w.r.t. x in } D(\rho-\beta t) \\ &\qquad \text{and bounded w.r.t. } t, \text{ where } i \le l,\ j \le l-1\}, \\ K^{l,\rho,\theta,\mu} &\equiv \{f : D(\rho)\times\Sigma(\theta) \to \mathbb{R} \mid \mathrm{e}^{\mu Y}\,\partial_x^i\,\partial_Y^j f \quad \text{is analytic w.r.t. } x \text{ and } Y,\ L^2 \text{ w.r.t. } x, \\ &\qquad \text{bounded w.r.t. } Y, \text{ where } i+j \le l \text{ and } j \le 2\}, \\ K^{l,\rho,\theta,\mu}_{\beta,T} &\equiv \{f \mid f \in K^{l,\rho-\beta t,\theta-\beta t,\mu-\beta t}, \qquad \partial_t\,\partial_x^i f \in K^{0,\rho-\beta t,\theta-\beta t,\mu-\beta t}, \quad \text{where} \quad i \le l-2\}\,. \end{aligned}$$

4.2.1 The hypotheses

In the rest of this Section we suppose that the following hypotheses hold true:

$$\tilde{u}_0 \in K^{l,\rho,\theta,\mu}\,, \tag{H.1}$$

$$\partial_t^i\,\partial_x^j U \quad \text{analytic w.r.t. } x \text{ in } D(\rho), \text{ and bounded w.r.t. } t, \text{ with } i+j \le l\,, \tag{H.2}$$

$$\tilde{u}_0|_{Y=0} = -U\,. \tag{H.3}$$

4.2.2 The main result

Theorem 4.4: *Suppose the hypotheses* (H.1)–(H.3) *are verified. Then there exist* $T > 0$ *and* $\beta > 0$, *depending on the data* U *and* $\tilde{u}_0$, *such that the Prandtl equations admit a unique solution of the form* $u = \tilde{u} + U$ *with* $\tilde{u} \in K^{l,\rho,\theta,\mu}_{\beta,T}$.

The rest of this Section is devoted to a brief account of the proof of the above result. Its proof is based on the abstract Cauchy-Kowalewski Theorem, which is stated in the next paragraph.

4.2.3 Sketch of the proof: The ACK Theorem

Suppose that in a scale of Banach spaces $\{X_\rho\}_{0<\rho\le\rho_0}$ the following equation is given:

$$\begin{aligned}&\partial_t u + F(u,t) = 0\,,\\ &u|_{t=0} = u_0\,.\end{aligned} \tag{ACK.1}$$

Assume that for some $\beta_0 > 0$ and $R > 0$, the operator F satisfies the following three conditions $\forall 0 < \rho' < \rho < \rho_0$:

$$F(u,t) : X_\rho \times [0, \rho_0/\beta_0) \to X_{\rho'} \quad \text{is continuous}\,, \tag{ACK.2}$$

$$\|F(u,t) - F(v,t)\|_{\rho'} \le C\,\frac{\|u-v\|_\rho}{\rho-\rho'} \;\;\forall u,v \quad \text{with} \quad \|u\|_\rho \le R,\; \|v\|_\rho \le R\,, \tag{ACK.3}$$

where C does not depend on u, v, ρ, ρ', t.

$$F(0,t) : [0,\rho_0/\beta_0) \to X_\rho \quad \text{is continuous with} \quad \|F(0,t)\|_\rho \le K\,, \tag{ACK.4}$$

with K fixed constant.

Theorem 4.5: (**ACK**) *Suppose the assumptions* (ACK.2)$-$(ACK.4) *are satisfied. Then there exists $\beta > 0$ such that the problem* (ACK.1) *admits a unique solution $u(t) : [0, (\rho_0-\rho)/\beta) \to X_\rho$ continuous and differentiable $\forall \rho < \rho_0$.*

4.2.4 Sketch of the proof: The heat operator

Consider the following heat equation in the half space with source, boundary data, and initial data:

$$(\partial_t - \partial_{YY})\, u = f \quad \text{for} \quad Y > 0\,, \tag{4.30}$$

$$u|_{Y=0} = g\,, \tag{4.31}$$

$$u|_{t=0} = u_0\,. \tag{4.32}$$

Denote the operator E which solves the above problem, as

$$u = E(f, g, u_0)\,. \tag{4.33}$$

It is possible to give an explicit representation of E in terms of the heat kernel (see [14]). Here we give the following estimate on E in an analytic function space.

