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Zero Viscosity Limit for Analytic Solutions of the Navier-Stokes Equation on a Half-Space. II. Construction of the Navier-Stokes Solution

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Abstract: This is the second of two papers on the zero-viscosity limit for the incompressible Navier-Stokes equations in a half-space in either 2D or 3D. Under the assumption of analytic initial data, we construct solutions of Navier-Stokes for a short time which is independent of the viscosity. The Navier-Stokes solution is constructed through a composite asymptotic expansion involving the solutions of the Euler and Prandtl equations, which were constructed in the first paper, plus an error term. This shows that the Navier-Stokes solution goes to an Euler solution outside a boundary layer and to a solution of the Prandtl equations within the boundary layer. The error term is written as a sum of first order Euler and Prandtl corrections plus a further error term. The equation for the error term is weakly nonlinear; its linear part is the time dependent Stokes equation. This error equation is solved by inversion of the Stokes equation, through expressing the solution as a regular (Euler-like) part plus a boundary layer (Prandtl-like) part. The main technical tool in this analysis is the Abstract Cauchy-Kowalewski Theorem.

1. Introduction

This is the second of two papers on the zero viscosity limit of the incompressible Navier-Stokes equations in a half-space with analytic initial data, and in either two or three spatial dimensions. Under the analyticity restriction and for small viscosity, we prove that the Navier-Stokes equations have a solution for a short time (independent of the viscosity). In the zero-viscosity limit, we show that this Navier-Stokes solution goes to an Euler solution outside a boundary layer and to a solution of the Prandtl equations within the boundary layer. As argued in the Introduction of Part I [6], we believe that the imposition of analyticity is needed to make this problem well-posed, by preventing boundary layer separation, but there is no proof of this.

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In the first paper [6], we proved short time existence of solutions for the Euler equations and the Prandtl equations with analytic initial data. In this second paper, we construct the Navier-Stokes solution as a sum of the Euler solution, the Prandtl solution and an error term. Existence and bounds of size ε (the square root of the viscosity) for the error term are the main results of this paper. The error equation is weakly nonlinear, since its solution is small. Its linear part is exactly the time-dependent Stokes equation, with forcing terms and with boundary and initial data. As for the solution of the Euler equations in [6], the incompressibility of the solution is ensured by use of the projection method in order to avoid dealing directly with the pressure.

The main technical tool here is the Abstract Cauchy-Kowalewski (ACK) Theorem, which is invoked to establish existence for the error equation. As discussed in the Introduction to Part I, the abstract version of this theorem applies to dissipative equations, even though the classical version does not.

A discussion of related references from the literature is presented in the Introduction to Part I.

In Sect. 2 we state the Navier-Stokes equations and discuss how the Euler equations and Prandtl equations, in the limit of small viscosity, can be formally derived from Navier-Stokes through different scalings and asymptotic expansions. The introduction of two different scalings, typical in singular perturbation theory, is formally necessary to describe two different regimes of the flow: the inviscid regime (far away from the boundary) and the viscous regime (close to the boundary) where the viscous forces cannot be neglected even for small viscosity. The meaning of Theorem 1, which is the main result of this paper, is to rigorously establish this formal result; i.e. to show that the Euler and Prandtl equations are each a good approximation of the Navier-Stokes equations in their respective domains of validity. In particular, the solution of the full Navier-Stokes equations is divided into Euler, Prandtl and error terms, and the error term is further divided into first order Euler, first order Prandtl and a higher order correction.

Section 3 contains an analysis of the time-dependent Stokes equations with prescribed boundary data. For this linear problem, which we shall solve explicitly, we also show that the solution is the superposition of an inviscid part, a boundary layer part, and a small correction. Section 4 contains the decomposition of the error equation Eqs. (4.1)–(4.4) into first order Euler and Prandtl equations, which are solved in Sections 5 and 6. The analysis of the equations for the remaining error takes all of Sect. 7. These "Navier-Stokes error equations" contain terms of size $O(\varepsilon^{-1})$ due to the generation of vorticity at the boundary. They are solved using what we call the " Navier-Stokes operator," which solves Stokes equations with a forcing term (see Eqs. (7.22)–(7.25)). It is suitable for solving the error equation (and thus the original Navier-Stokes equations) with an iterative procedure. With the bounds on this operator, and with the use of the abstract version of the Cauchy-Kowalewski Theorem, we can prove existence, uniqueness and boundedness (in a suitable norm) for the error.

Final conclusions are stated in Sect. 8. The function spaces that are used in this paper are all defined in Part I. For convenience, tables of function spaces and operators are presented there. As in Part I, the exposition is presented for the two-dimensional problem, but the results are all expressed for 3D as well as 2D.

2. Navier-Stokes Equations

2.1. A singular perturbation problem. The Navier-Stokes equations on the half plane for a velocity field $\boldsymbol{u}^{NS} = (u^{NS}, v^{NS})$ are

$$(\partial_t - \nu\Delta)\,\boldsymbol{u}^{NS} + \boldsymbol{u}^{NS}\cdot\nabla\boldsymbol{u}^{NS} + \nabla p^{NS} = 0, \tag{2.1}$$
$$\nabla\cdot\boldsymbol{u}^{NS} = 0, \tag{2.2}$$
$$\gamma\boldsymbol{u}^{NS} = 0, \tag{2.3}$$
$$\boldsymbol{u}^{NS}(t=0) = \boldsymbol{u}_0^{NS}. \tag{2.4}$$

Here, $\nu = \varepsilon^2$ is the viscosity coefficient, and γ is the trace operator, i.e. $\gamma f(x,t) = f(x, y = 0, t)$. The initial velocity $\boldsymbol{u}_0^{NS}(x,y)$ must satisfy the incompressibility condition and the compatibility condition with the BC Eq. (2.3):

$$\nabla\cdot\boldsymbol{u}_0^{NS} = 0, \tag{2.5}$$
$$\gamma\boldsymbol{u}_0^{NS} = 0. \tag{2.6}$$

In this paper we are interested in the behavior of the solution of N-S equations in the limit of small viscosity $\nu << 1$. As usual in perturbation theory, it is natural to write the solution as an asymptotic series of the form

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^0 + \varepsilon\boldsymbol{u}^1 + \varepsilon^2\boldsymbol{u}^2 + \ldots, \tag{2.7}$$

where all the terms $\boldsymbol{u}^i$ satisfy equations that are independent of ε (the reason for expanding in $\varepsilon = \sqrt{\nu}$ comes from the boundary layer expansion, which is described below). The equation for the leading order term $\boldsymbol{u}^0$ comes from just neglecting the viscous term in the Navier-Stokes equations, which yields the Euler equations

$$\partial_t\boldsymbol{u}^E + \boldsymbol{u}^E\cdot\nabla\boldsymbol{u}^E + \nabla p^E = 0, \tag{2.8}$$
$$\nabla\cdot\boldsymbol{u}^E = 0, \tag{2.9}$$
$$\gamma_n\boldsymbol{u}^E = v^E(x, y = 0, t) = 0, \tag{2.10}$$
$$\boldsymbol{u}^E(x, y, t = 0) = \boldsymbol{u}_0^E(x,y). \tag{2.11}$$

This procedure works well, at least for short times far away from the boundary, but gives unsatisfactory answers close to the boundary. Comparison of the boundary conditions Eqs. (2.3) and (2.10) for the Navier-Stokes and Euler equations, respectively, shows the cause of the failure. For Euler equations we can only impose zero normal velocity, since the equations are first order; while for Navier-Stokes the no-slip condition requires both normal and tangential velocities to vanish. We must therefore allow a region in the vicinity of the boundary where viscous forces are comparable to inertial forces, and where there is an adjustment of the tangential velocity from zero at the boundary to the value predicted by the Euler equations. This boundary layer should have size $\varepsilon = \sqrt{\nu}$, so that the viscous term νu_{yy} is of size $O(1)$.

Thus it is natural to write all quantities in terms of a rescaled normal variable $Y = y/\varepsilon$. Next, the incompressibility condition requires that $v_y = \varepsilon^{-1}v_Y = O(1)$, which requires the vertical velocity v to be size $O(\varepsilon)$. Imposing this scaling in the Navier-Stokes equations, and again neglecting terms which are first order in ε, one gets Prandtl's equations for the fluid velocity $\boldsymbol{u}^P(x, Y, t) = (u^P, \varepsilon v^P)$ in the vicinity of the boundary; i.e.

$$(\partial_t - \partial_{YY}) u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p^P = 0, \tag{2.12}$$
$$\partial_Y p^P = 0, \tag{2.13}$$
$$\partial_x u^P + \partial_Y v^P = 0, \tag{2.14}$$
$$\gamma u^P = \gamma v^P = 0, \tag{2.15}$$
$$u^P (x, Y \to \infty) \longrightarrow \gamma u^E, \tag{2.16}$$
$$u^P (x, Y, t = 0) = u_0^P (x, Y). \tag{2.17}$$

Equation (2.16) is the matching condition between the inner (viscous) flow and the outer (inviscid) flow. This condition is equivalent to the existence of an intermediate region (e.g. a region where $y = O(\varepsilon^\alpha)$ with $0 < \alpha < 1$), where there is a smooth transition between the viscous and inviscid regimes.

As already noticed (see Subsect. 5.2 of [6]), it is natural to introduce the new variable $\tilde{\boldsymbol{u}}^P = (\tilde{u}^P, \varepsilon\tilde{v}^P)$ defined as

$$\tilde{u}^P = u^P - \gamma u^E, \tag{2.18}$$
$$\tilde{v}^P = v^P + Y \partial_x \gamma u^E = -\int_0^Y dY' \partial_x \tilde{u}^P, \tag{2.19}$$

and write Prandtl equations in terms of $\tilde{u}^P$ as

$$\begin{aligned}(\partial_t - \partial_{YY}) \tilde{u}^P + \tilde{u}^P \partial_x \gamma u^E + \gamma u^E \partial_x \tilde{u}^P + \tilde{u}^P \partial_x \tilde{u}^P \\ + \left[\tilde{v}^P - Y \partial_x \gamma u^E\right] \partial_Y \tilde{u}^P = 0,\end{aligned} \tag{2.20}$$
$$\gamma \tilde{u}^P = -\gamma u^E, \tag{2.21}$$
$$\tilde{u}^P (x, Y \to \infty) \longrightarrow 0, \tag{2.22}$$
$$\tilde{u}^P (x, Y, t = 0) = u_0^P (x, Y) - \gamma u_0^E = \tilde{u}_0^P. \tag{2.23}$$

We also define the normal velocity $\overline{v}^P$ to be the velocity $\tilde{v}^P$ minus its value at infinity; i.e.

$$\overline{v}^P(Y) = \tilde{v}^P(Y) - \tilde{v}^P(Y = \infty) = \int_Y^\infty dY' \partial_x \tilde{u}^P. \tag{2.24}$$

In [6] we have proved that, under suitable hypotheses on the initial conditions, i.e. analyticity, incompressibility and compatibility with boundary conditions, both the Euler and Prandtl equations admit a unique solution in the appropriate space of analytic functions (see Theorems 4.1 and 5.1 in [6]). To be more specific, we found the existence and the uniqueness of an analytic solution for Euler equations which is L^2 in both the x and y variable. For Prandtl, on the other hand, we proved existence and uniqueness for a solution $\tilde{u}^P$ which is L^2 in the x variable, and exponentially decaying in the Y variable (i.e. outside the boundary layer); the normal component $\tilde{v}^P$ of the velocity is $O(\varepsilon)$, but not decaying in Y, and in fact goes to a constant outside the boundary layer.

At this point, a natural question is whether one can use the solutions of the Euler and Prandtl equations to build a zero$^{\text{th}}$ order approximation to the solution of Navier-Stokes equations. The following theorem, which is the main result of this paper, gives a positive answer to this question:

Theorem 1 (Informal Statement). *Suppose that $\boldsymbol{u}^E(x, y, t)$ and $\boldsymbol{u}^P(x, Y, t)$ are solutions of the Euler and Prandtl equations, respectively, which are analytic in the spatial variables x, y, Y. Then for a short time T, independent of ε, there is a solution $\boldsymbol{u}^{NS}(x, y, t)$ of the Navier-Stokes equations with*

$$\boldsymbol{u}^{NS} = \begin{cases} \boldsymbol{u}^{E} + O(\varepsilon) \textit{ outside boundary layer} \\ \boldsymbol{u}^{P} + O(\varepsilon) \textit{ inside boundary layer.} \end{cases} \tag{2.25}$$

A formal version of this result, with a complete specification of the possible initial data for the Navier-Stokes solution is given in the following theorem:

Theorem 1. *Suppose the initial condition for the Navier-Stokes equations is given in the following form*

$$\boldsymbol{u}_0^{NS} = \boldsymbol{u}_0^{E}(x,y) + \overline{\boldsymbol{u}}_0^{P}(x,Y) + \varepsilon\left[\boldsymbol{\omega}_0(x,y) + \boldsymbol{\Omega}_0(x,Y) + \boldsymbol{e}_0(x,Y)\right], \tag{2.26}$$

where

(i) $\boldsymbol{u}_0^{E} = (u_0^E, v_0^E) \in H^{l,\rho,\theta}$ *and*

$$\nabla \cdot \boldsymbol{u}_0^{E} = 0\,, \quad \gamma_n \boldsymbol{u}_0^{E} = 0,$$

(ii) $\overline{\boldsymbol{u}}_0^{P} = (\tilde{u}_0^P, \varepsilon \overline{v}_0^P) \in K^{l,\rho,\theta,\mu}$ *and*

$$\overline{v}_0^P = \int_Y^\infty dY' \partial_x \tilde{u}_0^P\,, \quad \gamma \tilde{u}_0^P = -\gamma u_0^E,$$

