Lagrangian Theory for 3D Vortex Sheets with Axial or Helical Symmetry*

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February 22, 2005

Abstract

Consider a three-dimensional vortex sheet in inviscid, incompressible flow which is irrotational away from the sheet. We derive an equation for the evolution of a vortex sheet in Lagrangian coordinates, i.e. an equation that is restricted to the sheet itself and is analogous to the Birkhoff-Rott equation for a two-dimensional (planar) sheet. This general equation is specialized to sheets with axial or helical symmetry, with or without swirl.

1 Introduction

A vortex sheet in a three-dimensional inviscid, incompressible fluid flow is a two-dimensional surface across which the tangential velocity is discontinuous. Vortex sheets occur in a wide variety of flows, such as flow behind an obstacle with a sharp trailing edge, or serve as approximations to thin shear layers in such flows. If the flow is irrotational away from the sheet, as is often the case, the sheet's motion and the dynamics of the entire flow are determined solely by the strength and shape of the sheet itself. Such a three dimensional flow can thus be described through a two-dimensional problem. Under the additional assumption of helical or axial symmetry, the problem is further reduced to being one-dimensional.

The most important three-dimensional effect from the point of view of vortex dynamics is vortex stretching, which is described by the term $\omega \cdot \nabla \mathbf{u}$ in the

*Research supported in part by the National Science Foundation under grant NSF-DMS-90-05881

Helmholtz equation

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} \tag{1.1}$$

for the vorticity $\omega = \nabla \times \mathbf{u}$. In a two dimensional flow or in axially or helically symmetric flow without swirl, the vortex stretching terms vanish; while for axi-symmetric flow without swirl, the vortex stretching terms vanish in the equation for ω_θ / r^2. For flow with helical symmetry and no swirl, there is a corresponding equation without vortex stretching terms. In this sense axially or helically symmetric flows with swirl represent the simplest three-dimensional flow with vortex stretching.

Vortex sheets with axial or helical symmetry and with swirl provide a relatively simple set of examples from which to develop an understanding of general three-dimensional flows. For example we hope that such flows may provide evidence for the formation of singularities (or lack thereof) in 3D inviscid flows.

For a general three-dimensional vortex sheet, a Lagrangian description is derived in the next section. Flows with axial or helical symmetry are discussed in Section 3. The vortex sheet formulation from Section 2 is then specialized to the cases of helical symmetry in section 4 and axial symmetry in section 5, both with or without swirl. Simple examples are analyzed in section 7.

The general vortex sheet equation (2.14) derived in Section 2 has been derived previously in [3, 6, 10]. The equations for an axisymmetric vortex sheet without swirl have been derived and numerically solved in [2, 3, 4, 5, 7, 8, 12, 13].

2 Lagrangian Theory for a General 3D Vortex Sheet

Assume that the vortex sheet S is a smooth surface dividing $\mathbf{R}^3$ into two parts Ω_+ and Ω_- with normal vector $\mathbf{n}$ on S pointing into Ω_+. Denote $(\mathbf{u}_+, p_+)$ and $(\mathbf{u}_-, p_-)$ as the velocity and pressure in Ω_+ and Ω_- respectively. The Eulerian equations for the flow field are

$$\begin{aligned} &\nabla \cdot \mathbf{u}_\pm = 0 \\ &\partial_t \mathbf{u}_\pm + \mathbf{u}_\pm \cdot \nabla \mathbf{u}_\pm + \nabla p_\pm = 0 \end{aligned} \tag{2.1}$$

in $\Omega_\pm$ and

$$\begin{aligned} \mathbf{n} \cdot \mathbf{u}_+ &= \mathbf{n} \cdot \mathbf{u}_- \\ p_+ &= p_- \end{aligned} \tag{2.2}$$

on S. In addition we assume that u is irrotational away from S, i.e.

$$\nabla \times \mathbf{u}_\pm = 0. \tag{2.3}$$

These are the Eulerian equations for flow due to a vortex sheet.

