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Équations aux dérivées partielles/*Partial Differential Equations*

Navier-Stokes equations on an exterior circular domain: construction of the solution and the zero viscosity limit

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Abstract. In this Note, we consider the limit of Navier-Stokes equations on a circular domain. By an explicit construction of the solution, it is proved that, when viscosity goes to zero, solution converges to the Euler solution outside the boundary layer and to the Prandtl solution inside the boundary layer.

La limite pour la solution des équations de Navier-Stokes quand la viscosité tend vers zéro

Résumé. *Nous étudions le problème de la limite quand la viscosité tend vers zéro pour la solution des équations de Navier-Stokes. Nous prouvons que la solution tend vers la solution des équations d'Euler loin de la frontière et vers la solution des équations de Prandtl dans la couche limite.*

Version française abrégée

La limite des équations de Navier-Stokes pour la viscosité tendant vers zéro est un des problèmes les plus difficiles de la théorie mathématique de la mécanique des fluides.

Récemment (*voir* [6]) ce problème a été étudié dans le cas où le fluide se trouve dans un demi-espace. Dans ce cas on a prouvé que les solutions analytiques des équations de Navier-Stokes sont la superposition d'une solution des équations d'Euler, d'une solution des équations de Prandtl qui tend vers zéro en dehors de la couche limite, et d'une correction. On a montré que cette correction est du même ordre de grandeur que la racine carrée de la viscosité.

Dans ce contexte on se demande si le même résultat est valide quand la frontière est courbe. En effet, les preuves les plus convaincantes de la séparation de la couche limite ont été trouvées quand la frontière est courbe (*voir* [2] et [3]). Donc on pouvait conjecturer que le résultat de [6] est seulement valable quand la frontière est rectiligne, et que la courbure de la frontière, en causant la séparation de la couche limite, peut invalider la construction de [6].

Note présentée par Yvonne Choquet-Bruhat.

Au contraire nous démontrons ici que, dans le cas où le fluide est confiné en dehors d'un domaine circulaire, les solutions analytiques des équations de Navier-Stokes convergent, quand la viscosité tend vers zéro, vers la solution des équations d'Euler en dehors de la couche limite et vers la solution des équations de Prandtl dans la couche limite.

Les équations de Navier-Stokes dans un domaine circulaire extérieur sont :

$$\partial_t u_r + \mathbf{u}\cdot\nabla u_r - r^{-1}u_\phi^2 + \partial_r p = \nu\big(\Delta u_r - r^{-2}u_r - 2r^{-2}\partial_\phi u_\phi\big), \tag{1}$$

$$\partial_t u_\phi + \mathbf{u}\cdot\nabla u_\phi + r^{-1}u_r u_\phi + r^{-1}\partial_\phi p = \nu\big(\Delta u_\phi + 2r^{-2}\partial_\phi u_r - r^{-2}u_\phi\big), \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3}$$

$$\gamma_R \mathbf{u} = 0, \tag{4}$$

$$\mathbf{u}(r,\phi,t=0) = \mathbf{u}_0(r,\phi), \tag{5}$$

où $\mathbf{u} = (u_r(r,\phi,t), u_\phi(r,\phi,t))$ est la vitesse du fluide dépendant de la variable radiale r avec $r \geq R$, de la variable angulaire ϕ avec $0 \leq \phi \leq 2\pi$ et du temps $t \geq 0$. Dans les équations (1)-(5) $\mathbf{u}\cdot\nabla = u_r\partial_r + r^{-1}u_\phi\partial_\phi$, $\Delta = r^{-1}\partial_r(r\partial_r)) + r^{-2}\partial_{\phi\phi}$, $\nabla\cdot\mathbf{u} = r^{-1}\partial_r(ru_r) + \partial_\phi u_\phi$, $p = p(r,\phi,t)$ est la pression, $\nu = \varepsilon^2$ la viscosité et γ_R l'opérateur de trace défini par $\gamma_R f(r,\phi,t) = f(R,\phi,t)$.