(iii) $\boldsymbol{\omega}_0 = (\omega_0^1, \omega_0^2) \in N^{l,\rho,\theta}$*,*

$$\nabla \cdot \boldsymbol{\omega}_0 = 0\,, \quad \gamma \omega_0^2 = -\gamma \overline{v}_0^P,$$

(iv) $\boldsymbol{\Omega} = (\Omega_0^1, \varepsilon\Omega_0^2) \in K^{l,\rho,\theta,\mu}$ *and*

$$\Omega_0^2 = \int_Y^\infty dY' \partial_x \Omega_0^1\,, \quad \gamma \Omega_0^1 = -\gamma \omega_0^1,$$

(v) $\boldsymbol{e}_0 = (e_0^1, e_0^2) \in L^{l,\rho,\theta}$ *and*

$$\nabla \cdot \boldsymbol{e}_0 = 0\,, \quad \gamma \boldsymbol{e}_0 = (0, -\gamma\Omega_0^2),$$

with $l \geq 6$*. Then there exist* $\overline{\rho} < \rho$*,* $\overline{\theta} < \theta$*,* $\overline{\mu} < \mu$*,* $\overline{\beta} > 0$*, and* $T > 0$*, all independent of* ε*, such that the solution of the Navier-Stokes equations can be written in the form*

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^{E}(x,y,t) + \overline{\boldsymbol{u}}^{P}(x,Y,t) + \varepsilon\left[\boldsymbol{\omega}(x,y,t) + \boldsymbol{\Omega}(x,Y,t) + \boldsymbol{e}(x,Y,t)\right] \tag{2.27}$$

in which

(i) $\boldsymbol{u}^{E} \in H^{l,\overline{\rho},\overline{\theta}}_{\overline{\beta},T}$ *is the solution of the Euler equations (2.1)–(2.4),*

(ii) $\overline{\boldsymbol{u}}^{P} = (\tilde{u}^P, \varepsilon\overline{v}^P) \in K^{l,\overline{\rho},\overline{\theta},\overline{\mu}}_{\overline{\beta},T}$ *is the modified Prandtl solution as defined in (2.18) and (2.24), exponentially decaying outside the boundary layer,*

(iii) $\boldsymbol{\omega} \in N^{l,\overline{\rho},\overline{\theta}}_{\overline{\beta},T}$ *is the first order correction to the inviscid flow; it solves Eqs. (4.7)–(4.10) below,*

(iv) $\boldsymbol{\Omega} \in K^{l,\overline{\rho},\overline{\theta},\overline{\mu}}_{\overline{\beta},T}$ *is the first order correction to the boundary layer flow; it solves Eqs. (4.11)–(4.14) below,*

(v) $\boldsymbol{e} \in L^{l,\overline{\rho},\overline{\theta}}_{\overline{\beta},T}$ *is an overall correction; it solves Eqs. (4.15)–(4.18) below.*

The norms of $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ *and* $\boldsymbol{e}$ *in the above spaces are bounded by a constant that does not depend on the viscosity.*

2.2. Discussion of the Theorem. Since $\overline{\boldsymbol{u}}^P$ is exponentially decaying for large $Y = y/\varepsilon$, then the expression (2.27) shows that $\boldsymbol{u}^{NS} = \boldsymbol{u}^E + O(\varepsilon)$ for y outside of the boundary layer (i.e. $y >> \varepsilon$). For y inside the boundary layer (i.e. $y \leq \varepsilon$), $\boldsymbol{u}^E = (\gamma u^E, 0) + O(\varepsilon)$, so that $\boldsymbol{u}^{NS} = \boldsymbol{u}^P + O(\varepsilon)$. This shows that the informal statement of the theorem follows from the rigorous statement.

In this theorem the Navier-Stokes solution is represented in terms of a composite expansion of the form (2.27), which includes a regular (Euler) term $\boldsymbol{u}^E$, a boundary layer term $\overline{\boldsymbol{u}}^P$ and a correction term. Since the Euler solution has non-zero boundary values, the Prandtl solution must be modified so that the sum of the two is zero at the boundary and approaches the Euler solution at the outer edge of the boundary layer. The theorem says that if the initial condition is a function L^2 in transversal and normal component (together with its derivatives up to order l), then the solution of the Navier-Stokes equations will have the composite expansion form given in Eq. (2.27), at least for a short time.

There are several other ways to represent the Navier-Stokes solution for small viscosity. The most common method in perturbation theory [3] is to write the solution as a matched asymptotic expansion in which

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^P + O(\varepsilon) \qquad \text{for } y \text{ small enough,} \tag{2.28}$$

$$\boldsymbol{u}^{NS} = \boldsymbol{u}^E + O(\varepsilon) \qquad \text{for } y \text{ not too small.} \tag{2.29}$$

The formal validity of this representation is usually demonstrated by showing that the $O(\varepsilon)$ terms are small, and that there is a region of overlap for the validity of the two expansions. While this representation is more easily understood than the composite expansion, it is much more difficult to rigorously analyze due to the two spatial regimes.

A second method for representing the solution, which has been used for example in [4, 8], is to introduce a cut off function $m = m(y/\varepsilon^\alpha)$ with $m(0) = 1$, $m(\infty) = 0$, and $0 < \alpha < 1$. The solution is then written as

$$\boldsymbol{u}^{NS} = m\boldsymbol{u}^P + (1 - m)\boldsymbol{u}^E + O(\varepsilon^\alpha). \tag{2.30}$$

This method has two difficulties: It introduces an artificial length scale ε^α which makes the error terms artificially large. It also requires error terms in the incompressibility equation, since $m\boldsymbol{u}^P + (1 - m)\boldsymbol{u}^E$ is not divergence-free. For these reasons we have found the composite expansion method to be the most convenient for analysis.

The rest of this paper is devoted to proving Theorem 1. Unless otherwise stated, $l \geq 6$ throughout.

2.3. The error equation. If we pose

$$\begin{aligned} u^{NS} &= u^E + \tilde{u}^P + \varepsilon w^1, \\ v^{NS} &= v^E + \varepsilon \int_Y^\infty dY' \partial_x \tilde{u}^P + \varepsilon w^2 = v^E + \varepsilon \overline{v}^P + \varepsilon w^2, \\ p^{NS} &= p^E + \varepsilon p^w, \end{aligned} \tag{2.31}$$

and use these expressions in the N-S equations, we get the following equation for the error $\boldsymbol{w} = (w^1, w^2)$:

$$\left(\partial_t - \varepsilon^2\Delta\right)\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u}^0 + \boldsymbol{u}^0\cdot\nabla\boldsymbol{w} + \varepsilon\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \nabla p^w = \boldsymbol{f} + \left(g\cdot\partial_y\tilde{u}^P, 0\right), \tag{2.32}$$

$$\nabla\cdot\boldsymbol{w} = 0, \tag{2.33}$$

$$\gamma\boldsymbol{w} = (0, g), \tag{2.34}$$

$$\boldsymbol{w}(t=0) = \boldsymbol{\omega}_0 + \boldsymbol{\Omega}_0 + \boldsymbol{e}_0, \tag{2.35}$$

in which $\boldsymbol{u}^0 = (u^0, v^0)$ is defined by

$$\begin{aligned} u^0 &= u^E + \tilde{u}^P, \\ v^0 &= v^E + \varepsilon\overline{v}^P = v^E + \varepsilon\int_Y^\infty dY'\partial_x\tilde{u}^P. \end{aligned} \tag{2.36}$$

The forcing term is $\boldsymbol{f} = (f^1, f^2)$ given by

$$\begin{aligned} f^1 = &-\varepsilon^{-1}\left\{\tilde{u}^P\left(\partial_x u^E - \partial_x\gamma u^E\right) + (\partial_x\tilde{u}^P)\left(u^E - \gamma u^E\right) + (\partial_y\tilde{u}^P)\left(v^E + y\partial_x\gamma u^E\right)\right\} \\ &- \overline{v}^P\partial_y u^E + \varepsilon\Delta u^E + \varepsilon\partial_x^2\tilde{u}^P, \end{aligned} \tag{2.37}$$

$$f^2 = -\left[\partial_t\overline{v}^P + u^0\partial_x\overline{v}^P + v^0\partial_y\overline{v}^P + \overline{v}^P\partial_y v^E\right] - \varepsilon^{-1}\tilde{u}^P\partial_x v^E + \varepsilon\Delta v^0, \tag{2.38}$$

and also

$$g = \int_0^\infty dY'\partial_x\tilde{u}^P. \tag{2.39}$$

We want to show that the forcing term $\boldsymbol{f}$ is in $L^{l-2,\rho_1,\theta_1}_{\beta_1,T}$, and that in this space it has $O(1)$ norm, namely that

$$|\boldsymbol{f}|_{l-2,\rho_1,\theta_1,\beta_1,T} \le c\left(|\boldsymbol{u}_0^E|_{l,\rho,\theta} + |\tilde{u}_0^P|_{l,\rho,\theta,\mu} + 1\right)^2, \tag{2.40}$$

where the constant c does not depend on ε. Let us consider f^1. From Theorems 4.1 and 5.1 of Part I [6], it is clear that the terms $\varepsilon\Delta u^E$ and $\varepsilon\partial_x^2\tilde{u}^P$ satisfy the estimate (2.40). Each of the remaining terms in $\boldsymbol{f}$ has a similar form: They are each ε^{-1} times the product of a function which is exponentially decaying (with respect to $Y = y/\epsilon$) outside the boundary layer (terms containing $\tilde{u}^P$ and $\overline{v}^P$), and a function that is $O(\varepsilon)$ inside the boundary layer (e.g. $u^E - \gamma u^E$). It follows that they all satisfy (2.40). In an analogous way one can see that f^2 is $O(1)$ and satisfies the estimate (2.40).

Thus Eqs. (2.32)–(2.35) for the error term $\boldsymbol{w}(x, Y, t)$ have bounded forcing terms. In Sects. 4–7 we shall prove that this system admits a solution $\boldsymbol{w}$ which can be represented in the following form:

$$\boldsymbol{w} = \boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{e}, \tag{2.41}$$

where the norms (in the appropriate function spaces) of $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ and $\boldsymbol{e}$ remain bounded by a constant independent of ε. The difficulty of this proof is the presence in Eq. (2.32) of terms like $\partial_y\tilde{u}^P$, which are $O(\varepsilon^{-1})$ inside the boundary layer.

3. The Boundary Layer Analysis for Stokes Equations

Before addressing the problem of solving Eqs. (2.32)–(2.35), it is useful to consider a somewhat simpler problem, the Stokes equations with zero initial condition and boundary data $\boldsymbol{g}$. This problem is of intrinsic interest, and the results will be used in the analysis of the Navier-Stokes equations. The time-dependent Stokes equations are

$$(\partial_t - \nu\Delta)\,\boldsymbol{u}^S + \nabla p^S = 0, \tag{3.1}$$
$$\nabla\cdot\boldsymbol{u}^S = 0, \tag{3.2}$$
$$\gamma\boldsymbol{u}^S = \boldsymbol{g}(x,t), \tag{3.3}$$
$$\boldsymbol{u}^S(x,y,t=0) = 0. \tag{3.4}$$

Here $\boldsymbol{g}$ is a vectorial function $\boldsymbol{g} = (g', g_n)$. Primed quantities denote the tangential components of a vector, while the subscript n denotes the normal component. The compatibility condition $\boldsymbol{g}(x, t = 0) = 0$ is required for the boundary data.

In this section we shall show that the solution of the above problem has a structure similar to that for the Navier-Stokes solution Eq. (2.27); i.e. it is the superposition of an inviscid (Euler) part, a boundary layer (Prandtl) part which exponentially decays to zero outside a region of size $\varepsilon = \sqrt{\nu}$, and a correction term which is size $O(\varepsilon)$ everywhere. The Stokes problem Eqs. (3.1)–(3.4) has already been addressed by Ukai in [7], (where even the case of non-zero initial data was considered), without making the distinction between inviscid part, boundary layer part and correction term.

We seek a solution of the form

$$u^S = u^E + \tilde{u}^P + w^1,\quad v^S = v^E + \varepsilon\overline{v}^P + w^2,\quad p^S = p^E + p^w, \tag{3.5}$$

so that (u^E, v^E) represents an inviscid solution, $(\tilde{u}^P, \overline{v}^P)$ is a boundary layer solution decaying (in both components) outside the boundary layer, (w^1, w^2) is a small correction, and the pressures p^E and p^w are bounded at infinity. Please note that in this section u^E, $\tilde{u}^P$ and w refer to the "Euler", "Prandtl" and correction components of the Stokes solution; everywhere else in the paper, this notation is used for the usual Euler and Prandtl solutions and for the correction in the Navier-Stokes solution. These quantities solve the following equations:

$$\partial_t\boldsymbol{u}^E + \nabla p^E = 0, \tag{3.6}$$
$$\nabla\cdot\boldsymbol{u}^E = 0, \tag{3.7}$$
$$\gamma_n\boldsymbol{u}^E = g_n, \tag{3.8}$$
$$\boldsymbol{u}^E(x,y,t=0) = 0, \tag{3.9}$$

$$(\partial_t - \nu\Delta)\,\tilde{u}^P = 0, \tag{3.10}$$
$$\partial_x\tilde{u}^P + \partial_Y\overline{v}^P = 0, \tag{3.11}$$
$$\overline{v}^P \to 0 \text{ as} Y \to \infty,$$
$$\gamma\tilde{u}^P = g' - \gamma u^E, \tag{3.12}$$
$$\tilde{u}^P(x,y,t=0) = 0, \tag{3.13}$$

$$(\partial_t - \nu\Delta)\,\boldsymbol{w} + \nabla p_w = 0, \tag{3.14}$$
$$\nabla\cdot\boldsymbol{w} = 0, \tag{3.15}$$
$$\gamma\boldsymbol{w} = (0, -\varepsilon\gamma\overline{v}^P), \tag{3.16}$$
$$\boldsymbol{w}(x,y,t=0) = 0. \tag{3.17}$$

Note that Eq. (3.10)–Eq. (3.13) use the fast variable $Y = y/\epsilon$ with $\nu = \epsilon^2$, in terms of which $\Delta = \epsilon^2\partial_{xx} + \partial_{YY}$. Also, there is no term $\Delta\boldsymbol{u}^E$, since it is identically zero. We now solve explicitly these equations.