Proposition 4.6: *Suppose $f \in K^{l,\rho,\theta,\mu}_{\beta,T}$ with $f|_{Y=0} = 0$, $g \in K'^{l,\rho}_{\beta,T}$ and $u_0 \in K^{l,\rho,\theta,\mu}$ with $g|_{t=0} = u_0|_{Y=0}$. Then $E(f,g,u_0) \in K^{l,\rho,\theta,\mu}_{\beta,T}$ and the following estimate holds:*

$$\|E(f,g,u_0)\|_{l,\rho,\theta,\mu,\beta,T} \le c\Big(\|f\|_{l,\rho,\theta,\mu,\beta,T} + \|g\|_{l,\rho,\beta,T} + \|u_0\|_{l,\rho,\theta,\mu}\Big)\,.$$

4.2.5 Sketch of the proof: The convection operator

Define the following operator:

$$K[u] \equiv U\,\partial_x u - u\,\partial_x U + u\,\partial_x u - \left(\int_0^Y \partial_x u\, dY' + Y\,\partial_x U\right)\partial_Y u\,. \tag{4.34}$$

The following Proposition is a consequence of the Cauchy estimate for analytic functions:

Proposition 4.7: *Suppose $u, v \in K^{l,\rho,\theta,\mu}$. Then $\forall 0 < \rho' < \rho$, $\forall 0 < \theta' < \theta$, and $\forall 0 < \mu' < \mu$,*

$$\|K[u]-K[v]\|_{l,\rho',\theta',\mu'} \le c\left[\frac{\|u-v\|_{l,\rho,\theta',\mu'}}{\rho-\rho'} + \frac{\|u-v\|_{l,\rho',\theta,\mu'}}{\theta-\theta'} + \frac{\|u-v\|_{l,\rho',\theta',\mu}}{\mu-\mu'}\right].$$

4.2.6 Sketch of the proof: Conclusion

With the help of the operators defined above, the Prandtl equations (4.25)$-$(4.29) can be put in the following form:

$$\tilde{u} + F(\tilde{u}, t) = 0\,, \tag{4.35}$$

in which

$$F(\tilde{u}, t) \equiv E(K[\tilde{u}],\, U, -\tilde{u}_0)\,. \tag{4.36}$$

Eq. (4.35) is suitable for the application of the ACK Theorem (in the integrated form).

Using the estimates on the operators E and K, given in Propositions 4.6 and 4.7 one can prove that the operator F, as defined by (4.36), satisfies the hypotheses of the ACK Theorem. The proof of Theorem 4.4 is thus achieved.

5. Validity of boundary layer theory: Infinite Reynolds number limit for the Navier-Stokes equations

This section describes the behavior of solutions of the Navier-Stokes equations in the limit of infinite Reynolds number, which we refer to as the zero viscosity limit.

The behavior of a dissipative fluid in the zero viscosity limit is one of the most difficult and interesting problems of the mathematical theory of fluid dynamics. It is noteworthy that the problem is not merely mathematical. In fact, since the experiments of Prandtl and Reynolds, it is well known that for increasing Reynolds number the fluid can experience dramatic changes in its behavior. Many possible mechanisms have been proposed for the explanation of the phenomenon of transition to turbulence. It is widely believed though that a comprehensive theory is still lacking. On the mathematical side a major shortcoming is the fact that no uniqueness theorem for the 3D Navier-Stokes (as well for Euler) equations is available. This problem is related to the fact that at this stage of the mathematical theory it is impossible to say if the solutions of the Navier-Stokes equations stay regular or develop a singularity. The estimated time for the existence of a regular solution is in fact dependent on the data and on the viscosity.

In this section we address the question of existence of a regular solution of the Navier-Stokes equations in the presence of boundaries for a time which is independent of the viscosity. In [18] Swann proved that a unique solution to the Navier-Stokes equation in $\mathbb{R}^3$ exists for a time that is small but independent of the viscosity. Moreover Swann proved that the Navier-Stokes solutions in $\mathbb{R}^3$ converge to Euler solutions. See also the related results of Kato in [7] and [8].

When boundaries are present the problem is harder because of the formation of the boundary layer which can lead to the formation of singularities, as discussed in Section 3. Significant levels of vorticity are ejected from within the boundary layer into the external inviscid flow.