Si la viscosité est très petite on peut en négliger les effets loin de la frontière, et on arrive aux équations d'Euler (3.1)-(3.5). Si, dans la couche limite, on introduit la variable $Y = \varepsilon^{-1}(r-R)$ et on suppose que les dérivées de la vitesse du fluide par rapport à Y sont $O(1)$, on obtient les équations de Prandtl (4.1)-(4.5).

Il est naturel de se demander si les équations d'Euler et les équations de Prandtl sont, dans leurs domaines respectifs de validité, des bonnes approximations des équations de Navier-Stokes.

Nous prouvons le théorème suivant :

THÉORÈME 1. – *Soit* $\mathbf{u}_0$ *la donnée initiale des équations de Navier-Stokes de la forme suivante :*

$$\mathbf{u}_0 = \mathbf{u}_0^E + \bar{\mathbf{u}}_0^P + \varepsilon\mathbf{w}_0,$$

où $\mathbf{u}_0^E \in H^{l,\rho_0,\theta_0}$, $\tilde{u}^P_{\phi\,0} \in K^{l,\rho_0,\theta_0,\mu_0}$, $\mathbf{w}_0 \in L^{l,\rho_0,\theta_0}$, *avec les conditions d'incompressibilité :*

$$\nabla\cdot\mathbf{u}_0^E = 0\,,\quad \bar{u}^P_{r\,0} = \frac{1}{R+\varepsilon Y}\int_Y^\infty (R+\varepsilon Y')\partial_\phi \tilde{u}^P_{\phi\,0}(Y',\phi)dY'\,,\quad \nabla\cdot\mathbf{w}_0 = 0,$$

et les conditions de compatibilité :

$$\gamma_R u^E_{r\,0} = 0\,,\quad \tilde{u}^P_{\phi\,0}(Y=0,\phi) = -\gamma_R u^E_{\phi\,0},\quad \gamma_R\mathbf{w}_0 = \big(-\bar{u}^P_{r\,0}(Y=0,\phi),0\big).$$

Alors il existe $\rho < \rho_0$, $\theta < \theta_0$, $\mu < \mu_0$ *et* $\beta > \beta_0$ *tels que la solution des équations de Navier-Stokes soient de la forme suivante :*

$$\mathbf{u} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon\mathbf{w},$$

où $\mathbf{u}^E \in H^{l,\rho,\theta}_{\beta,T}$ *est la solution des équations d'Euler* (3.1)-(3.5), $\tilde{u}^P_\phi \in K^{l,\rho,\theta,\mu}_{\beta,T}$ *la solution des équations de Prandtl* (4.1)-(4.5) *et* $\mathbf{w}$ *la solution des équations de la correction* (5.1)-(5.5).

1. Introduction

A classical problem in fluid dynamics is the zero viscosity limit of Navier-Stokes equations in the presence of boundaries. It is well known that in this limit case the fluid shows two different regimes. Far away from boundaries the fluid behaves as a non-viscous fluid. In a small region close to the boundaries (size the square root of the viscosity), on the other hand, the effect of viscosity cannot be neglected.

The Euler equations are therefore believed to govern the fluid away from boundaries. One can get these equations simply by neglecting the viscosity effect in the Navier-Stokes equations, and imposing the no-flux boundary condition.

Prandtl equations are believed to govern the fluid inside the boundary layer. One can obtain these equations by rescaling the normal variable with the square root of the viscosity and through a formal asymptotic expansion.

In a recent paper (*see* [6]), it was proved that, when the fluid is confined on a half plane, these approximations are indeed correct. In [6] the authors constructed the solution of Navier-Stokes equations as a the sum of the Euler solution, the Prandtl solution and a correction which they proved to be vanishingly small with the square root of the viscosity. The initial data were restricted to be analytic.

In this paper we shall be concerned with the incompressible Navier-Stokes equations outside a circular domain. Under suitable hypotheses (*see* Theorems 3.1, 4.1, and 5.1), we shall prove that the solution $\mathbf{u}(r, \phi, t)$ of the Navier-Stokes equations outside a circular domain has the following form:

$$\mathbf{u} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon \mathbf{w}. \tag{1.1}$$

where $\mathbf{u}^E$ is the solution of the incompressible Euler equations, $\bar{\mathbf{u}}^P$ is the Prandtl solution exponentially decaying outside the boundary layer, and $\varepsilon\mathbf{w}$ is a small correction. See [4] for some related examples.