3.1. Convective equation. Take the divergence of (3.6) to obtain $\Delta p^E = 0$. Then apply Δ to (3.6) and use the initial condition $\boldsymbol{u}^{\,E} = 0$, to obtain

$$\Delta \boldsymbol{u}^{\,E} = 0. \tag{3.18}$$

Therefore the solution of Euler problem is

$$\boldsymbol{u}^{\,E} = \nabla N g_n, \tag{3.19}$$

where the operator $N = -1/|\xi'| \exp(-|\xi'|y)$ solves the Laplace equation with Neumann boundary condition; i.e.

$$\begin{aligned} \Delta N g_n &= 0, \\ \gamma \partial_y N g_n &= g_n. \end{aligned} \tag{3.20}$$

3.2. Boundary Layer Problem. To solve Eqs. (3.10)–(3.13) it is useful to introduce the operator $\tilde{E}_1$ acting on functions $f(x,t)$ defined on the boundary

$$\begin{aligned} \tilde{E}_1 f(x,Y,t) =& 2\int_0^t ds \frac{Y}{t-s} \frac{\exp\left[-Y^2/4(t-s)\right]}{(4\pi(t-s))^{1/2}} \\ & \int_{-\infty}^{\infty} dx' \frac{\exp\left[-(x-x')^2/4\varepsilon^2(t-s)\right]}{\left(4\pi\varepsilon^2(t-s)\right)^{1/2}} f(x',s). \end{aligned} \tag{3.21}$$

This operator solves the heat equation with boundary conditions f and zero initial conditions

$$\begin{aligned} \left(\partial_t - \varepsilon^2\partial_{xx} - \partial_{YY}\right)\tilde{E}_1 f &= 0, \\ \gamma \tilde{E}_1 f &= f, \\ \tilde{E}_1 f(x,Y,t=0) &= 0. \end{aligned} \tag{3.22}$$

Note that the operator $\tilde{E}_1$ differs from the operator E_1 (defined in Sect. 5.1 of Part I) by the fact that it involves an integration on the transversal component x also. Define

$$M\boldsymbol{g} = g' + N' g_n. \tag{3.23}$$

The solution of the boundary layer equations is written as

$$\tilde{u}^P = \tilde{E}_1 M \boldsymbol{g}\ . \tag{3.24}$$

Using the incompressibility condition and the limiting condition, the normal component is

$$\overline{v}^P = \int_Y^{\infty} dY' \partial_x \tilde{u}^P. \tag{3.25}$$

3.3. The Correction Term. Here we shall use the Fourier transform variable with respect to x. As in Part I, the corresponding transform variable is denoted ξ'. As in Subsect. 3.1, $\Delta p^w = 0$. Since p^w is bounded at ∞, then

$$\left(\partial_y + |\xi'|\right) p^w = 0. \tag{3.26}$$

Define $\tau = \left(\partial_y + |\xi'|\right) w^2$, so that Eqs. (3.14)–(3.16) imply

$$\left(\partial_t - \varepsilon^2\Delta\right)\tau = 0, \tag{3.27}$$

$$\begin{aligned}\gamma\tau &= \gamma\left(\partial_y + |\xi'|\right) w^2,\\ &= \gamma\left(-\nabla' w^1 + |\xi'| w^2\right),\\ &= |\xi'|\alpha,\end{aligned} \tag{3.28}$$

in which

$$\alpha = -\varepsilon\int_0^\infty dY'\partial_x \tilde{u}^P. \tag{3.29}$$

Since τ solves the heat equation with the above boundary condition, then

$$\tau = |\xi'|\tilde{E}_1\alpha. \tag{3.30}$$

From the definition of τ, w^2 satisfies

$$\partial_y w^2 + |\xi'|w^2 = |\xi'|\tilde{E}_1\alpha, \tag{3.31}$$

which leads to

$$w^2(x, Y, t) = e^{-|\xi'|y}\alpha + \overline{U}\tilde{E}_1\alpha \tag{3.32}$$

in which $\overline{U}$ is defined as

$$\overline{U}f(\xi', Y) = \varepsilon|\xi'|\int_0^Y e^{-\varepsilon|\xi'|(Y-Y')}f(\xi', Y')dY'. \tag{3.33}$$

Notice that a similar operator occurs in Eq.(4.12) in [6]. Finally, the incompressibility condition implies that

$$w^1 = -N'e^{-|\xi'|y}\alpha + N'(1-\overline{U})\tilde{E}_1\alpha. \tag{3.34}$$

These above results can be summarized as follows: The solution of the Stokes problem Eqs. (3.1)–(3.4) is denoted by $\mathcal{S}\boldsymbol{g}$, with

$$\begin{aligned}\boldsymbol{u}^S &= \mathcal{S}\boldsymbol{g} = \mathcal{S}^E\boldsymbol{g} + \mathcal{S}^P\boldsymbol{g} + \mathcal{S}^C\boldsymbol{g}\\ &= \begin{pmatrix} -N'Dg_n \\ Dg_n \end{pmatrix} + \begin{pmatrix} \tilde{E}_1 M\boldsymbol{g} \\ \varepsilon\int_Y^\infty dY'\partial_x\tilde{E}_1 M\boldsymbol{g} \end{pmatrix} + \begin{pmatrix} -N'e^{-|\xi'|y} + N'(1-\overline{U})\tilde{E}_1 \\ e^{-|\xi'|y} + \overline{U}\tilde{E}_1 \end{pmatrix}\alpha.\end{aligned} \tag{3.35}$$

After some manipulation, this can be simplified, as in [7], to

$$\boldsymbol{u}^S = \mathcal{S}\boldsymbol{g} = \begin{pmatrix} -N'e^{-|\xi'|y}g_n + N'(1-\overline{U})\tilde{E}_1 V_1\boldsymbol{g} \\ e^{-|\xi'|y}g_n + \overline{U}\tilde{E}_1 V_1\boldsymbol{g} \end{pmatrix}, \tag{3.36}$$

in which

$$V_1\boldsymbol{g} = g_n - N'g'. \tag{3.37}$$

3.4. Estimates. In this subsection we prove some basic simple estimates on the operators $\mathcal{S}^E$, $\mathcal{S}^P$, and $\mathcal{S}^C$. Propositions 3.1, 3.2 and 3.3 are presented as results on the time-dependent Stokes equations, but are not used in the sequel. For analysis of the Navier-Stokes equations, only Proposition 3.4 and Lemma 3.2 will be used.

We cannot in general give an estimate for the operator $\mathcal{S}^E$ in a space involving the L^2 norm in y. Nevertheless it is possible to give such an estimate for a special class of boundary data.

Proposition 3.1. *Suppose that* $\boldsymbol{g}$ *satisfies*

$$g_n = |\xi'| \int_0^\infty dy' f(\xi', y', t) k(\xi', y') \tag{3.38}$$

with $|\xi'| \int_0^\infty dy' |k(\xi', y')| \le 1$ *and* $f \in H^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{S}^E \boldsymbol{g} \in H^{l,\rho,\theta}_{\beta,T}$, *and the following estimate holds:*

$$|\mathcal{S}^E \boldsymbol{g}|_{l,\rho,\theta,\beta,T} \le c|f|_{l,\rho,\theta,\beta,T}. \tag{3.39}$$

Using Jensen's inequality and replacing a factor of $|\xi'| \int_0^\infty dy' |k(\xi', y')|$ by 1,

$$\begin{aligned}
\sup_{\theta' \le \theta} & \int_{\Gamma(\theta')} dy \int d\xi' e^{2\rho|\xi'|} \left[\int_0^\infty dy' |\xi'| e^{-|\xi'| y} k(\xi', y') f(\xi', y') \right]^2 \\
&\le \sup_{\theta' \le \theta} \int_{\Gamma(\theta')} dy \int d\xi' e^{2\rho|\xi'|} \int_0^\infty dy' |\xi'| e^{-2|\xi'| y} k(\xi', y') \left[f(\xi', y') \right]^2 \\
&= \int d\xi' e^{2\rho|\xi'|} \int_0^\infty dy' k(\xi', y') \left[f(\xi', y') \right]^2 \\
&\le |f|^2_{0,\rho,\theta,\beta,T}.
\end{aligned} \tag{3.40}$$

Analogous bounds can be proved for the differentiated terms in the norm.

Now consider the "Prandtl" part. We first state an estimate for the operator $\tilde{E}_1$.

Lemma 3.1. *Let* $f \in K'^{l,\rho}_{\beta,T}$ *with* $f(t = 0) = 0$. *Then* $\tilde{E}_1 f \in L^{l,\rho,\theta}_{\beta,T}$ *for some* θ, *and the following estimate holds in* $L^{l,\rho,\theta}_{\beta,T}$:

$$|\tilde{E}_1 f|_{l,\rho,\theta,\beta,T} \le c|f|_{l,\rho,\beta,T}. \tag{3.41}$$

A much stronger estimate actually holds. One can in fact prove the exponential decay of $\tilde{E}_1 f$ in the normal variable away from the boundary; see the proof given in the Appendix.

Using Lemma 3.1, the following estimates on $\mathcal{S}^{P\prime}$ and $\mathcal{S}^P_n$ (respectively the transversal and the normal components of the operator $\mathcal{S}^P$) are obvious:

Proposition 3.2. *Suppose* $\boldsymbol{g} \in K'^{l,\rho}_{\beta,T}$ *with* $\boldsymbol{g}(t = 0) = 0$. *Then* $\mathcal{S}^{P\prime}\boldsymbol{g} \in L^{l,\rho,\theta}_{\beta,T}$ *and* $\mathcal{S}^P_n \boldsymbol{g} \in L^{l-1,\rho,\theta}_{\beta,T}$ *for some* θ, *and*

$$|\mathcal{S}^{P\prime}\boldsymbol{g}|_{l,\rho,\theta,\beta,T} \le c|\boldsymbol{g}|_{l,\rho,\beta,T}, \tag{3.42}$$

$$|\mathcal{S}^P_n \boldsymbol{g}|_{l-1,\rho,\theta,\beta,T} \le c|\boldsymbol{g}|_{l,\rho,\beta,T}. \tag{3.43}$$

Again a stronger estimate could be proved, namely that $\mathcal{S}^P \boldsymbol{g}$ is exponentially decaying when $Y \longrightarrow \infty$ (i.e. outside the boundary layer). The loss of one derivative in the normal component is due to the incompressibility condition (see e.g., Eq. (3.25)).

The estimate on $\mathcal{S}^C$ will be a consequence of the following bound on the operator $\overline{U}$:

Lemma 3.2. *Let* $f \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\overline{U} f \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\overline{U} f|_{l,\rho,\theta,\beta,T} \le c|f|_{l,\rho,\theta,\beta,T}. \tag{3.44}$$

The proof of Lemma 3.2 is like the proof of Proposition 3.1, and is based on the fact that $\overline{U}f$ can be written as a derivative with respect to the normal variable. Lemma 3.2 leads to the following proposition for $\mathcal{S}^C$:

Proposition 3.3. *Suppose $\boldsymbol{g} \in K'^{l,\rho}_{\beta,T}$. Then $\mathcal{S}^C\boldsymbol{g} \in L^{l-1,\rho,\theta}_{\beta,T}$ for some θ, and*

$$|\mathcal{S}^C\boldsymbol{g}\,|_{l-1,\rho,\theta,\beta,T} \leq \varepsilon^{1/2}c|\boldsymbol{g}\,|_{l,\rho,\beta,T}. \tag{3.45}$$

This estimate on the size of the error is not optimal. In fact, a more careful analysis of $\mathcal{S}^C$ would reveal that the error term is made up of two parts: a Eulerian part (namely $e^{-|\xi'|y}\alpha$) depending on the unscaled variable y, which is of size ε in $H^{l-1,\rho,\theta}_{\beta,T}$, and a part (namely $\overline{U}\tilde{E}_1\alpha$) depending on the scaled variable Y, which is of size ε in $L^{l-1,\rho,\theta}_{\beta,T}$. Something similar occurs in the analysis of the error for the Navier-Stokes equations (see Sect. 4); to prove that the error $\boldsymbol{w}$ is size ε we shall break it up in several parts (see Eq. (4.6) below) and estimate them in the appropriate function spaces.

We now give an estimate on the Stokes operator $\mathcal{S}$. Combine Lemma 3.1, 3.2 with the representation (3.36) to obtain the following bound on $\mathcal{S}$:

Proposition 3.4. *Suppose that $\boldsymbol{g} \in K'^{l,\rho}_{\beta,T}$, with $\boldsymbol{g}\,(t = 0) = 0$ and $g_n = |\xi'|\int_0^\infty dy'\, f(\xi', y', t)k(\xi', y')$ with $|\xi'|\int_0^\infty dy'|k(\xi', y')| \leq 1$ and $f \in L^{l,\rho,\theta}_{\beta,T}$. Then $\mathcal{S}\boldsymbol{g} \in L^{l,\rho,\theta}_{\beta,T}$, and*

$$|\mathcal{S}\boldsymbol{g}\,|_{l,\rho,\theta,\beta,T} \leq c\left(|g'|_{l,\rho,\beta,T} + |f|_{l,\rho,\theta,\beta,T}\right). \tag{3.46}$$

In addition, for each $t \leq T$, $\mathcal{S}\boldsymbol{g} \in K^{l,\rho',\theta'}$, and satisfies

$$\sup_{0\leq t\leq T}|\mathcal{S}\boldsymbol{g}\,|_{l,\rho',\theta'} \leq c\left(|g'|_{l,\rho,\beta,T} + |f|_{l,\rho,\theta,\beta,T}\right) \tag{3.47}$$

in which $0 < \rho' < \rho - \beta T$ and $0 < \theta' < \theta - \beta t$.