Here we want discuss some situations for which it is possible to control the phenomenon of separation and establish both existence of the Navier-Stokes solution for a time independent of the viscosity and convergence, away from boundaries, of that solution to the Euler solution.

5.1 Kato's criterion

In [9] Kato gave a criterion to establish when, in the zero viscosity limit, a weak solution of the Navier-Stokes equations converges to a solution of the Euler equations.

Let $\boldsymbol{u}^{NS}$ and $\boldsymbol{u}^{E}$ denote the solutions of the Navier-Stokes and Euler equations in a bounded domain $\Omega \subset \mathbb{R}^m$. Suppose that

$$\boldsymbol{u}_0^{NS} \to \boldsymbol{u}_0^{E} \quad \text{when} \quad \nu \to 0\,, \tag{5.1}$$

$$\int_0^T \|\boldsymbol{f}^{NS} - \boldsymbol{f}^{E}\|\, dt \to 0 \quad \text{when} \quad \nu \to 0\,, \tag{5.2}$$

where $\boldsymbol{u}_0^{NS}$ and $\boldsymbol{f}^{NS}$ denote the initial condition and the forcing term of the Navier-Stokes equations; similarly for $\boldsymbol{u}^{E}$ and $\boldsymbol{f}^{E}$. The time T in (5.2) is the time up to which there exists a regular solution for the Euler equations.

The main result of [9] is the following.

Theorem 5.1: *Suppose* (5.1) *and* (5.2) *are verified. Then the following conditions are equivalent:*

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^{E}\|_{L^2} \to 0 \quad \textit{uniformly in } t \in [0,T]\,, \tag{K.1}$$

$$\nu \int_0^T \|\nabla \boldsymbol{u}^{NS}\|_{\Gamma_\nu}^2\, dt \to 0\,. \tag{K.2}$$

In (K.2), $\|\cdot\|_{\Gamma_\nu}$ denotes the L^2-norm restricted to a strip of width $O(\nu)$ close to the boundary.

It is usually hard to verify that condition (K.2) holds. Therefore this criterion is not easily applicable to concrete situations. The merit of Theorem 5.1 is that it sheds some light on the situations in which convergence does not occur. In fact (K.2) is a condition on the energy dissipation in a strip whose size is smaller than the boundary layer. If this energy dissipation does not go to zero, this means the occurrence of some kind of singularity in the lower part of the boundary layer.

We now briefly give the main step of the proof of the Theorem. We shall focus on proving that (K.2) implies (K.1), the converse being a straightforward application of the energy estimate. In giving the sketch of the proof, for simplicity, we shall suppose that the forcing terms in Navier-Stokes and Euler equations are the same, i.e. $\boldsymbol{f}^{NS} = \boldsymbol{f}^{E}$.

5.1.1 Sketch of the proof

The main idea of the proof of the Theorem is to construct a smooth divergence free boundary layer corrector $\boldsymbol{v}$ so that $\boldsymbol{u}^E - \boldsymbol{v}$ satisfies the no slip boundary condition. Moreover, through the use of a cut-off function, Kato managed to construct the boundary layer corrector so that its support is contained in a strip of size $O(\nu)$ close to the boundary. This allows approximation (in the L^2 sense) of the Euler solution $\boldsymbol{u}^E$ by $\boldsymbol{u}^E - \boldsymbol{v}$. Therefore, with the use of basic energy estimates, one gets

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^2}^2 \le 2\int_0^T \big[-(\boldsymbol{u}^{NS} \otimes \boldsymbol{u}^{NS}, \nabla(\boldsymbol{u}^E - \boldsymbol{v})) + (\boldsymbol{u}^{NS}, \boldsymbol{u}^E \cdot \nabla \boldsymbol{u}^E) + \nu(\nabla \boldsymbol{u}^{NS}, \nabla(\boldsymbol{u}^E - \boldsymbol{v}))\big]\, dt + o(1)\,. \tag{5.3}$$

In the above inequality, $(\cdot,\cdot)$ denotes the scalar product in L^2. The symbol $o(1)$ denotes terms that goes to zero with ν. Some simple manipulations of (5.3) lead to the following:

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^2}^2 \le 2\int_0^T \big[-\big((\boldsymbol{u}^{NS} - \boldsymbol{u}^E) \otimes (\boldsymbol{u}^{NS} - \boldsymbol{u}^E), \nabla \boldsymbol{u}^E\big) + \big(\boldsymbol{u}^{NS} \otimes \boldsymbol{u}^{NS}, \nabla \boldsymbol{v}\big) + \nu(\nabla \boldsymbol{u}^{NS}, \nabla u^E - \boldsymbol{v}))\big]\, dt + o(1)\,. \tag{5.4}$$

Given that $\boldsymbol{u}^E$ is a regular solution of the Euler equations, one has the control on the L^∞ norm of $\nabla \boldsymbol{u}^E$. The first term of the r.h.s. of (5.4) is easily estimated. The third term is estimated using the fact that $\boldsymbol{v}$ is supported in a strip of width $O(\nu)$. To estimate the second term one has to use also the Hardy inequality. Therefore

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^2}^2 \le c\int_0^T \|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^2}^2\, dt + c'\int_0^T [\nu\|\nabla \boldsymbol{u}^{NS}\|_{\Gamma_\nu}^2 + \nu^{1/2}\|\nabla \boldsymbol{u}^{NS}\|_{\Gamma_\nu}]\, dt\,. \tag{5.5}$$

Combined with the Gronwall inequality, this shows that (K.2) implies (K.1).

5.1.2 Temam and Wang's criterion

In the same spirit of Kato, in [19] TEMAM and WANG proved that if the gradient of the pressure at the wall Γ of a channel, does not behave too badly (see condition (TW.1) below), than convergence of the Navier-Stokes solution to the Euler solution follows:

Theorem 5.2: Let $0 \le \delta < 1/2$. *Suppose that*

$$\text{either} \quad \nu^\delta \int_0^T \|p^{NS}\|_{H^{1/2}(\Gamma)}\, dt \le c, \quad \text{or} \quad \nu^{\delta+1/4}\int_0^T \|\nabla p^{NS}\|_{L^2(\Gamma)}\, dt \le c\,. \tag{TW.1}$$

Then

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^2} \le c\nu^{(1-2\delta)/5}\,. \tag{TW.2}$$

5.2 Geophysical fluids

In [10] MASMOUDI analyzed the case, relevant in geophysical applications, in which the viscosity in the vertical (in the sense of being perpendicular to the boundary) direction is much smaller than the viscosity in the horizontal directions. The viscous force $\boldsymbol{F}_{\text{visc}}$ considered in [10] is in fact of the form

$$\boldsymbol{F}_{\text{visc}} = \nu\, \partial_{zz} \boldsymbol{u}^{NS} + \eta\, \Delta_{\boldsymbol{x}} \boldsymbol{u}^{NS}\,, \tag{5.6}$$

where z is the vertical direction and $\boldsymbol{x}$ are the horizontal directions. The main hypothesis in [10] is that

$$\frac{\nu}{\eta} \to 0 \quad \text{when} \quad \eta \to 0\,. \tag{5.7}$$

Theorem 5.3: *Suppose* (5.1) *and* (5.2) *are verified, and that the viscous force in the Navier-Stokes equations has the form* (5.6) *where the viscosity coefficients satisfy* (5.7). *Then*

$$\|\boldsymbol{u}^{NS} - \boldsymbol{u}^E\|_{L^\infty([0,T], L^2)} \to 0 \quad \textit{when} \quad \eta \to 0\,.$$

The time T is again the time up to which one has the existence of a regular solution of the Euler equations.

5.2.1 Sketch of the proof

Masmoudi constructs a smooth divergence-free boundary layer corrector $\boldsymbol{u}^{BL}$ whose support is contained in a strip of width $O(\sqrt{\nu\eta})$. He then writes the Navier-Stokes solution as

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^E + u^{BL} + \boldsymbol{w}\,, \tag{5.8}$$

where $\boldsymbol{w}$ is an error to be estimated. Notice that the error satisfies the no-slip boundary condition. Given that the size of the boundary layer has been chosen to be very small, then the L^2 norm of $\boldsymbol{u}^{BL}$ goes to zero in the zero viscosity limit. Therefore the Theorem will be proved if one can prove that $\|\boldsymbol{w}\|_{L^2} \to 0$ when $\eta \to 0$.