The plan of the paper is the following: In Section 2 we shall introduce the spaces of analytic functions we shall be using through the rest of this paper. In Sections 3 and 4 we state existence and uniqueness theorems for Euler equations on the exterior of a circular domain, and for Prandtl equations in the boundary layer. In Section 5 we use Euler and Prandtl solutions and a correction $\varepsilon\mathbf{w}$ to construct the Navier-Stokes solution. Theorem 5.1 states that the norm of $\mathbf{w}$, in the appropriate function space, stays bounded for a time T which is independent of the viscosity.

2. Function spaces

In this section, we introduce the function spaces used in the proof of existence and uniqueness for Euler, Prandtl, and error equations. We first define the domain of analyticity of our functions. We use polar coordinates (r, ϕ):

$$\begin{aligned}
D(\rho) &= \{\phi \in \mathbf{C} \; : \; \Re\phi \in (0, 2\pi) \; , \; |\Im\phi| \leq \rho\} \\
\Sigma(\theta) &= \{Y \in \mathbf{C} \; : \; \Re Y \geq 0 \; , \; |\Im Y| \leq \Re Y \tan\theta\} \\
\Sigma(\theta, a, R) &= \{r \in \mathbf{C} \; : \; R \leq \Re r \leq a, \; |\Im r| \leq (\Re r - R)\tan\theta\} \\
&\quad \cup \{r \in \mathbf{C} \; : \; \Re r \geq a, \; |\Im r| \leq (a - R)\tan\theta\}
\end{aligned}$$

We next define the paths of integration:

$$\begin{aligned}
\Gamma(b) &= \{\phi \in D(\rho) \; : \; \Im\phi = b\} \\
\Gamma(\theta', a, R) &= \{r \in \Sigma(\theta, a) \; : \; \Im r = (\Re r - R)\tan\theta'\} \cup \{r \in \Sigma(\theta, a) \; : \; \Im r = (a - R)\tan\theta'\} \\
&= \{r \in \Sigma(\theta, a, R) \; : \; R \leq \Re r \leq a, \; \Im r = (\Re r - R)\tan\theta'\} \\
&\quad \cup \{r \in \Sigma(\theta, a, R) \; : \; \Re r \geq a, \; \Im r = (a - R)\tan\theta'\}.
\end{aligned}$$

In the rest of this paper, we shall use $\Sigma(\theta, a)$ and $\Gamma(\theta')$ to denote $\Sigma(\theta, a, 0)$ and $\Gamma(\theta', 0)$ respectively. Moreover, all functions depending on the angular variable ϕ will be periodic in this variable.

The ambient spaces for the inviscid equation are as follows:

DEFINITION 2.1. – $H'^{l,\rho}$ is the set of all functions $f(\phi)$ such that:
- f is analytic inside $D(\rho)$;
- $\partial_\phi^m f \in L^2(\Gamma(\Im\phi))$ for $m \le l$;
- $|f|_{l,\rho} = \sum_{m\le l} \sup_{|\Im\phi|\le\rho} \|\partial_\phi f(\cdot + i\Im\phi)\|_{L^2(\Gamma(\Im\phi))} < \infty$.

DEFINITION 2.2. – $H^{l,\rho,\theta}$ is the set of all functions $f(r,\phi)$ such that:
- f is analytic inside $D(\rho) \times \Sigma(\theta, a, R)$;
- $r^{-m}\partial_\phi^m \partial_r^i f \in L^2(\Sigma(\theta, a, R); H'^{0,\rho})$ with $m + i \le l$;
- $|f|_{l,\rho,\theta} = \sum_{m+i\le l} \sup_{\theta'<\theta} \||r^{-m}\partial_\phi^m \partial_r^i f(r,\cdot)|_{0,\rho}\|_{L^2(\Gamma(\theta',a,R))} < \infty$.