The proof of this proposition uses Jensen's inequality as in the proof of Proposition 3.1. Proposition 3.4 and Lemma 3.2 are the only results from this section that will be used in the rest of this paper.

4. The Error Equation

The equation for the error is

$$\begin{aligned}\left(\partial_t - \varepsilon^2\Delta\right)\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u}^0 + \boldsymbol{u}^0\cdot\nabla\boldsymbol{w}&\\ +\varepsilon\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \nabla p^w &= \boldsymbol{f} + \left(g\cdot\partial_y\tilde{u}^P, 0\right), \quad (4.1)\\ \nabla\cdot\boldsymbol{w} &= 0, \quad (4.2)\\ \gamma\boldsymbol{w} &= (0, g), \quad (4.3)\\ \boldsymbol{w}\,(t = 0) &= \boldsymbol{\omega}_0 + \boldsymbol{\Omega}_0 + \boldsymbol{e}_0, \quad (4.4)\end{aligned}$$

in which the forcing term $\boldsymbol{f}$ is in $L^{l-2,\rho_1,\theta_1}_{\beta_1,T}$, and is $O(1)$ (see Eq. (2.40)). Notice that in Eqs. (4.1) and (4.2), and in the rest of this paper, the divergence and the gradient are taken with respect to the unscaled variable y; i.e.

$$\nabla = \left(\partial_x, \partial_y\right). \tag{4.5}$$

The rest of this paper is concerned with proving that equations (4.1)–(4.4) admit a unique solution, and that this solution is $O(1)$. We shall prove the following Theorem:

Theorem 2. *Suppose that* $\boldsymbol{u}^{E} \in H^{l,\rho,\theta}_{\beta,T}$*, that* $\tilde{u}^P \in K^{l,\rho,\theta,\mu}_{\beta,T}$*, so that* $\boldsymbol{f}$ *has norm in* $L^{l-2,\rho,\theta}_{\beta,T}$ *bounded by a constant independent of* ε*. Then there exist* $\rho_2 < \rho$*,* $\theta_2 < \theta$ *and* $\beta_2 > \beta$ *and* $\mu_2 > 0$ *such that Eqs. (4.1)–(4.4) admit a solution which can be written in the form:*

$$\boldsymbol{w} = \boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{e}\,, \tag{4.6}$$

where

- $\boldsymbol{\omega} \in N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ *satisfies Eqs. (4.7)–(4.10);*
- $\boldsymbol{\Omega} \in K^{l-2,\rho_2,\theta_2,\mu_2}_{\beta_2,T}$ *satisfies Eqs. (4.11)–(4.14), and*
- $\boldsymbol{e} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ *satisfies Eqs. (4.15)–(4.18).*

The quantity $\boldsymbol{\omega}$ represents the first order correction to the Euler flow. It satisfies the following equations:

$$\partial_t \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}^{E} + \boldsymbol{u}^{E} \cdot \nabla \boldsymbol{\omega} + \nabla p^{\omega} = 0, \tag{4.7}$$

$$\nabla \cdot \boldsymbol{\omega} = 0, \tag{4.8}$$

$$\gamma_n \boldsymbol{\omega} = g, \tag{4.9}$$

$$\boldsymbol{\omega}(t = 0) = \boldsymbol{\omega}_0. \tag{4.10}$$

In addition the initial data $\boldsymbol{\omega}_0$ is required to satisfy the condition (iii) of Theorem 2.1.

The quantity $\boldsymbol{\Omega} = \left(\Omega^1, \Omega^2\right)$ represents the first order correction inside the boundary layer, with the convective terms omitted. It satisfies the following equations:

$$\left(\partial_t - \partial_{YY}\right) \Omega^1 = 0, \tag{4.11}$$

$$\Omega^2 = \varepsilon \int_Y^{\infty} dY' \partial_x \Omega^1, \tag{4.12}$$

$$\gamma \Omega^1 = -\gamma \omega^1, \tag{4.13}$$

$$\Omega^1(t = 0) = \Omega^1_0. \tag{4.14}$$

The third part of the error $\boldsymbol{e}$ satisfies the following equations:

$$\begin{aligned}&\left(\partial_t - \varepsilon^2 \Delta\right) \boldsymbol{e} + \boldsymbol{e} \cdot \nabla \left[\boldsymbol{u}^{0} + \varepsilon\left(\boldsymbol{\omega} + \boldsymbol{\Omega}\right)\right] \\ &+ \left[\boldsymbol{u}^{0} + \varepsilon\left(\boldsymbol{\omega} + \boldsymbol{\Omega}\right)\right] \cdot \nabla \boldsymbol{e} + \varepsilon \boldsymbol{e} \cdot \nabla \boldsymbol{e} + \nabla p^{e} = \Xi,\end{aligned} \tag{4.15}$$

$$\nabla \cdot \boldsymbol{e} = 0, \tag{4.16}$$

$$\gamma \boldsymbol{e} = \left(0, -\gamma \Omega^2\right), \tag{4.17}$$

$$\boldsymbol{e}\,(t = 0) = \boldsymbol{e}_{0}. \tag{4.18}$$

The forcing term Ξ is given by:

$$\begin{aligned}\Xi = &- \left[\boldsymbol{u}^{0} \cdot \nabla \boldsymbol{\Omega} + \boldsymbol{\omega} \cdot \nabla \left(\overline{\boldsymbol{u}}^{P} + \varepsilon \boldsymbol{\Omega}\right) + \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}^{0} + \left(\overline{\boldsymbol{u}}^{P} + \varepsilon \boldsymbol{\Omega} + \varepsilon \boldsymbol{\omega}\right) \cdot \nabla \boldsymbol{\omega} + \varepsilon \boldsymbol{\Omega} \cdot \nabla \boldsymbol{\Omega}\right] \\ &+ \varepsilon^2 \left[\Delta \boldsymbol{\omega} + \left(\partial_{xx} \Omega^1, 0\right)\right] - \left(0, (\partial_t - \varepsilon^2 \Delta) \Omega^2\right) + \boldsymbol{f} + \left(g \cdot \partial_y \tilde{u}^P, 0\right).\end{aligned} \tag{4.19}$$

The initial data Ω^1_0 and $\boldsymbol{e}_{0}$ are required to satisfy conditions (iv) and (v) of Theorem 2.1.

The reason for the complicated representation Eq. (4.6) for the error $\boldsymbol{w}$ is the following: To solve Eqs. (4.1)–(4.4) one has to use the projection operator due to the incompressibility condition. The natural ambient space is therefore the space of functions which are L^2 in both transversal and normal components. In the right-hand side of

Eq. (4.1), there are terms which are rapidly varying inside the boundary layer, and thus depend on the rescaled variable Y. So, in taking the L^2 norm with respect to the normal variable we are forced to use the variable Y instead of y. The boundary condition (4.3), on the other hand, gives rise to terms which depend on the variable y. Their L^2 norm evaluated using the rescaled variable Y would be $O(\varepsilon^{-1/2})$. To avoid such a catastrophic error, we use the decomposition (4.6): $\boldsymbol{\omega}$, which is L^2 in y, takes care of the boundary condition (4.3) (see Eq. (4.9)); $\boldsymbol{e}$, which is L^2 in Y, takes care of the rapidly varying forcing term; $\boldsymbol{\Omega}$ cancels the transversal component of $\boldsymbol{\omega}$ at the boundary (see Eq. (4.13)).

5. The Correction to the Euler Flow

In this section we shall prove the following theorem:

Theorem 3. *Suppose that $g \in K'^{l-1,\rho}_{\beta,T}$. Then there exist $\rho_2 < \rho$, $\theta_2 < \theta$ and $\beta_2 > \beta$ such that Eqs. (4.7)–(4.10) admit a unique solution $\boldsymbol{\omega} \in N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. The following estimate in $N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ holds:*

$$|\boldsymbol{\omega}|^{l-2,\rho_2,\theta_2}_{\beta_2,T} \leq c\left(|\boldsymbol{u}\,^E_0|_{l,\rho,\theta} + |\tilde{u}^P_0|_{l,\rho,\theta,\mu} + |\boldsymbol{\omega}_0|_{l,\rho,\theta}\right), \tag{5.1}$$

where the norms of $\boldsymbol{u}\,^E_0$, $\tilde{u}^P_0$ and $\boldsymbol{\omega}_0$ are taken in $H^{l,\rho,\theta}$, $K^{l,\rho,\theta,\mu}$ and $N^{l,\rho,\theta}$ respectively.

The structure of Eqs. (4.7)–(4.10) is somewhat similar to the structure of Euler equations and the proof of the above theorem closely follows the proof of Theorem 4.1 in [6]. The functional setting here is slightly different; in fact Theorem 3 above is stated in the space $N^{l,\rho,\theta}_{\beta,T}$, where only the first derivative with respect to time is taken, instead of the space $H^{l,\rho,\theta}_{\beta,T}$, where time derivatives up to order l are allowed. This is due to the presence of the boundary condition g deriving from Prandtl equations. We shall prove the above theorem using the ACK Theorem.

The solution of Eqs. (4.7)–(4.10) can be written as

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \left(-N', 1\right) e^{-|\xi'|y}\left(g - g_0\right) + P_t\boldsymbol{\omega}^*, \tag{5.2}$$

where the operator P_t is the integrated (with respect to time) half space projection operator defined in Eq.(4.35) of [6]. The first term in this expression provides the correct initial data, the second term the correct boundary data, and the third term the correct forcing terms.

The projection operator P_t satisfies the following bounds in $N^{l,\rho,\theta}_{\beta,T}$:

Proposition 5.1. *Let $\boldsymbol{u}\,^* \in N^{l,\rho,\theta}_{\beta,T}$. Then $P_t\boldsymbol{u}\,^* \in N^{l,\rho,\theta}_{\beta,T}$ and*

$$|P_t\boldsymbol{u}\,^*|_{l,\rho,\theta,\beta,T} \leq c|u|_{l,\rho,\theta,\beta,T}. \tag{5.3}$$

Proposition 5.2. *Let $\boldsymbol{u}\,^* \in N^{l,\rho,\theta}_{\beta,T}$. Let $\rho' < \rho - \beta T$ and $\theta' < \theta - \beta T$. Then $P_t\boldsymbol{u}\,^* \in N^{l,\rho',\theta'}$ for all $0 \leq t \leq T$, and*

$$|P_t\boldsymbol{u}\,^*|_{l,\rho',\theta'} \leq c\int_0^t ds|u(\cdot,\cdot,s)|_{l,\rho',\theta'} \leq c|u|_{l,\rho,\theta,\beta,T}. \tag{5.4}$$

Using Eq. (5.5) one sees that (4.7)–(4.10) are equivalent to the following equation for $\boldsymbol{\omega}^*$:

$$\boldsymbol{\omega}^* + \boldsymbol{H}'(\boldsymbol{\omega}^*, t) = 0, \tag{5.5}$$

where

$$\begin{aligned}\boldsymbol{H}'(\boldsymbol{\omega}^*, t) = &\left[\boldsymbol{\omega}_0 + (-N', 1)e^{-|\xi'|y}(g - g_0) + P_t\boldsymbol{\omega}^*\right] \cdot \nabla \boldsymbol{u}^E \\ &+\boldsymbol{u}^E \cdot \nabla \left[\boldsymbol{\omega}_0 + (-N', 1)e^{-|\xi'|y}(g - g_0) + P_t\boldsymbol{\omega}^*\right].\end{aligned} \tag{5.6}$$

Using the Cauchy estimate, and with the same procedure we used to prove existence and uniqueness for Euler equations in [6], one can see that the operator $\boldsymbol{H}'$ satisfies all the hypotheses of the ACK Theorem; therefore there exist $\rho_2 < \rho$, $\theta_2 < \theta$ and $\beta_2 > \beta$ such that Eq. (5.5) admits a unique solution $\boldsymbol{\omega}^* \in N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. Equation (5.2) and Proposition 5.1 also imply $\boldsymbol{\omega} \in N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. Theorem 3 is thus proved.

6. The Boundary Layer Correction

We prove the following theorem:

Theorem 4. *Let $\boldsymbol{\omega}$ be the solution of Eqs. (4.7)–(4.10) found in Theorem 5.1. Then there exist $\rho_2' > \rho_2$, $\theta_2' > \theta_2$, $\beta_2' > \beta_2$, and $\mu_2 > 0$ such that Eqs. (4.11)–(4.14) admit a unique solution $\boldsymbol{\Omega} \in K^{l-2,\rho_2',\theta_2',\mu_2}_{\beta_2',T}$. It satisfies the following estimate in $K^{l-2,\rho_2',\theta_2',\mu_2}_{\beta_2',T}$:*

$$|\boldsymbol{\Omega}|_{l-2,\rho_2',\theta_2',\mu_2,\beta_2',T} \leq c\left(|\boldsymbol{u}^E_0|_{l,\rho,\theta} + |\tilde{u}^P_0|_{l,\rho,\theta,\mu} + |\boldsymbol{\omega}_0|_{l,\rho,\theta} + |\boldsymbol{\Omega}_0|_{l,\rho,\theta,\mu_2}\right), \tag{6.1}$$

where the norms of $\boldsymbol{u}^E_0$, $\tilde{u}^P_0$, $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}_0$ are taken in $H^{l,\rho,\theta}$, $K^{l,\rho,\theta,\mu}$, $N^{l,\rho,\theta}$ and $K^{l,\rho,\theta,\mu}$ respectively.