Inserting the ansatz (5.8) into the Navier-Stokes equations, and using the Euler equations, one gets the equations for the error $\boldsymbol{w}$. The structure of this error equations is very similar to the error equations found by Sammartino and Caflisch in [15]. More difficult are the terms involving generation of vorticity at the boundary, i.e. terms involving the derivative with respect to the vertical direction of the boundary layer corrector $\boldsymbol{u}^{BL}$.

Masmoudi performs an energy estimate on the equations for the error. Here we just show how he deals with the most difficult term, i.e. $\int (w_z\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}}) \cdot \boldsymbol{w}_{\boldsymbol{x}}$, in which w_z and $\boldsymbol{w}_{\boldsymbol{x}}$ denote respectively the vertical and horizontal components of the error $\boldsymbol{w}$, and $\boldsymbol{u}^{BL}_{\boldsymbol{x}}$ denotes the horizontal components of $\boldsymbol{u}^{BL}$. His estimate is

$$\int (w_z\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}}) \cdot \boldsymbol{w}_{\boldsymbol{x}} \le \int \frac{w_z}{z} (z^2\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}}) \cdot \frac{\boldsymbol{w}_{\boldsymbol{x}}}{z} \le c \left\| \frac{w_z}{z} \right\|_{L^2} \|(z^2\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}})\|_{L^\infty} \left\| \frac{\boldsymbol{w}_{\boldsymbol{x}}}{z} \right\|_{L^2} . \tag{5.9}$$

Given that $\boldsymbol{u}^{BL}$ is different from zero only in the strip of width $\sqrt{\nu\eta}$ leads to

$$\|(z^2\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}})\|_{L^\infty} = O(\sqrt{\nu\eta}) .$$

Moreover, using the fact that $\boldsymbol{w}|_{z=0} = 0$ and the the Hardy inequality, one can write the estimates

$$\left\| \frac{w_z}{z} \right\|_{L^2} \le c\|\partial_z w_z\|_{L^2} , \qquad \left\| \frac{\boldsymbol{w}_{\boldsymbol{x}}}{z} \right\|_{L^2} \le c\|\partial_z \boldsymbol{w}_{\boldsymbol{x}}\|_{L^2} .$$

Given that $\boldsymbol{w}$ satisfies the incompressibility condition, one can substitute $\partial_z w_z$ with $\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{w}_{\boldsymbol{x}}$. Therefore one can continue the estimate (5.9) and get that

$$\int (w_z\, \partial_z \boldsymbol{u}^{BL}_{\boldsymbol{x}}) \cdot \boldsymbol{w}_{\boldsymbol{x}} \le c\eta \|\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{w}_{\boldsymbol{x}}\|^2_{L^2} + c'\nu \|\partial_z \boldsymbol{w}_{\boldsymbol{x}}\|^2_{L^2} ;$$

both terms in the right hand side can be finally dominated by the usual dissipative terms of the energy estimate.

Notice that in the above estimate we have not used the hypothesis (5.7). This hypothesis is necessary only to estimate the terms coming from the Laplacian acting on $\boldsymbol{u}^{BL}$.

5.3 The analyticity result

Both Kato and Masmoudi in their analyses used an *artificial* boundary layer corrector with no attempt of reproducing the behavior of the Navier-Stokes solution close to the boundary. In [15] the authors instead used the Prandtl solution $\boldsymbol{u}^P$ from [14] to analyze the zero viscosity limit of the Navier-Stokes equations in the half space. Supposing the data to be analytic, they made the ansatz ($\varepsilon = \sqrt{\nu}$)

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^E + \tilde{\boldsymbol{u}}^P + \varepsilon \boldsymbol{u}^E_1 + \varepsilon \boldsymbol{u}^P_1 + \varepsilon \boldsymbol{w} . \tag{5.10}$$

In (5.10) $\boldsymbol{u}^E$ and $\boldsymbol{u}^P = \tilde{\boldsymbol{u}}^P + \boldsymbol{u}^E|_{y=0}$ are the analytic solutions of the Euler and Prandtl equations from [14] (the existence of analytic solutions of the Euler equations with analytic data was known since the paper of Bardos and Benachour [2]). The terms $\boldsymbol{u}^E_1$ and $\boldsymbol{u}^P_1$ are the solutions of the first order Euler and Prandtl equations. These equations are linear and are easily shown to admit analytic solutions. The problem is therefore reduced to proving that the error $\boldsymbol{w}$ exists and stays bounded.