DEFINITION 2.3. – $H^{l,\rho,\theta}_{\beta,T}$ is the set of all functions $f(r,\phi,t)$ such that:
- $f(r,\phi,t) \in H^{l,\rho-\beta t,\theta-\beta t}$ for all $t \in [0,T]$;
- $|f|_{l,\rho,\theta,\beta,T} = \sum_{j\le l} \sup_{0\le t\le T} |\partial_t^j f(\cdot,\cdot,t)|_{l-j,\rho-\beta t,\theta-\beta t} < \infty$.

We now introduce the ambient spaces for the boundary layer equation.

DEFINITION 2.4. – $K^{l,\rho,\theta,\mu}$ is the set of all functions $f(Y,\phi)$ such that:
- f is analytic inside $\Sigma(\theta) \times D(\rho)$;
- $\partial_\phi^m \partial_Y^i f \in C^0(\Sigma(\theta); H'^{0,\rho})$ with $m + i \le l$ and $i \le 2$;
- $|f|_{l,\rho,\theta,\mu} = \sum_{i\le 2} \sum_{m\le l-i} \sup_{Y\in\Sigma(\theta)} e^{\mu\Re Y} |\partial_\phi^m \partial_r^i f(Y,\cdot)|_{0,\rho} < \infty$.

DEFINITION 2.5. – $K^{l,\rho,\theta,\mu}_{\beta,T}$ is the set of all functions $f(Y,\phi,t)$ such that:
- $f(Y,\phi,t) \in K^{l,\rho-\beta t,\theta-\beta t,\mu-\beta t}$ for all $t \in [0,T]$;
- $\partial_t \partial_\phi^m f(Y,\phi,t) \in H^{0,\rho,\theta}$ with $m \le l-1$ for all $t \in [0,T]$;
- $$|f|_{l,\rho,\theta,\mu,\beta,T} = \sum_{i\le 2} \sum_{m\le l-i} \sup_{0\le t\le T} |\partial_Y^i \partial_\phi^m f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} + \sum_{m\le l-1} \sup_{0\le t\le T} |\partial_t \partial_\phi^m f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty.$$

We can finally introduce the ambient spaces for the error equation.

DEFINITION 2.6. – $L^{l,\rho,\theta}$ is the space of all functions $f(Y,\phi)$ such that:
- f is analytic inside $\Sigma(\theta, a/\varepsilon) \times D(\rho)$;
- $\partial_\phi^m \partial_Y^i f \in L^2(\Sigma(\theta, a/\varepsilon); H'^{0,\rho})$ with $m + i \le l$ and $i \le 2$;
- $|f|_{l,\rho,\theta} = \sum_{i\le 2} \sum_{m\le l-i} \sup_{\theta'\le\theta} \||(R+\varepsilon Y)^{-m} \partial_\phi^m \partial_Y^i f(Y,\cdot)|_{0,\rho}\|_{L^2(\Gamma(\theta',a/\varepsilon))} < \infty$.

DEFINITION 2.7. – $L^{l,\rho,\theta}_{\beta,T}$ is the space of all functions $f(Y,\phi,t)$ such that:
- $f(Y,\phi,t) \in L^{l,\rho-\beta t,\theta-\beta t}$ for all $t \in [0,T]$;
- $\partial_t \partial_\phi^m f(Y,\phi,t) \in H^{0,\rho,\theta}$ with $m \le l-1$ for all $t \in [0,T]$;
- $$|f|_{l,\rho,\theta,\mu,\beta,T} = \sum_{i\le 2} \sum_{m\le l-i} \sup_{0\le t\le T} |\partial_Y^i \partial_\phi^m f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} + \sum_{m\le l-1} \sup_{0\le t\le T} |\partial_t \partial_\phi^m f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty.$$