The proof of this theorem uses the following lemma:

Lemma 6.1. *There exists $\rho_2' < \rho_2$ such that the boundary data $\gamma\omega^1$ is in $K'^{l-2,\rho_2'}_{\beta_2,T}$. The following estimate holds in $K'^{l-2,\rho_2'}_{\beta_2,T}$:*

$$|\gamma\omega^1|_{l-2,\rho_2',\beta_2,T} \leq c|\boldsymbol{\omega}|_{l-2,\rho_2,\theta_2,\beta_2,T}. \tag{6.2}$$

The above lemma can be proved using a Sobolev estimate to bound the sup with respect to y of ω^1, and then a Cauchy estimate on the x derivative to bound the term $\partial_y\partial_x^{l-2}\omega^1$.

The solution of Eqs. (4.11)–(4.14) can be explicitly written as

$$\Omega^1 = E_0(t)\Omega^1_0 - E_1\gamma\omega^1 = E_0(t)\left(\Omega^1_0 + \gamma\omega^1_0\right) - E_1\gamma\left(\omega^1 - \omega^1_0\right) - \gamma\omega^1_0, \tag{6.3}$$

where the operator $E_0(t)$ and E_1 have been defined in [6]. Proposition 5.1 and Proposition 5.3 of [6], imply that $\Omega^1 \in K^{l-2,\rho_2',\theta_2',\mu_2}_{\beta_2',T}$. Using the expression (4.12) for Ω^2 and again shrinking the domain of analyticity in x, and renaming ρ_2', we obtain also $\Omega^2 \in K^{l-2,\rho_2',\theta_2',\mu_2}_{\beta_2',T}$. The proof of Theorem 4 is thus complete.

By a redefinition of $\rho_2, \theta_2, \beta_2$, we may take $\rho_2' = \rho_2$, $\theta_2' = \theta_2$, $\beta_2' = \beta_2$.

7. The Navier-Stokes Operator

In this section we shall prove the following theorem

Theorem 5. *Under the hypotheses of Theorem 2, there exist ρ_2, θ_2, β_2, such that Eqs. (4.15)–(4.18) admit a unique solution $\boldsymbol{e} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. This solution satisfies the following estimate in $L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$:*

$$|\boldsymbol{e}|_{l-2,\rho_2,\theta_2,\beta_2,T} \leq c\left(|\boldsymbol{u}_0^E|_{l,\rho,\theta} + |\tilde{u}_0^P|_{l,\rho,\theta,\mu} + |\boldsymbol{\omega}_0|_{l,\rho,\theta} + |\boldsymbol{\Omega}_0|_{l,\rho,\theta,\mu} + |\boldsymbol{e}_0|_{l,\rho,\theta}\right), \tag{7.1}$$

where the norms of $\boldsymbol{u}_0^E$, $\tilde{u}_0^P$, $\boldsymbol{\omega}_0$, $\boldsymbol{\Omega}_0$ and $\boldsymbol{e}_0$ are taken in $H^{l,\rho,\theta}$, $K^{l,\rho,\theta,\mu}$, $N^{l,\rho,\theta}$, $K^{l,\rho,\theta,\mu}$ and $L^{l,\rho,\theta}$ respectively.

We shall prove this theorem using the ACK Theorem. In the same way as for the Euler and Prandtl equations, we first invert the second order heat operator, taking into account the incompressibility condition and the BC and IC. This is performed using the heat operator,defined in Subsect. 7.1, which inverts $\left(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx}\right)$. Then in Subsect. 7.2 we insert the divergence-free projection and obtain the operator $\mathcal{N}_0$. Using the Stokes operator from Sect. 3 to handle the boundary data, in Subsect. 7.3 we define the operator $\mathcal{N}^*$, which is suitable for the iterative solution of the Navier-Stokes equations (i.e. treating initial data and nonlinearities as forcing terms). Bounds on this operator are given in Propositions 7.6 and 7.7. With the use of this Navier-Stokes operator, and taking into account initial and boundary data Eq. (4.17) and Eq. (4.18), in Subsect. 7.4 we finally solve the error equation. In Subsects. 7.5 and 7.6 we prove by the ACK Theorem that this iterative procedure converges to a unique solution.

7.1. The heat operator. We have already introduced the operator $\tilde{E}_1$ in (3.21) which solves the heat equation with boundary data. We now want to solve the heat equation with a source and with zero initial and boundary data on the half plane $Y \geq 0$; i.e.

$$\begin{array}{rcl} \left(\partial_t - \varepsilon^2\partial_{xx} - \partial_{YY}\right) u & = & w(x,Y,t), \\ u(x,Y,t=0) & = & 0, \\ \gamma u & = & 0. \end{array} \tag{7.2}$$

First introduce the heat kernel $\tilde{E}_0(x,Y,t)$, defined by

$$\tilde{E}_0(x,Y,t) = \frac{e^{-x^2/4t\varepsilon^2}}{\sqrt{4\pi t\varepsilon^2}}\frac{e^{-Y^2/4t}}{\sqrt{4\pi t}}. \tag{7.3}$$

We solve the problem (7.2) on the half plane with the following operator:

$$\begin{aligned} u(x,Y,t) &= \tilde{E}_2 w \\ &= \int_0^t ds \int_0^\infty dY' \int_{-\infty}^\infty dx' \left[\tilde{E}_0(x-x',Y-Y',t-s)\right. \\ &\quad \left. -\tilde{E}_0(x-x',Y+Y',t-s)\right] w(x',Y',s). \end{aligned} \tag{7.4}$$

We now state some estimates on this operator. In these estimates w is defined for $Y \geq 0$.

Proposition 7.1. *Let $w \in L^{l,\rho,\theta}_{\beta,T}$. Then $\tilde{E}_2 w \in L^{l,\rho,\theta}_{\beta,T}$ and*

$$|\tilde{E}_2 w|_{l,\rho,\theta,\beta,T} \leq c|w|_{l,\rho,\theta,\beta,T}. \tag{7.5}$$

Proposition 7.2. *Suppose* $w \in L^{l,\rho,\theta}_{\beta,T}$ *with* $\gamma w = 0$ *and that* $\rho' \leq \rho - \beta t$, $\theta' \leq \theta - \beta t$. *Then* $\tilde{E}_2 w(t) \in L^{l,\rho',\theta'}$ *and*

$$|\tilde{E}_2 w|_{l,\rho',\theta'} \leq c \int_0^t ds |w(\cdot,\cdot,s)|_{l,\rho',\theta'} \leq c|w|_{l,\rho,\theta,\beta,T}. \tag{7.6}$$

The proofs of these two propositions are given in the Appendix.

7.2. The projected heat operator. In [6] we introduced the divergence-free projection operator P^∞. Here we employ a similar operator with the normal variable rescaled by a factor ε. The projection operator in the x and Y variable, $\overline{P}^\infty$, is the pseudodifferential operator whose symbol is

$$\overline{P}^\infty = \frac{1}{\varepsilon^2 \xi'^2 + \xi_n^2} \begin{pmatrix} \xi_n^2 & -\varepsilon\xi'\xi_n \\ -\varepsilon\xi'\xi_n & \varepsilon^2\xi'^2 \end{pmatrix}, \tag{7.7}$$

where ξ' and ξ_n denote the Fourier variables corresponding to x and Y respectively. For all $\boldsymbol{w}$ this operator satisfies

$$\nabla \cdot \overline{P}^\infty \boldsymbol{w} = \partial_x \overline{P}^\infty{}' \boldsymbol{w} + \varepsilon \partial_Y \overline{P}^\infty{}_n \boldsymbol{w} = 0. \tag{7.8}$$

In [6] to avoid Fourier transform in y we expressed P^∞ as an integration in the normal variable. For $\overline{P}^\infty$ one can similarly see that

$$\begin{aligned} \overline{P}^\infty{}_n \boldsymbol{w} = \frac{1}{2} \Big[&\varepsilon|\xi'| \int_{-\infty}^Y dY' e^{-\varepsilon|\xi'|(Y-Y')}(-N'w^1 + w^2) \\ &+ \varepsilon|\xi'| \int_Y^\infty dY' e^{\varepsilon|\xi'|(Y-Y')}(N'w^1 + w^2) \Big], \end{aligned} \tag{7.9}$$

$$\begin{aligned} \overline{P}^\infty{}' \boldsymbol{w} = w^1 + \frac{1}{2} \Big[&-\varepsilon|\xi'| \int_{-\infty}^Y dY' e^{-\varepsilon|\xi'|(Y-Y')}(w^1 + N'w^2) \\ &- \varepsilon|\xi'| \int_Y^\infty dY' e^{\varepsilon|\xi'|(Y-Y')}(w^1 - N'w^2) \Big]. \end{aligned} \tag{7.10}$$

Next we present estimates on the projection operator. In these estimates w is defined on $Y \geq 0$, but we write $\overline{P}^\infty w$ to mean the following: First extend w oddly to $Y < 0$, i.e.

$$w(x,Y) = -w(x,-Y) \quad \text{when} \quad Y \leq 0\,; \tag{7.11}$$

then apply $\overline{P}^\infty$, and finally restrict the result to $Y \geq 0$ for application of the norm. The resulting expressions for $\overline{P}^\infty$ are

$$\begin{aligned} \overline{P}^\infty{}_n \boldsymbol{w} = \frac{1}{2}\varepsilon|\xi'| \Big[&\int_0^Y dY' \left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')} \right) (-N'w^1 + w^2) \\ + &\int_Y^\infty dY' \left(e^{\varepsilon|\xi'|(Y-Y')}(N'w^1 + w^2) - e^{\varepsilon|\xi'|(-Y-Y')}(-N'w^1 + w^2) \right) \Big], \end{aligned} \tag{7.12}$$

$$\overline{P}^{\infty\prime}\boldsymbol{w} = w^1 - \frac{1}{2}\varepsilon|\xi'|\left[\int_0^Y dY'\left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')}\right)(w^1 + N'w^2)\right.$$
$$\left. + \int_Y^\infty dY'\left(e^{\varepsilon|\xi'|(Y-Y')}(w^1 - N'w^2) - e^{\varepsilon|\xi'|(-Y-Y')}(w^1 + N'w^2)\right)\right]. \tag{7.13}$$

The following estimate is easily proved

Proposition 7.3. *Let* $\boldsymbol{w} \in L^{l,\rho,\theta}$ *with* $\gamma\boldsymbol{w} = 0$. *Then* $\overline{P}^{\infty}\boldsymbol{w} \in L^{l,\rho,\theta}$ *and*

$$|\overline{P}^{\infty}\boldsymbol{w}|_{l,\rho,\theta} \leq c|\boldsymbol{w}|_{l,\rho,\theta}. \tag{7.14}$$

We are now ready to introduce the projected heat operator $\mathcal{N}_0$, acting on vectorial functions, defined as

$$\mathcal{N}_0 = \overline{P}^{\infty}\tilde{E}_2. \tag{7.15}$$

One can easily show that $\overline{P}^{\infty}$ commutes with the heat operator $\left(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx}\right)$. It then follows that for each $\boldsymbol{w}$ such that $\gamma\boldsymbol{w} = 0$,

$$\nabla\cdot\mathcal{N}_0\boldsymbol{w} = 0, \tag{7.16}$$
$$\left(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx}\right)\mathcal{N}_0\boldsymbol{w} = \overline{P}^{\infty}\boldsymbol{w}. \tag{7.17}$$

The following estimates are a consequence of the properties of $\overline{P}^{\infty}$ and $\tilde{E}_2$ separately:

Proposition 7.4. *Suppose* $\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{N}_0\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{N}_0\boldsymbol{w}|_{l,\rho,\theta,\beta,T} \leq c|\boldsymbol{w}|_{l,\rho,\theta,\beta,T}. \tag{7.18}$$

Proposition 7.5. *Suppose* $\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$ *with* $\gamma\boldsymbol{w} = 0$ *and that* $\rho' \leq \rho - \beta t$, $\theta' \leq \theta - \beta t$. *Then* $\boldsymbol{w}$ *and* $\mathcal{N}_0\boldsymbol{w}$ *are in* $L^{l,\rho',\theta'}$ *for each* t, *and*

$$|\mathcal{N}_0\boldsymbol{w}|_{l,\rho',\theta'} \leq c\int_0^t ds|\boldsymbol{w}(\cdot,\cdot,s)|_{l,\rho',\theta'} \leq c|w|_{l,\rho,\theta,\beta,T}. \tag{7.19}$$

Note that $\tilde{E}_2$ has zero boundary data; thus the conditions in Proposition 7.3 are all satisfied.