5.3.1 The main result

Here we give an informal statement of the main result of [15].

Theorem 5.4: (Informal Statement) *Suppose that $\boldsymbol{u}^E(x, y, t)$ and $\boldsymbol{u}^P(x, Y, t)$ are the solutions of the Euler and Prandtl equations, respectively, which are analytic in the spatial variables x, y, Y. Then for a short time T, independent of $\varepsilon = \sqrt{\nu}$, there is a solution $\boldsymbol{u}^{NS}(x, y, t)$ of the Navier-Stokes equations with*

$$\boldsymbol{u}^{NS} = \begin{cases} \boldsymbol{u}^E + O(\varepsilon) & \text{outside the boundary layer,} \\ \boldsymbol{u}^P + O(\varepsilon) & \text{inside the boundary layer .} \end{cases} \tag{5.11}$$

In the above statement Y denotes the rescaled normal variable as in subsection 2.2. For simplicity the above Theorem is stated in the half space with x denoting the tangential variable.

The proof of the above Theorem is achieved if one proves that $\boldsymbol{u}^{NS}$ admits the representation (5.10) where $\boldsymbol{w}$ stays bounded for a time independent of the viscosity.

5.3.2 Sketch of the proof: The error equation

Denote

$$\boldsymbol{u}^{APP} \equiv \boldsymbol{u}^E + \tilde{\boldsymbol{u}}^P + \varepsilon \boldsymbol{u}^E_1 + \varepsilon \boldsymbol{u}^P_1 ,$$

and insert the representation (5.10) into the Navier-Stokes equations, to get the following equation for $\boldsymbol{w}$:

$$(\partial_t - \varepsilon^2 \Delta)\, \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{APP} + \boldsymbol{u}^{APP} \cdot \boldsymbol{\nabla} \boldsymbol{w} + \varepsilon \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{w} + \boldsymbol{\nabla} p^w = \boldsymbol{F} , \tag{5.12}$$

where $\boldsymbol{F}$ is a bounded forcing term. Eq. (5.12) must be supplemented with the appropriate boundary and initial conditions. Notice that $\boldsymbol{w}$ satisfies the incompressibility condition.

The difficulty in solving (5.12) is mostly the term $w_y\,\partial_y u^P$, since $\partial_y u^P = O(\varepsilon^{-1})$ inside the boundary layer. The analyticity hypothesis and the ACK Theorem allow use of the boundary condition $w_y|_{y=0} = 0$ so that $w_y = O(\varepsilon)$ inside the boundary layer.

5.3.3 Sketch of the proof: The Navier-Stokes operator

The idea to solve the above equation is to recast it in a form suitable for application of the ACK Theorem. This is accomplished in three steps:

1. invert the heat operator $(\partial_t - \varepsilon^2\Delta)$ through the inverse heat operator $\mathcal{E}$;
2. take into account the incompressibility conditions through the Leray's projection operator P;
3. take into account the no-slip boundary conditions through the Stokes operator $\mathcal{S}$, which solves Stokes equations with boundary conditions.

Combine these three operators to define the following operator $\mathcal{N}$, which we call the Navier-Stokes operator:

$$\mathcal{N} = P\mathcal{E} - \mathcal{S}\gamma P\mathcal{E}\,, \tag{5.13}$$

where γ is the trace operator at the boundary $y = 0$. The above operator solves the Stokes equations with prescribed source term and with zero initial and boundary data. Also define the convective operator

$$\mathcal{K}[\boldsymbol{w}] \equiv \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{APP} + \boldsymbol{u}^{APP}\cdot\boldsymbol{\nabla}\boldsymbol{w} + \varepsilon\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{w} - \boldsymbol{F}\,,$$

in terms of which the equation for the error becomes

$$\boldsymbol{w} + \mathcal{N}\mathcal{K}[\boldsymbol{w}] = 0\,. \tag{5.14}$$

Eq. (5.14) is of the form (ACK.1) (integrated in time). The rest of this section is devoted to describing how to prove that the operator $\mathcal{N}\mathcal{K}$ satisfies the hypotheses of the ACK Theorem.