3. Euler equations: existence and uniqueness

Euler equations outside a circular domain are:

$$\partial_t u_r^E + \mathbf{u}^E \cdot \nabla u_r^E - r^{-1}(u_\phi^E)^2 + \partial_r p^E = 0. \tag{3.1}$$

$$\partial_t u_\phi^E + \mathbf{u}^E \cdot \nabla u_\phi^E + r^{-1}u_r^E u_\phi^E + r^{-1}\partial_\phi p^E = 0. \tag{3.2}$$

$$\nabla \cdot \mathbf{u}^E = 0 \tag{3.3}$$

$$\gamma_R {u_r}^E = 0 \tag{3.4}$$

$$\mathbf{u}^E(r, \phi, t = 0) = \mathbf{u}_0^E \tag{3.5}$$

The following theorem holds:

THEOREM 3.1. – *Suppose* $\mathbf{u}_0^E \in H^{l,\rho_0,\theta_0}$ *satisfy* $\nabla \cdot \mathbf{u}_0^E = 0$ *and* $\gamma_R {u_r}_0^E = 0$. *Then there exist* $\rho < \rho_0$, $\theta < \theta_0$ *and* $\beta > \beta_0$ *such that* (3.1)-(3.5) *admit a unique solution* $\mathbf{u}^E \in H^{l,\rho,\theta}_{\beta,T}$.

The proof of the above theorem is achieved using the abstract version of the Cauchy-Kowalewski theorem (*see* [5] and references therein), and an estimate, in $H^{l,\rho,\theta}$, for the projection (on the divergence free part of a vector) operator (for the details of the proof *see* [1].)

4. Boundary layer equations

In a small layer close to the boundary, there is a transition between the viscous regime and the inviscid regime. Introducing the scaled variable $Y = \varepsilon^{-1}(r - R)$ into Navier-Stokes equations, one obtains the following equations for $\bar{\mathbf{u}}^P = (\varepsilon \bar{u}_r^P(Y, \phi, t), \tilde{u}_\phi^P(Y, \phi, t))$ to first order in $\varepsilon = \sqrt{\nu}$:

$$\partial_t \tilde{u}_\phi^P + R^{-1}(\tilde{u}_\phi^P \partial_\phi \tilde{u}_\phi^P + \tilde{u}_\phi^P \partial_\phi \gamma_R u_\phi^E + \gamma_R u_\phi^E \partial_\phi \tilde{u}_\phi^P) + (u_r^P - \gamma \tilde{u}_r^P - Y \partial_\phi \gamma_R u_\phi^E)\partial_Y \tilde{u}_\phi^P = \partial_{YY} \tilde{u}_\phi^P. \tag{4.1}$$

$$u_r^P = -\frac{1}{R + \varepsilon Y} \int_0^Y (R + \varepsilon Y')\partial_\phi \tilde{u}_\phi^P(Y', \phi, t)dY'. \tag{4.2}$$

$$\gamma \tilde{u}_\phi^P = -\gamma_R {u_r}^E, \tag{4.3}$$

$$\tilde{u}_\phi^P(Y \to \infty, \phi, t) = 0. \tag{4.4}$$

$$\tilde{u}_\phi^P(Y, \phi, t = 0) = \tilde{u}_{\phi 0}^P. \tag{4.5}$$

where $\gamma \tilde{u}_\phi^P = \tilde{u}_\phi^P\ (Y = 0, \phi, t)$. Equation (4.1) is an equation for $\tilde{u}_\phi^P$, where $\bar{u}_r^P$ is given by the incompressibility condition (4.2). Equations (4.3) is the condition ensuring that the sum of the Euler and Prandtl transversal components $u_\phi^E + \tilde{u}_\phi^P$ is zero. Equation (4.4) is the matching condition between the boundary layer solution $\tilde{u}_\phi^P$ and the outer inviscid solution u_ϕ^E. Notice that $\bar{\mathbf{u}}^P$ differs from the usual Prandtl velocity $\mathbf{u}^P$ by its value at infinity, *i.e.* $\bar{\mathbf{u}}^P = \mathbf{u}^P - \mathbf{u}^P(Y = \infty, \phi, t)$. The following theorem holds:

THEOREM 4.1. – *Suppose* $\tilde{u}_{\phi 0}^P \in K^{l,\rho_0,\theta_0,\mu_0}$ *satisfy* $\gamma \tilde{u}_{\phi 0}^P = -\gamma_R u_0^E$. *Then, there exist* $\rho < \rho_0$, $\theta < \theta_0$, $\mu < \mu_0$ *and* $\beta > \beta_0$ *such that equations* (4.1)-(4.5) *admit a unique solution* $\tilde{u}_\phi^P \in K^{l,\rho,\theta,\mu}_{\beta,T}$.