A "Lagrangian" description of the motion of S is given in terms of a parameterization $\mathbf{X}(\alpha, \beta, t)$ of S. There is some arbitrariness in the parameterization of S. For example the motion of S is determined only by the normal component of the velocity $\partial_t \mathbf{X}$; any tangential component can be considered as a shift in the parameterization. Moreover since $\mathbf{u}$ is not continuous across S, a Lagrangian description following the motion of fluid particles is impossible. On the other hand it turns out that the average velocity

$$\mathbf{U} = \frac{1}{2}(\mathbf{u}_+ + \mathbf{u}_-) \tag{2.4}$$

is the natural choice for the velocity of the vortex sheet, i.e. we set

$$\frac{\partial}{\partial t}\mathbf{X}(\alpha, \beta, t) = \mathbf{U}(\alpha, \beta, t). \tag{2.5}$$

Since this equation describes the sheet through its parameterization $\mathbf{X}(\alpha, \beta, t)$, we refer to it as a "Lagrangian" theory.

First we use the Biot-Savart law to find $\mathbf{U}$ in terms of S. Since $\nabla^2 \mathbf{u} = -\nabla \times \omega$, then

$$\mathbf{u}(\mathbf{x}) = \int \omega(\mathbf{x}') \times \nabla_x G(\mathbf{x} - \mathbf{x}') d\mathbf{x}' \tag{2.6}$$

in which $G = -(4\pi|\mathbf{x} - \mathbf{x}'|)^{-1}$ is the free space Green's function. For a vortex sheet flow, $\omega(\mathbf{x}) = \sigma(\mathbf{x})\delta_S(\mathbf{x})$ so that

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\int_S \sigma(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} ds(\mathbf{x}') \tag{2.7}$$

away from the sheet, i.e. in Ω_+ and Ω_-. Now take the limit as $\mathbf{x}$ approaches a point on S from either side. According to the Plemejl formulas (derived in Appendix A)

$$\mathbf{u}_\pm(\mathbf{x}) = \frac{1}{4\pi} PV \int_S \sigma(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} ds(\mathbf{x}') \pm \frac{1}{2}\sigma(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) \tag{2.8}$$

in which the integral is a Cauchy principle value integral. It follows that

$$\mathbf{U}(\mathbf{x}) = \frac{1}{4\pi} PV \int_S \sigma(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} ds(\mathbf{x}') \tag{2.9}$$

for $\mathbf{x}$ on S. Since $\partial_t \mathbf{X} = \mathbf{U}$, then this can be rewritten as

$$\partial_t \mathbf{X} = \frac{1}{4\pi} PV \int_S \sigma(\mathbf{X}') \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3} ds(\mathbf{X}'). \tag{2.10}$$

Next we find an expression for $\sigma(\mathbf{x})$. In Ω_+ and Ω_- there are potential functions ϕ_+ and ϕ_- satisfying

$$\mathbf{u}_\pm = -\nabla \phi_\pm. \tag{2.11}$$

Since $\mathbf{u}$ has a jump across S, $\omega = \mathbf{n} \times (\mathbf{u}_+ - \mathbf{u}_-)\ \delta_S$, i.e. $\sigma = \mathbf{n} \times (\mathbf{u}_+ - \mathbf{u}_-)$. Then since $\mathbf{n} \cdot \nabla\phi_+ = \mathbf{n} \cdot \nabla\phi_-$,

$$\begin{aligned} \sigma &= \mathbf{n} \times (\mathbf{u}_+ - \mathbf{u}_-) \\ &= -\mathbf{n} \times (\nabla\phi_+ - \nabla\phi_-) \\ &= -\mathbf{n} \times \nabla_\tau \Phi \end{aligned} \tag{2.12}$$

in which $\Phi = \phi_+ - \phi_-$ on S and ∇_τ is the gradient along S, i.e.

$$\nabla_\tau \Phi = \nabla\Phi - (\mathbf{n} \cdot \nabla\Phi)\mathbf{n}. \tag{2.13}$$

Therefore the evolution equation for the sheet is

$$\partial_t \mathbf{X} = -\frac{1}{4\pi} PV \int_S (\mathbf{n}' \times \nabla_\tau \Phi') \times \frac{\mathbf{X} - \mathbf{X}'}{|\mathbf{X} - \mathbf{X}'|^3} ds(\mathbf{X}') \tag{2.14}$$

In which $\Phi' = \Phi(\mathbf{X}')$, etc.