7.3. The Navier-Stokes operator. With the Stokes operator defined in Sect. 3 and the projected heat operator of the previous subsection, we now introduce the Navier-Stokes operator $\mathcal{N}^*$ defined as

$$\mathcal{N}^* = \mathcal{N}_0 - \mathcal{S}\gamma\mathcal{N}_0. \tag{7.20}$$

This operator is used to solve the time-dependent Stokes equations with forcing, which is equivalent to the Navier-Stokes equations if the nonlinear terms are put into the forcing. In fact

$$\boldsymbol{w} = \mathcal{N}^*\boldsymbol{w}^\star \tag{7.21}$$

solves the system

$$\left(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx}\right)\boldsymbol{w} + \nabla p^w = \boldsymbol{w}^\star, \tag{7.22}$$
$$\nabla\cdot\boldsymbol{w} = 0, \tag{7.23}$$
$$\gamma\boldsymbol{w} = 0, \tag{7.24}$$
$$\boldsymbol{w}(t = 0) = 0, \tag{7.25}$$

and satisfies the following bound:

Proposition 7.6. *Suppose* $\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{N}^*\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{N}^*\boldsymbol{w}|_{l,\rho,\theta,\beta,T} \leq c|\boldsymbol{w}|_{l,\rho,\theta,\beta,T}. \tag{7.26}$$

We already know, from Proposition 7.4, that $\mathcal{N}_0$ obeys an estimate like (7.26). Therefore the only part of $\mathcal{N}^*$ which has to be estimated is that involving the Stokes operator $\mathcal{S}$. To bound this term it is enough to notice that $\gamma\mathcal{N}_0\boldsymbol{w}$ is a boundary data for which the assumptions of Proposition 3.4 hold. In fact since $\tilde{E}_2\boldsymbol{w}$ has been extended oddly for $Y < 0$, then $\gamma_n\mathcal{N}_0\boldsymbol{w} = \gamma\overline{P}^{\infty}{}_n\tilde{E}_2\boldsymbol{w}$ is (see Eq. (7.12))

$$\gamma\overline{P}^{\infty}{}_n\tilde{E}_2\boldsymbol{w} = \varepsilon|\xi'|\int_0^{\infty} dY' e^{-\varepsilon|\xi'|Y'} N'\tilde{E}_2 w^1. \tag{7.27}$$

According to Proposition 7.1, this is of the form required in Proposition 3.4 for the normal part g_n of $\boldsymbol{g} = \mathcal{N}_0\boldsymbol{w}$. The tangential part g' satisfies the bound

$$|g'|_{l,\rho,\theta,\beta,T} \leq c|\tilde{E}_2\boldsymbol{w}|_{l,\rho,\theta,\beta,T}. \tag{7.28}$$

Therefore

$$|\mathcal{S}\gamma\mathcal{N}_0\boldsymbol{w}|_{l,\rho,\theta,\beta,T} \leq c|\tilde{E}_2\boldsymbol{w}|_{l,\rho,\theta,\beta,T} \leq c|\boldsymbol{w}|_{l,\rho,\theta,\beta,T}, \tag{7.29}$$

which concludes the proof of Proposition 7.6.

We shall also use the following Proposition, which is proved in the same way as the previous result, using Proposition 7.5:

Proposition 7.7. *Suppose* $\boldsymbol{w} \in L^{l,\rho,\theta}_{\beta,T}$ *with* $\gamma\boldsymbol{w} = 0$ *and that* $\rho' \leq \rho - \beta t$, $\theta' \leq \theta - \beta t$. *Then in* $L^{l,\rho',\theta'}$,

$$|\mathcal{N}^*\boldsymbol{w}|_{l,\rho',\theta'} \leq c\int_0^t ds|\boldsymbol{w}(\cdot,\cdot,s)|_{l,\rho',\theta'} \leq c|w|_{l,\rho,\theta,\beta,T}. \tag{7.30}$$

7.4. The solution of the error equation. We can now solve Eqs. (4.15)–(4.18). If one looks at these equations one sees that they are of the form (7.22)–(7.25) (where all forcing and nonlinear terms are in $\boldsymbol{w}^{\star}$, see Eq. (7.37) below) plus boundary and initial data. We therefore express $\boldsymbol{e}$ as the sum of two terms: the first involving the Navier-Stokes operator and the second where all boundary and initial data are. In fact we write

$$\boldsymbol{e} = \mathcal{N}^*\boldsymbol{e}^* + \boldsymbol{\sigma}, \tag{7.31}$$

where $\boldsymbol{\sigma}$ solves the following time-dependent Stokes problem with initial and boundary data:

$$(\partial_t - \partial_{YY})\boldsymbol{\sigma} + \nabla\phi, = 0 \tag{7.32}$$

$$\nabla\cdot\boldsymbol{\sigma} = 0, \tag{7.33}$$

$$\gamma\boldsymbol{\sigma} = (0, \varepsilon G), \tag{7.34}$$

$$\boldsymbol{\sigma}(t = 0) = \boldsymbol{e}_0, \tag{7.35}$$

having denoted:

$$G = -\int_0^{\infty} dY'\partial_x\Omega^1, \tag{7.36}$$

and where $\boldsymbol{e}^*$ satisfies the following equation:

$$\begin{aligned} \boldsymbol{e}^{*} = \Xi - \Big\{ &\boldsymbol{e} \cdot \nabla \left[\boldsymbol{u}^{0} + \varepsilon (\boldsymbol{\omega} + \boldsymbol{\Omega}) \right] \\ &+ \left[\boldsymbol{u}^{0} + \varepsilon (\boldsymbol{\omega} + \boldsymbol{\Omega}) \right] \cdot \nabla \boldsymbol{e} + \varepsilon \boldsymbol{e} \cdot \nabla \boldsymbol{e} - \varepsilon^2 \partial_{xx} \boldsymbol{\sigma} \Big\} . \end{aligned} \tag{7.37}$$

Equations (7.32)–(7.35) can be solved explicitly. First note that ϕ is harmonic, so that, imposing it to be bounded at infinity,

$$\left(\partial_y + |\xi'| \right) \phi = 0. \tag{7.38}$$

Apply $(\partial_y + |\xi'|)$ to the normal component of Eq. (7.32), and define

$$\tau = \left(\partial_y + |\xi'| \right) \sigma^2 \tag{7.39}$$

which satisfies

$$(\partial_t - \partial_{YY}) \tau = 0, \tag{7.40}$$

$$\gamma \tau = \varepsilon |\xi'| G, \tag{7.41}$$

$$\tau(t = 0) = |\xi'| V_1 \boldsymbol{e}_0, \tag{7.42}$$

in which $V_1 \boldsymbol{e}_0 = e_0^2 - N' e_0^1$. Denote $G_0 = G(t = 0)$. Then the solution of the system (7.40)–(7.42) is

$$\begin{aligned} \tau &= E_0(t) \left(|\xi'| V_1 \boldsymbol{e}_0 - \varepsilon |\xi'| G_0 \right) + E_1 \left[\gamma \varepsilon |\xi'| G - \varepsilon |\xi'| G_0 \right] + \varepsilon |\xi'| G_0 \\ &= |\xi'| \tilde{\tau}. \end{aligned} \tag{7.43}$$

The initial condition $\boldsymbol{e}_0$ is in $L^{l,\rho,\theta}$; this obviously implies $\boldsymbol{e}_0 \in L^{l-2,\rho_2'',\theta_2''}$. One has the following proposition:

Proposition 7.8. *Given that* $\boldsymbol{e}_0 \in L^{l-2,\rho_2,\theta_2}$, *that* $G \in K'^{l-2,\rho_2}_{\beta_2,T}$, *and the compatibility condition* $\gamma_n \boldsymbol{e}_0 = \varepsilon G_0$, *then* $\tilde{\tau} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ *and*

$$|\tilde{\tau}|_{l-2,\rho_2,\theta_2,\beta_2,T} \leq c \left(|\boldsymbol{e}_0|_{l-2,\rho_2,\theta_2} + |G|_{l-2,\rho_2,\beta_2,T} \right) . \tag{7.44}$$

The proof of this proposition is based on the estimates on the operators $E_0(t)$ and E_1 given in Propositions 5.2 and 5.3 of [6]; regarding the estimate in Proposition 5.3, we notice in fact that if a function is in $K^{l,\rho,\theta,\mu}_{\beta,T}$ it is *a fortiori* in $L^{l,\rho,\theta}_{\beta,T}$.

Now, the expression (7.43) for τ in (7.39) and the boundary condition (7.34) on σ^2 imply that

$$\sigma^2 = \varepsilon e^{-\varepsilon |\xi'| Y} G + \overline{U} \tilde{\tau}, \tag{7.45}$$

where $\overline{U}$ has been defined in (3.33). The incompressibility condition then leads to

$$\sigma^1 = -\varepsilon N' e^{-\varepsilon |\xi'| Y} G + N' (1 - \overline{U}) \tilde{\tau}. \tag{7.46}$$

A bound for σ is given by

Proposition 7.9. *Suppose that* $G = |\xi'| \tilde{G}$, *with* $\tilde{G} \in K^{l-2,\rho_2}_{\beta_2,T}$, *then* $\boldsymbol{\sigma} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ *and*

$$\begin{aligned} |\boldsymbol{\sigma}|_{l-2,\rho_2,\theta_2,\beta_2,T} &\leq c \left(|\boldsymbol{e}_0|_{l-2,\rho_2,\theta_2} + |\tilde{G}|_{l-2,\rho_2,\beta_2,T} \right) \\ &\leq c \left(|\boldsymbol{u}_0^E|_{l,\rho,\theta} + |\tilde{u}_0^P|_{l,\rho,\theta,\mu} + |\boldsymbol{\omega}_0|_{l,\rho,\theta} + |\boldsymbol{\Omega}_0|_{l,\rho,\theta,\mu} + |\boldsymbol{e}_0|_{l,\rho,\theta} \right) . \end{aligned} \tag{7.47}$$

The proof of this proposition is based on Lemma 3.2 and Proposition 7.8 for the estimate of the terms involving $\tilde{\tau}$, and on the fact that if $\tilde{G} \in K'^{l-2,\rho_2}_{\beta_2,T}$, then $\varepsilon|\xi'|e^{-\varepsilon|\xi'|Y}\tilde{G} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$.

We are now ready to prove existence and uniqueness for Eqs. (4.15)–(4.18). Use Eq. (7.31) in (7.37), interpret this equation as an equation for $\boldsymbol{e}^*$, and use the abstract version of the Cauchy–Kowalewski Theorem, in the function spaces $X_\rho = L^{l,\rho,\theta}$ and $Y_{\rho,\beta,T} = L^{l,\rho,\theta}_{\beta,T}$, to prove existence and uniqueness for the solution. This is similar to the procedure used in [6] to prove existence and uniqueness for the Euler and Prandtl equations. Rewrite Eq. (7.37) as

$$\boldsymbol{e}^* = \boldsymbol{F}(\boldsymbol{e}^*, t), \tag{7.48}$$

where $\boldsymbol{F}(\boldsymbol{e}^*, t)$ is

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{e}^*, t) = \mathbf{k} - \Big\{ & \left[\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma})\right] \cdot \nabla\mathcal{N}^*\boldsymbol{e}^* \\ & + \mathcal{N}^*\boldsymbol{e}^* \cdot \nabla\left[\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma})\right] \\ & + \varepsilon\mathcal{N}^*\boldsymbol{e}^* \cdot \nabla\mathcal{N}^*\boldsymbol{e}^* \Big\} \end{aligned} \tag{7.49}$$

and $\boldsymbol{k}$ is the forcing term

$$\begin{aligned} \boldsymbol{k} &= \Xi - \Big\{ \left[\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega})\right] \cdot \nabla\boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \nabla\left[\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega})\right] + \varepsilon\boldsymbol{\sigma} \cdot \nabla\boldsymbol{\sigma} \Big\} \\ &= \boldsymbol{f} - \Big\{ \Big[\big(\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma})\big) \cdot \nabla\boldsymbol{\Omega} + \big(\boldsymbol{\omega} \cdot \nabla\overline{\boldsymbol{u}}^P - (g\partial_y\tilde{u}^P, 0)\big) + (\boldsymbol{\Omega} + \boldsymbol{\sigma}) \cdot \nabla\overline{\boldsymbol{u}}^P \Big] \\ & \Big[(\boldsymbol{\Omega} + \boldsymbol{\sigma}) \cdot \nabla\boldsymbol{u}^E + \big(\overline{\boldsymbol{u}}^P + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma})\big) \cdot \nabla\boldsymbol{\omega} + \big(\boldsymbol{u}^0 + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma})\big) \cdot \nabla\boldsymbol{\sigma} \Big] \Big\} \\ & + \varepsilon^2 \left[\Delta\boldsymbol{\omega} + \partial_{xx}\left(\Omega^1, 0\right) + \partial_{xx}\boldsymbol{\sigma}\right] - \left(0, (\partial_t - \varepsilon^2\Delta)\Omega^2\right). \end{aligned} \tag{7.50}$$

The rest of this section is concerned with proving that the operator $\boldsymbol{F}$ satisfies all the hypotheses of ACK Theorem.

7.5. The forcing term. In this subsection we shall prove the following proposition, asserting that the forcing term is bounded in $L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ and $O(1)$:

Proposition 7.10. *There exists a constant R_0 such that*

$$|\boldsymbol{F}(0,t)|_{l-2,\rho_2-\beta_2 t,\theta_2-\beta_2 t} \le R_0. \tag{7.51}$$

Equation (7.49) shows that

$$\boldsymbol{F}(0,t) = \boldsymbol{k} \tag{7.52}$$

with $\boldsymbol{k}$ given by (7.50). We already know that $\boldsymbol{f} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ (see the discussion after Eq. (2.40)). The terms in the first square brackets are exponentially decaying outside the boundary layer. Inside the boundary layer they can be shown to be $O(1)$ with a Cauchy estimate on the terms where ∂_y is present: this is possible because they go linearly fast to zero at the boundary. All terms inside the second square brackets are more easily handled because no $O(\varepsilon^{-1})$ appear. Proposition 7.10 is thus proved.