The proof of the above theorem is again based on ACK theorem, and on an estimate of the inverse heat operator in $K^{l,\rho,\theta,\mu}_{\beta_1}$ (for more details *see* [1]).

5. The error equation

If one introduces the expression (1.1) in Navier-Stokes equations, and uses (3.1)-(3.5) and (4.1)-(4.5) for $\mathbf{u}^E$ and $\bar{\mathbf{u}}^P$, one obtains the following equations for the error $\mathbf{w}$.

$$(5.1)\quad \partial_t w_\phi + \mathbf{u}^0\cdot\nabla w_\phi + \mathbf{w}\cdot\nabla u_\phi^0 + \varepsilon\mathbf{w}\cdot\nabla w_\phi + r^{-1}\left(u_r^0 w_\phi + u_\phi^0 w_r + \varepsilon w_r w_\phi\right) + r^{-1}\partial_\phi p^w = \varepsilon^2\left[\Delta w_\phi + 2r^{-2}\partial_\phi w_r - r^{-2}w_\phi\right] + h_\phi$$

$$(5.2)\quad \partial_t w_r + \mathbf{u}^0\cdot\nabla w_r + \mathbf{w}\cdot\nabla u_r^0 + \varepsilon\mathbf{w}\cdot\nabla w_r - r^{-1}\left(2w_\phi u_\phi^0 + w_\phi^2\right) + \partial_r p^w = \varepsilon^2\left[\Delta w_r - r^{-2}w_r - 2r^{-2}\partial_\phi w_\phi\right] + h_r$$

$$(5.3)\quad \nabla\cdot\mathbf{w} = 0$$

$$(5.4)\quad \gamma_R\mathbf{w} = (-\gamma\bar{u}_{r,\phi}^P, 0)$$

$$(5.5)\quad \mathbf{w}(r,\phi,t=0) = \mathbf{w}_0.$$

where we have defined $\mathbf{u}^0 = \left(u_r{}^E + u_r^P, u_\phi^E + \varepsilon\tilde{u}_\phi^P\right)$, and $\mathbf{h} \in L_{\beta_2,T}^{l-2,\rho_2,\theta_2}$ is an $O(1)$ source depending on $\mathbf{u}^E$ and $\bar{\mathbf{u}}^P$ (*see* [1]). The following theorem holds (*see* [1]):

THEOREM 5.1. – *Suppose* $\mathbf{w}_0 \in L^{l,\rho_0,\theta_0}$ *satisfy* $\nabla\cdot\mathbf{w}_0$ *and* $\gamma_R\mathbf{w}_0 = 0$. *Then there exist* $\rho < \rho_0, \theta < \theta_0$ *and* $\beta > \beta_0$ *such that equations* (5.1)-(5.5) *admit a unique solution* $\mathbf{w}$. *This solution can be written as:*

$$\mathbf{w} = \mathbf{u}_1^E + \mathbf{\Omega} + \mathbf{e},$$

where $\mathbf{u}_1^E \in N_{\beta,T}^{l,\rho,\theta}$ *is the solution of the first order Euler equations,* $\mathbf{\Omega} \in K_{\beta,T}^{l,\rho,\theta,\mu}$ *a boundary layer correction, and* $\mathbf{e} \in L_{\beta,T}^{l,\rho,\theta}$ *an overall correction.*

The space $N_{\beta,T}^{l,\rho,\theta}$ in the above Theorem differs from the space $H_{\beta,T}^{l,\rho,\theta}$ for the fact that derivatives with respect to r are taken only up to second order. The ACK theorem and an estimate in $L_{\beta,T}^{l,\rho,\theta}$ for the Stokes operator (*see* [6]) are the ingredients for the proof of the above theorem (for details *see* [1].)

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