Finally we express this integral using the parameters α and β. First consider Φ. Bernoulli's law holds in Ω_+ and Ω_- separately, i.e.

$$-\partial_t \phi_\pm + \frac{1}{2} |\nabla\phi_\pm|^2 + p_\pm = c_\pm. \tag{2.15}$$

On S, $\mathbf{n} \cdot \nabla\phi_+ = \mathbf{n} \cdot \nabla\phi_-$, so that

$$\begin{aligned} |\nabla\phi_+|^2 - |\nabla\phi_-|^2 &= (\nabla\phi_+ + \nabla\phi_-) \cdot (\nabla_\tau\phi_+ - \nabla_\tau\phi_-) \\ &= -2\mathbf{U} \cdot \nabla_\tau \Phi. \end{aligned} \tag{2.16}$$

Subtract the equation (2.15) for ϕ_+ from (2.15) for ϕ_- and use $p_+ = p_-$ to obtain

$$\partial_t \Phi + \mathbf{U} \cdot \nabla_\tau \Phi = c \equiv c_- - c_+. \tag{2.17}$$

This says that on a point moving with speed $\mathbf{U}$, $\Phi(t) = \Phi(0) + ct$ in which c is constant. Since we are only interested in $\nabla_\tau \Phi$, we may set this constant to zero for simplicity; i.e. $c = 0$. Moreover since $\mathbf{X}(\alpha, \beta, t)$ moves at speed $\mathbf{U}$ for fixed α and β, we find that $\partial_t \Phi(\alpha, \beta, t) = 0$. This says that

$$\Phi = \Phi(\alpha, \beta) \tag{2.18}$$

so that Φ should be part of the prescribed initial data and then does not change in time.

Assume that $\mathbf{X}_\beta, \mathbf{X}_\alpha$ and $\mathbf{n}$ form a right hand system so that $\mathbf{X}_\alpha \times \mathbf{X}_\beta = -|\mathbf{X}_\alpha \times \mathbf{X}_\beta|\mathbf{n}$. Then on S

$$
\begin{aligned}
\Phi_\alpha &= \mathbf{X}_\alpha \cdot \nabla_\tau \Phi \\
\Phi_\beta &= \mathbf{X}_\beta \cdot \nabla_\tau \Phi
\end{aligned} \tag{2.19}
$$

and

$$
\begin{aligned}
\sigma &= -\mathbf{n} \times \nabla_\tau \Phi \\
&= |\mathbf{X}_\alpha \times \mathbf{X}_\beta|^{-1}(\mathbf{X}_\alpha \times \mathbf{X}_\beta) \times \nabla_\tau \Phi \\
&= -|\mathbf{X}_\alpha \times \mathbf{X}_\beta|^{-1}((\mathbf{X}_\beta \cdot \nabla_\tau \Phi)\mathbf{X}_\alpha - (\mathbf{X}_\alpha \cdot \nabla_\tau \Phi)\mathbf{X}_\beta) \\
&= -|\mathbf{X}_\alpha \times \mathbf{X}_\beta|^{-1}(\Phi_\beta \mathbf{X}_\alpha - \Phi_\alpha \mathbf{X}_\beta).
\end{aligned} \tag{2.20}
$$

Also

$$
ds(\mathbf{X}) = |\mathbf{X}_\alpha \times \mathbf{X}_\beta| d\alpha d\beta. \tag{2.21}
$$

Therefore the vortex sheet equation in parametric form becomes

$$
\partial_t \mathbf{X}(\alpha, \beta, t) = -\frac{1}{4\pi} PV \int (\Phi'_\beta \mathbf{X}'_\alpha - \Phi'_\alpha \mathbf{X}'_\beta) \times (\mathbf{X} - \mathbf{X}')|\mathbf{X} - \mathbf{X}'|^{-3} d\alpha' d\beta' \tag{2.22}
$$

in which $\mathbf{X} = \mathbf{X}(\alpha, \beta, t),\ \mathbf{X}' = \mathbf{X}(\alpha', \beta', t)$, etc. This equation, as well as the non-parametric form (2.14), was derived earlier in [3, 6, 8, 10].

3 Flow with Helical or Axial Symmetry

Fluid flows with helical or axial symmetry are briefly described in this section, following which they will be specialized to vortex sheets in subsequent sections. The discussion closely follows that of Landman [11] but is repeated here for convenience and completeness. In addition to several changes in notation, his parameter α is replaced by $\kappa = \alpha^{-1}$ so that axial symmetric flow is just the special case $\kappa = 0$. Furthermore we distinguish between flows with swirl and those without swirl for both helical and axial symmetry, since vortex stretching terms can be neglected in flow without swirl.