7.6. The Cauchy estimate. In this subsection we shall prove that the operator $\boldsymbol{F}$ satisfies the last hypothesis of the ACK Theorem. Here and in the rest of this section

$$\begin{aligned} \rho' < \rho(s) \le \rho_2 - \beta_2 s, \\ \theta' < \theta(s) \le \theta_2 - \beta_2 s. \end{aligned}$$

Proposition 7.11. *Suppose $\rho' < \rho(s) \le \rho_2 - \beta_2 s$ and $\theta' < \theta(s) \le \theta_2 - \beta_2 s$. If $\boldsymbol{e}^{*1}$ and $\boldsymbol{e}^{*2}$ are in $L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ with*

$$|\boldsymbol{e}^{*1}|_{l-2,\rho_2,\theta_2,\beta_2,T} \le R, \; |\boldsymbol{e}^{*2}|_{l-2,\rho_2,\theta_2,\beta_2,T} \le R, \tag{7.53}$$

then

$$\begin{aligned}
&|\boldsymbol{F}(\boldsymbol{e}^{*1},t) - \boldsymbol{F}(\boldsymbol{e}^{*2},t)|_{l-2,\rho',\theta'} \\
&\le C\int_0^t ds \left\{ \frac{|\boldsymbol{e}^{*1}-\boldsymbol{e}^{*2}|_{l-2,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|\boldsymbol{e}^{*1}-\boldsymbol{e}^{*2}|_{l-2,\rho',\theta(s)}}{\theta(s)-\theta'} \right\} \\
&+C|\boldsymbol{e}^{*1}-\boldsymbol{e}^{*2}|_{l-2,\rho',\theta',\beta_2,t} \int_0^t ds \sum_{i=1,2} \left\{ \frac{|\boldsymbol{e}^{*i}|_{l-2,\rho(s),\theta'}}{\rho(s)-\rho'} + \frac{|\boldsymbol{e}^{*i}|_{l-2,\rho',\theta(s)}}{\theta(s)-\theta'} \right\}
\end{aligned} \tag{7.54}$$

in which all the norms are in $L^{l,\rho,\theta}$ and $L^{l,\rho,\theta}_{\beta,T}$.

The proof of the above proposition occupies the remainder of this section. First introduce the Cauchy estimates in $L^{l,\rho,\theta}$.

Lemma 7.1. *Let $f(x,Y) \in L^{l,\rho,\theta}$. Then for $0 < \rho' < \rho$ and $0 < \theta' < \theta$,*

$$|\partial_x f|_{l,\rho',\theta} \le c\frac{|f|_{l,\rho,\theta}}{\rho-\rho'}, \tag{7.55}$$

$$|\chi(Y)\partial_Y f|_{l,\rho,\theta'} \le c\frac{|f|_{l,\rho,\theta}}{\theta-\theta'}. \tag{7.56}$$

In the above proposition $\chi(Y)$ is a monotone, bounded function, going to zero linearly fast near the origin (see e.g. Eq.(4.28)) of [6]. The Sobolev inequality implies the following lemmas:

Lemma 7.2. *Let $f(x,Y)$ and $g(x,Y)$ be in $L^{l,\rho,\theta}$. Then for $0 < \rho' < \rho$,*

$$|g\partial_x f|_{l,\rho',\theta} \le c|g|_{l,\rho',\theta}\frac{|f|_{l,\rho,\theta}}{\rho-\rho'}. \tag{7.57}$$

Lemma 7.3. *Let $f(x,Y)$ and $g(x,Y)$ be in $L^{l,\rho,\theta}$ with $g(x,Y=0)=0$. Then for $0 < \theta' < \theta$,*

$$|g\partial_Y f|_{l,\rho,\theta'} \le c|g|_{l,\rho,\theta'}\frac{|f|_{l,\rho,\theta}}{\theta-\theta'}. \tag{7.58}$$

Lemmas 7.2 and 7.3 then imply

Lemma 7.4. *Suppose $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ are in $L^{l-2,\rho,\theta}_{\beta,T}$ with $\gamma_n \boldsymbol{e}^1 = \gamma_n \boldsymbol{e}^2 = 0$. Then for $0 < \rho' < \rho$ and $0 < \theta' < \theta$,*

$$|\boldsymbol{e}^1\cdot\nabla\boldsymbol{e}^1 - \boldsymbol{e}^2\cdot\nabla\boldsymbol{e}^2|_{l-2,\rho',\theta'} \le c\left[\frac{|\boldsymbol{e}^1-\boldsymbol{e}^2|_{l-2,\rho,\theta'}}{\rho-\rho'} + \frac{|\boldsymbol{e}^1-\boldsymbol{e}^2|_{l-2,\rho',\theta}}{\theta-\theta'}\right], \tag{7.59}$$

where the constant c depends only on $|\boldsymbol{e}^1|_{l-2,\rho,\theta,\beta,T}$ and $|\boldsymbol{e}^2|_{l-2,\rho,\theta,\beta,T}$.

We are now ready to prove Proposition 7.11. We first take into consideration the nonlinear part $\mathcal{N}^* \boldsymbol{e}^{\,*} \cdot \nabla \mathcal{N}^* \boldsymbol{e}^{\,*}$. From the estimates (7.26) and (7.30) on the Navier-Stokes operator, the estimate (7.59) on the convective operator and the fact that $\gamma_n \mathcal{N}^* \boldsymbol{e}^{\,*} = 0$, it follows that

$$
\begin{aligned}
&|\mathcal{N}^* \boldsymbol{e}^{\,*1} \cdot \nabla \mathcal{N}^* \boldsymbol{e}^{\,*1} - \mathcal{N}^* \boldsymbol{e}^{\,*2} \cdot \nabla \mathcal{N}^* \boldsymbol{e}^{\,*2}|_{l-2,\rho',\theta'} \\
&\leq C \int_0^t ds \left[\frac{|\boldsymbol{e}^{\,*1}(\cdot,\cdot,s) - \boldsymbol{e}^{\,*2}(\cdot,\cdot,s)|_{l-2,\rho(s),\theta_2}}{\rho(s)-\rho'} \right. \\
&\quad \left. + \frac{|\boldsymbol{e}^{\,*1}(\cdot,\cdot,s) - \boldsymbol{e}^{\,*2}(\cdot,\cdot,s)|_{l-2,\rho_2,\theta(s)}}{\theta(s)-\theta'} \right] \\
&\quad + C|\boldsymbol{e}^{\,*1} - \boldsymbol{e}^{\,*2}|_{l-2,\rho_2,\theta_2,\beta_2,T} \int_0^t ds \sum_{i=1,2} \left[\frac{|\boldsymbol{e}^{\,*i}(\cdot,\cdot,s)|_{l-2,\rho(s),\theta_2}}{\rho(s)-\rho'} \right. \\
&\quad \left. + \frac{|\boldsymbol{e}^{\,*i}(\cdot,\cdot,s)|_{l-2,\rho_2,\theta(s)}}{\theta(s)-\theta'} \right] \\
&\leq C \int_0^t ds \left[\frac{|\boldsymbol{e}^{\,*1}(\cdot,\cdot,s) - \boldsymbol{e}^{\,*2}(\cdot,\cdot,s)|_{l-2,\rho(s),\theta_2}}{\rho(s)-\rho'} \right. \\
&\quad \left. + \frac{|\boldsymbol{e}^{\,*1}(\cdot,\cdot,s) - \boldsymbol{e}^{\,*2}(\cdot,\cdot,s)|_{l-2,\rho_2,\theta(s)}}{\theta(s)-\theta'} \right].
\end{aligned} \tag{7.60}
$$

Since $\gamma_n \left(\boldsymbol{u}^{\,0} + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma}\,) \right) = 0$, one can estimate the term $\left(\boldsymbol{u}^{\,0} + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma}\,) \right) \cdot \nabla \mathcal{N}^* \boldsymbol{w}^{\,\star}$ in a similar fashion. The term $\mathcal{N}^* \boldsymbol{w}^{\,\star} \cdot \nabla \left(\boldsymbol{u}^{\,0} + \varepsilon(\boldsymbol{\omega} + \boldsymbol{\Omega} + \boldsymbol{\sigma}\,) \right)$ is easily estimated. The proof of Proposition 7.11 is thus achieved.

7.7. Conclusion of the Proof of Theorem 5. The operator $\boldsymbol{F}\,(\boldsymbol{e}^{\,*}, t)$ satisfies all the hypotheses of the ACK Theorem. Therefore, there exists a $\beta_2 > 0$ such that Eq. (7.48) has a unique solution $\boldsymbol{e}^{\,*} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. Because of Proposition 7.6, then $\mathcal{N}^* \boldsymbol{e}^{\,*} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$. Given the expression (7.31) for the error $\boldsymbol{e}$ and Proposition [7.9] for $\boldsymbol{\sigma}$, the proof of Theorem 5 is achieved.

7.8. Conclusion of the Proof of Theorem 1. We have thus proved that $\boldsymbol{u}^{\,E} \in H^{l,\rho,\theta}_{\beta,T}$ (Theorem 4.1 of [6]), that $\overline{\boldsymbol{u}}^{P} \in K^{l-1,\rho,\theta,\mu}_{\beta,T}$ (Theorem 3 of [6]), that $\boldsymbol{\omega} \in N^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ (Theorem 5.1), that $\boldsymbol{\Omega} \in K^{l-2,\rho_2,\theta_2,\mu_2}_{\beta_2,T}$ (Theorem 4), and that $\boldsymbol{e} \in L^{l-2,\rho_2,\theta_2}_{\beta_2,T}$ (Theorem 5). By a redefinition of the parameters, we may take $(\rho_2, \theta_2, \beta_2, \mu_2) = (\rho, \theta, \beta, \mu)$, and the proof of Theorem 1 is achieved.

8. Conclusions

In the analysis above, we have proved existence of solutions of the Navier-Stokes equations in two and three dimensions for a time that is short but independent of the viscosity. As the viscosity goes to zero, the Navier-Stokes solution has been shown to approach an Euler solution away from the boundary and a Prandtl solution in a thin boundary layer. The initial data were assumed to be analytic: although this restriction is severe, we believe that it might be optimal. In fact separation of the boundary layer is related to development of a singularity in the solution of the time-dependent Prandtl equations, as discussed in [2]. We conjecture that the time of separation (and thus the singularity time) cannot be controlled by a Sobolev bound on the initial data, unless some positivity and monotonicity is assumed as in [5]. It would be very important to verify this by an explicit singularity construction, or to refute it by an existence theorem in Sobolev spaces for Prandtl.

This result suggest further work on several related problems: Analysis of the zero-viscosity limit for Navier-Stokes equations in the exterior of a ball is presented in [1]. An alternative derivation of this result may be possible by a more direct analysis of the Navier-Stokes solution. In two-dimensions, a solution is known to exist for a time that is independent of the viscosity. Thus by writing the solution as a Stokes operator times the nonlinear terms and analysis of the Stokes operator, it should be possible to recognize the regular (Euler) and boundary layer (Prandtl) parts directly.

We believe that the method of the present paper could be used to prove convergence of the Navier-Stokes solution to an Euler solution with a vortex sheet, in the zero viscosity limit outside a boundary layer around the sheet. Note that the problem with a vortex sheet should be easier because the boundary layer is weaker since tangential slip is allowed, but it is more complicated since the boundary is curved and moving.

Appendix A: The Estimates for the Heat Operators

Proof of Lemma 3.1. To prove Lemma 3.1 it is useful to introduce the following changes of variables into the expression (3.21) for the operator $\tilde{E}_1$:

$$\zeta = \frac{Y}{[4(t-s)]^{1/2}}\ ,\quad \eta = \frac{x'-x}{[4(t-s)]^{1/2}}. \tag{A.1}$$

One has

$$\tilde{E}_1 f = \frac{2}{\pi}\int_{Y/\sqrt{(4t)}}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2} f\left(x+\eta Y/\zeta, t-Y^2/4\zeta^2\right). \tag{A.2}$$

To get an estimate in $L^{l,\rho,\theta}_{\beta,T}$ one has to bound the appropriate L^2 norm in x and Y of $\partial_x^i\tilde{E}_1 f$ with $i\le l$, $\partial_t\partial_x^i\tilde{E}_1 f$ with $i\le l-2$ and $\partial_x^i\partial_Y^j E_2 f$ with $i\le l-2$, $j\le 2$. We shall in fact prove a stronger estimate; we shall in fact prove that these terms are exponentially decaying in the Y variable. Let us first bound $\partial_x^i\tilde{E}_1 f$:

$$\begin{aligned}
&\sup_{0\le t\le T}\ \sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}\|\partial_x^i\tilde{E}_1 f\|_{L^2(\Re x)}\\
&=\sup_{0\le t\le T}\ \sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}\left\{\int_{-\infty}^{\infty} d\Re x\right.\\
&\left.\left[\frac{2}{\pi}\int_{\frac{Y}{\sqrt{4t}}}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\partial_x^i f\left(x+\frac{\eta Y}{\zeta}, t-\frac{Y^2}{4\zeta^2}\right)\right]^2\right\}^{1/2}\\
&\le\sup_{0\le t\le T}\ \sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}\left\{\frac{2}{\pi}\int_{Y/\sqrt{(4t)}}^{\infty} d\zeta e^{-\zeta^2}\right.\\
&\qquad\left.\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\|\partial_x^i f\left(\cdot+i\Im x, t-Y^2/4\zeta^2\right)\|^2_{L^2(\Re x)}\right\}^{1/2}
\end{aligned}$$

$$\le \sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\left\{\left[\sup_{0\le t\le T}\sup_{|\Im x|\le\rho-\beta t}\|\partial_x^i f(\cdot+i\Im x,t)\|_{L^2(\Re x)}\right]^2\right.$$
$$\left.\frac{2}{\sqrt{\pi}}\int_{Y/\sqrt(4t)}^{\infty} d\zeta e^{-\zeta^2}\right\}^{1/2}$$
$$\le \sup_{0\le t\le T}\sup_{|\Im x|\le\rho-\beta t}\|\partial_x^i f(\cdot+i\Im x,t)\|_{L^2(\Re x)}$$
$$\sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\left\{\frac{2}{\sqrt{\pi}}\int_{Y/\sqrt(4t)}^{\infty} d\zeta e^{-\zeta^2}\right\}^{1/2}$$
$$\le |\partial_x^i f|_{0,\rho,\theta,\beta,T}. \tag{A.3}$$