5.3.4 Sketch of the proof: Functional setting

The functional setting for the error equation differs from the functional setting for the Prandtl equations in that it involves functions that are L^2 with respect to the normal variable. This is necessary because the projection operator is more naturally estimated in the space of L^2 functions. First define the domain of analyticity in the normal variable:

$$\begin{aligned}
&\boldsymbol{\Sigma}(\theta,a) \equiv \{y\in\mathbb{C} \mid |\Im y| \le \tan\theta\,\Re y \ \text{ for } \Re y \le a, |\Im y| \le \tan\theta a \ \text{ for } \Re y \ge a\},\\
&H^{l,\rho,\theta} \equiv \{f : D(\rho)\times\boldsymbol{\Sigma}(\theta,a/\varepsilon) \to \mathbb{R} \mid \partial_x^i\,\partial_Y^j\, f \text{ is analytic w.r.t. } x \text{ and } Y, L^2 \text{ w.r.t. } x\,,\\
&\qquad\qquad L^2 \text{ w.r.t. } Y, \text{ where } i+j\le l \text{ and } j\le 2\}\,,\\
&H^{l,\rho,\theta}_{\beta,T} \equiv \{f \mid f\in H^{l,\rho-\beta t,\theta-\beta t,\mu-\beta t},\ \partial_t\partial_x^i f \in H^{0,\rho-\beta t,\theta-\beta t,\mu-\beta t}, \text{ where } i \le l-2\}.
\end{aligned}$$

5.3.5 Sketch of the proof: The estimates

The operators $\mathcal{N}$ and $\mathcal{K}$ satisfy the following estimates:

Proposition 5.5: *Suppose $\boldsymbol{w}\in H^{l,\rho,\theta}_{\beta T}$. Then $\mathcal{N}\boldsymbol{w}\in \mathcal{H}^{l,\rho,\theta}_{\beta T}$ satisfies following estimate:*

$$|\mathcal{N}\boldsymbol{w}|_{l,\rho,\theta,\beta,T} \le c|\boldsymbol{w}|_{l,\rho,\theta,\beta,T}\,.$$

Proposition 5.6: *Suppose $\boldsymbol{w},\boldsymbol{v}\in H^{l,\rho,\theta}$ with $w_y|_{y=0} = 0$. Then $\forall 0<\rho'<\rho,\ 0<\theta'<\theta$ the following estimate holds:*

$$|\mathcal{K}[\boldsymbol{w}] - \mathcal{K}[\boldsymbol{v}]|_{l,\rho',\theta'} \le c\left[\frac{|\boldsymbol{w}-\boldsymbol{v}|_{l,\rho,\theta'}}{\rho-\rho'} + \frac{|\boldsymbol{w}-\boldsymbol{v}|_{l,\rho',\theta}}{\theta-\theta'}\right].$$

The estimate given in Proposition 5.3.5 is a consequence of the boundedness of the operators P, $\mathcal{E}$, and $\mathcal{S}$ in the space $H^{l,\rho,\theta}_{\beta,T}$.

The estimate given in Proposition 5.3.5 is crucial in the analysis and is a consequence of the Cauchy estimate for an analytic function. In fact by analyticity one can control the generation of vorticity in the boundary layer.

5.3.6 Sketch of the proof: Conclusion

The above Propositions show that the operator $\mathcal{N}\mathcal{K}$ satisfies the hypotheses of the ACK Theorem. It follows that (5.14) has a bounded solution w, and that the solution of the Navier-Stokes equation can be represented, for a time independent of the viscosity, in the form (5.10). The proof of the Theorem 5.3.1 is thus achieved.

6. Summary

Prandtl boundary layer theory is the canonical example of singular perturbation theory for PDEs. The results described above show that there has been a recent renewal of mathematical interest in Prandtl's equations, but that many open problems still remain. These include the following:

- Are the Prandtl equations well posed in a Sobolev norm?
- How well can the Prandtl singularities be characterized?
- Can the result on analytic solutions be extended to curved boundaries and to vortex sheets?
- Can these Prandtl results be extended to other singular perturbation problems?

These and other open problems should keep the Prandtl equations a subject of active research for some time.

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Addresses: Prof. Russel E. Caflisch, Department of Mathematics, University of California, Los Angeles, 405 Hilgard Avenue, Los Angeles, CA, USA, email: caflisch@math.ucla.edu; Prof. Marco Sammartino, Dipartimento di Matematica, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy, email: marco@dipmat.math.unipa.it