In passing from the second to the third line, we used the Jensen inequality to pass the square inside the integrals in ζ and η, and performed the integration in $\Re x$. We now bound $\partial_t\partial_x^i\tilde{E}_1 f$ with $i\le l-2$ by

$$\sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}\|\partial_t\partial_x^i\tilde{E}_1 f\|_{L^2(\Re x)}$$
$$\le \sup_{0\le t\le T}\sup_{|\Im x|\le\rho-\beta t}\|\partial_t\partial_x^i f(\cdot+i\Im x,t)\|_{L^2(\Re x)}$$
$$\sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\left\{\frac{2}{\sqrt{\pi}}\int_{Y/\sqrt(4t)}^{\infty} d\zeta e^{-\zeta^2}\right\}^{1/2}$$
$$\le |\partial_t\partial_x^i f|_{0,\rho,\theta,\beta,T}. \tag{A.4}$$

The procedure for the above bound is essentially the same that was used for $\partial_x^i\tilde{E}_1 f$. The only thing to note is that the derivative with respect to time passed through the integral in ζ because $f(x,t=0)=0$. We now bound $\partial_Y\partial_x^i\tilde{E}_1 f$ with $i\le l-2$ by

$$\sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}\|\partial_Y\partial_x^i\tilde{E}_1 f\|_{L^2(\Re x)}$$
$$= \sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}$$
$$\left\|\frac{2}{\pi}\int_{Y/\sqrt(4t)}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\right.$$
$$\left.\left[\frac{\eta}{\zeta}\partial_x^{i+1} f(x+\eta Y/\zeta, t-Y^2/4\zeta^2)-\frac{Y}{2\zeta^2}\partial_t\partial_x^i f(x+\eta Y/\zeta, t-Y^2/4\zeta^2)\right]\right\|$$
$$\le \sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\sup_{|\Im x|\le\rho-\beta t}$$
$$\left\{\frac{2}{\sqrt{\pi}}\int_{Y/\sqrt(4t)}^{\infty} d\zeta e^{-\zeta^2}\frac{Y}{2\zeta^2}\right.$$
$$\left.\left[\left\|\partial_x^{i+2} f(\cdot+\Im x, t-Y^2/4\zeta^2)\right\|^2+\left\|\partial_t\partial_x^i f(\cdot+\Im x, t-Y^2/4\zeta^2)\right\|^2\right]\right\}^{1/2}$$
$$\le \left(|\partial_x^{i+2} f|_{0,\rho,\theta,\beta,T}+|\partial_t\partial_x^i f|_{0,\rho,\theta,\beta,T}\right)$$

$$
\sup_{0\le t\le T}\sup_{Y\in\Sigma(\theta-\beta t)} e^{(\mu-\beta t)\Re Y}\left[\frac{2}{\sqrt{\pi}}\int_{Y/\sqrt{(4t)}}^{\infty} d\zeta e^{-\zeta^2}\frac{Y}{2\zeta^2}\right]^{1/2}
$$
$$
\le c|f|_{l,\rho,\theta,\beta,T}. \tag{A.5}
$$

In passing from the second to the third line of the above estimate, we first integrated by parts in η the term $\partial_x^{i+1} f$ and then used Jensen's inequality to pass the L^2 norm in $\Re x$ inside the integral in ζ and η. To bound $\partial_{YY}\partial_x^i\tilde{E}_1 f$ with $i\le l-2$, note that $\tilde{E}_1 f$ satisfies the heat equation and use the bounds above. The proof of Lemma 3.1 is thus achieved.

Proof of Proposition 7.1. To prove Proposition 7.1, it is useful to make the following changes of variables into the expression (7.4) for the operator $\tilde{E}_2$:

$$
\zeta=\frac{Y'-Y}{\sqrt{4(t-s)}}\ ,\quad z=\frac{Y'+Y}{\sqrt{4(t-s)}}\ ,\quad \eta=\frac{x'-x}{\sqrt{4(t-s)}}. \tag{A.6}
$$

These lead to

$$
\tilde{E}_2 f=\int_0^t ds\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\left[\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\zeta e^{-\zeta^2} f(x+\eta\sqrt{4(t-s)},Y+\zeta\sqrt{4(t-s)},s)\right.
$$
$$
\left.-\int_{Y/\sqrt{4(t-s)}}^{\infty} d\zeta e^{-\zeta^2} f(x+\eta\sqrt{4(t-s)},-Y+\zeta\sqrt{4(t-s)},s)\right]. \tag{A.7}
$$

To get an estimate in $L^{l,\rho,\theta}_{\beta,T}$, bound $\partial_x^i\tilde{E}_2 f$ with $i\le l$, $\partial_t\partial_x^i\tilde{E}_2 f$ with $i\le l-2$ and $\partial_x^i\partial_Y^j\tilde{E}_2 f$ with $i\le l-2$, $j\le 2$. First bound $\partial_x^i\tilde{E}_2 f$ by

$$
\sup_{0\le t\le T}\sup_{\theta'\le\theta-\beta t}\left\|\sup_{|\Im x|\le\rho-\beta t}\|\partial_x^i\tilde{E}_2 f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta',a/\varepsilon))}
$$
$$
\le\sup_{0\le t\le T}\sup_{\theta'\le\theta-\beta t}\left\{\int_{\Gamma(\theta',a/\varepsilon)} dY\sup_{|\Im x|\le\rho-\beta t}\int_{-\infty}^{\infty} d\Re x\right.
$$
$$
\left[\int_0^t ds\left(\int_{-Y/\sqrt{4(t-s)}}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\partial_x^i f(x+\eta\sqrt{4(t-s)},Y+\zeta\sqrt{4(t-s)},s)\right.\right.
$$
$$
\left.\left.\left.-\int_{Y/\sqrt{4(t-s)}}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\partial_x^i f(x+\eta\sqrt{4(t-s)},-Y+\zeta\sqrt{4(t-s)},s)\right)\right]^2\right\}^{1/2}
$$
$$
\le c\sup_{0\le t\le T}\sup_{\theta'\le\theta-\beta t}\left\{\int_{\Gamma(\theta',a/\varepsilon)} dY\int_0^T ds\right.
$$
$$
\left[\int_{-\infty}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\sup_{|\Im x|\le\rho-\beta t}\left\|f(\cdot+i\Im x,Y+\zeta\sqrt{4(t-s)})\right\|^2\right.
$$
$$
\left.\left.+\int_{-\infty}^{\infty} dz e^{-z^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\sup_{|\Im x|\le\rho-\beta t}\left\|f(\cdot+i\Im x,-Y+z\sqrt{4(t-s)})\right\|^2\right]\right\}^{1/2}
$$

$$\leq c \sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\|\sup_{|\Im x|\leq\rho-\beta t}\|\partial_x^i f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta',a/\varepsilon))}. \tag{A.8}$$

In passing from the second to the third line of the above estimate, we used Jensen's inequality and overestimated the integrals in s, ζ and z. Now bound $\partial_Y\partial_x^i\tilde{E}_1 f$ with $i\leq l-2$ by

$$\begin{aligned}
&\sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\|\sup_{|\Im x|\leq\rho-\beta t}\|\partial_Y\partial_x^i\tilde{E}_2 f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta',a/\varepsilon))}\\
&\leq \sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\{\int_{\Gamma(\theta',a/\varepsilon)}dY\sup_{|\Im x|\leq\rho-\beta t}\int_{-\infty}^{\infty}d\Re x\right.\\
&\left[\int_0^t ds\left(\int_{-Y/\sqrt{4(t-s)}}^{\infty}d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty}d\eta e^{-\eta^2}\partial_Y\partial_x^i f(x+\eta\sqrt{4(t-s)},Y+\zeta\sqrt{4(t-s)},s)\right.\right.\\
&\left.-\int_{Y/\sqrt{4(t-s)}}^{\infty}d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty}d\eta e^{-\eta^2}\partial_Y\partial_x^i f(x+\eta\sqrt{4(t-s)},-Y+\zeta\sqrt{4(t-s)},s)\right)\\
&\left.\left.+\ 2\int_0^t ds\frac{e^{-Y^2/4(t-s)}}{\sqrt{4(t-s)}}\int_{-\infty}^{\infty}d\eta e^{-\eta^2}\partial_x^i f(x+\eta\sqrt{4(t-s)},0,s)\right]^2\right\}^{1/2}\\
&\leq c\sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\|\sup_{|\Im x|\leq\rho-\beta t}\|\partial_Y\partial_x^i f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta',a/\varepsilon))}\\
&+\sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\{\int_{\Gamma(\theta',a/\varepsilon)}dY\sup_{|\Im x|\leq\rho-\beta t}\int_{-\infty}^{\infty}d\Re x\right.\\
&\left.\left[2\int_0^t ds\frac{e^{-Y^2/4t}}{\sqrt{4(t-s)}}\int_{-\infty}^{\infty}d\eta e^{-\eta^2}\partial_x^i f(x+\eta\sqrt{4(t-s)},0,s)\right]^2\right\}^{\frac{1}{2}}\\
&\leq c|\partial_Y\partial_x^i f|_{0,\rho,\theta,\beta,T}\\
+\ &c\sup_{0\leq t\leq T}\sup_{\theta'\leq\theta-\beta t}\left\{\sup_{0\leq t\leq T}\sup_{|\Im x|\leq\rho-\beta t}\left\|\partial_x^i f(\cdot+i\Im x,0,t)\right\|^2_{L^2(\Re x)}\right.\\
&\left.\int_{\Gamma(\theta',a/\varepsilon)}dYe^{-Y^2/4t}\int_0^t ds\frac{1}{\sqrt{4(t-s)}}\right\}^{1/2}\\
&\leq c|\partial_Y\partial_x^i f|_{0,\rho,\theta,\beta,T}+c\sum_{0\leq j\leq 1}\sup_{0\leq t\leq T}\left\|\sup_{|\Im x|\leq\rho-\beta t}\|\partial_Y^j\partial_x^i f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta'=0,a/\varepsilon))}\\
&\leq c|f|_{l,\rho,\theta,\beta,T}.
\end{aligned}\tag{A.9}$$

In passing from the third to the fourth line, we estimated the value of $\partial_x^i f$ at the boundary with the L^2 (in Y) estimate of $\partial_x^i f$ and $\partial_Y\partial_x^i f$. Now bound $\partial_{YY}\partial_x^i\tilde{E}_1 f$ with $i\leq l-2$ by

$$
\begin{aligned}
&\sup_{0\le t\le T}\ \sup_{\theta'\le\theta-\beta t}\left\| \sup_{|\Im x|\le\rho-\beta t} \|\partial_Y^2\partial_x^i \tilde{E}_2 f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta',a/\varepsilon))}\\
&\le c|\partial_{YY}\partial_x^i f|_{0,\rho,\theta,\beta,T} + c\sum_{1\le j\le 2}\ \sup_{0\le t\le T}\left\| \sup_{|\Im x|\le\rho-\beta t}\|\partial_Y^j\partial_x^i f\|_{L^2(\Re x)}\right\|_{L^2(\Gamma(\theta'=0,a/\varepsilon))}\\
&\quad + \sup_{0\le t\le T}\ \sup_{\theta'\le\theta-\beta t}\Bigg\{\int_{\Gamma(\theta',a/\varepsilon)} dY \sup_{|\Im x|\le\rho-\beta t}\int_{-\infty}^{\infty} d\Re x\\
&\quad\left[\int_0^t ds\frac{Ye^{-Y^2/4(t-s)}}{\sqrt{4(t-s)^3}}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\partial_x^i f(x+\eta\sqrt{4(t-s)},0,s)\right]^2\Bigg\}^{\frac{1}{2}}\\
&\le c|f|_{l,\rho,\theta,\beta,T}\\
&\quad + \sup_{0\le t\le T}\ \sup_{\theta'\le\theta-\beta t}\Bigg\{\int_{\Gamma(\theta',a/\varepsilon)} dY \sup_{|\Im x|\le\rho-\beta t}\int_{-\infty}^{\infty} d\Re x\\
&\quad\left[\int_{Y/\sqrt{4t}}^{\infty} d\zeta e^{-\zeta^2}\int_{-\infty}^{\infty} d\eta e^{-\eta^2}\partial_x^i f(x+\eta Y/\zeta,0,s)\right]^2\Bigg\}^{\frac{1}{2}}\\
&\le c|f|_{l,\rho,\theta,\beta,T}.
\end{aligned} \tag{A.10}
$$

In passing from the third to the fourth line, we used Jensen's inequality to pass the square inside the integral in ζ and η. Then we used the fact that the integral in ζ from $Y/\sqrt{4t}$ to infinity is an exponential decaying function of Y to perform the integration in Y. Finally we estimated the value of $\partial_x^i f$ at the boundary with the L^2 (in Y) estimate of $\partial_x^i f$ and $\partial_Y\partial_x^i f$.

Proof of Proposition 7.2. The proof of Proposition 7.2 uses the same calculations as in the previous proof, except that in Proposition 7.2 the boundary terms with $Y = 0$ are all zero. With these terms absent, the result (7.6) follows from the estimates (A.8)–(A.10).

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