A flow possesses helical symmetry if in cylindrical coordinates (r, θ, z), the physical quantities depend only on the variable r, ξ and t, in which

$$
\xi = z + \kappa\theta. \tag{3.1}
$$

The parameter κ is the pitch angle of the helix. In particular $\kappa = 0$ corresponds to axial symmetry.

We use the two unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\xi$ given by

$$
\mathbf{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|} = \frac{r\mathbf{e}_z + \kappa \mathbf{e}_\theta}{(r^2 + \kappa^2)^{1/2}} \tag{3.2}
$$

in addition to the Beltrami vector $\mathbf{h}$ defined as

$$\mathbf{h} = \frac{r\nabla r \times \nabla \xi}{\kappa^2 + r^2} = h^2(\kappa \mathbf{e}_z - r\mathbf{e}_\theta) \tag{3.3}$$

in which

$$h^2 = \frac{1}{\kappa^2 + r^2}. \tag{3.4}$$

This vector has the following properties:

$$\nabla \times \mathbf{h} = -2\kappa h^2 \mathbf{h} \quad \nabla \cdot \mathbf{h} = 0 \quad |\mathbf{h}|^2 = h^2 \tag{3.5}$$

and $\mathbf{h}$ is in the direction of symmetry of the flow, that is

$$\mathbf{h} \cdot \nabla g = 0 \text{ for } g = g(r, \xi, t). \tag{3.6}$$

Since they are divergence free, the velocity and vorticity fields can be written in helical variables as

$$\begin{aligned} \mathbf{u} &= \nu \mathbf{h} + \nabla \chi \times \mathbf{h} \\ \omega &= \zeta \mathbf{h} + \nabla \psi \times \mathbf{h} \end{aligned} \tag{3.7}$$

in which ν, ζ, χ and ψ are scalar functions of (r, ξ, t). When rewritten in cylindrical coordinates, the velocity field is

$$\begin{aligned} \mathbf{u} &= \frac{1}{r}\chi_\xi \mathbf{e}_r - h^2(r\nu + \kappa\chi_r)\mathbf{e}_\theta + h^2(\kappa\nu - r\chi_r)\mathbf{e}_z \\ &= \frac{1}{r}\chi_\xi \mathbf{e}_r + \nu\mathbf{h} - h\chi_r \mathbf{e}_\xi \end{aligned} \tag{3.8}$$

This shows that the cylindrical and helical components of $\mathbf{u}$ are

$$\begin{aligned} (u_r, u_\theta, u_z) &= (\frac{1}{r}\chi_\xi, -h^2(r\nu + \kappa\chi_r), h^2(\kappa\nu - r\chi_r)) \\ (u_r, u_h, u_\xi) &= (\frac{1}{r}\chi_\xi, h\nu, -h\chi_r) \end{aligned} \tag{3.9}$$

in which $u_h = h^{-1}\mathbf{h} \cdot \mathbf{u}$, $u_\xi = \mathbf{e}_\xi \cdot \mathbf{u}$. Note that the second expression for $\mathbf{u}$ is not written in a right hand order, since $\mathbf{h}$ goes to $-\mathbf{e}_\theta$ as κ goes to zero. Similarly the cylindrical and helical components of ω are

$$\begin{aligned} (\omega_r, \omega_\theta, \omega_z) &= (\frac{1}{r}\psi_\xi, -h^2(r\zeta + \kappa\psi_r), h^2(\kappa\zeta - r\psi_r)) \\ (\omega_r, \omega_h, \omega_\xi) &= (\frac{1}{r}\psi_\xi, h\zeta, -h\psi_r). \end{aligned} \tag{3.10}$$

Using this representation of $\mathbf{u}$ and ω, the fluid equations are the following: The kinematic condition $\omega = \nabla \times \mathbf{u}$ is

$$\begin{aligned} \psi &= \nu \\ \Delta^* \chi &= -h^2\zeta - 2\kappa h^4 \nu \end{aligned} \tag{3.11}$$

in which

$$\Delta^*\chi = \nabla\cdot h^2\nabla\chi = \frac{1}{r}\frac{\partial}{\partial r}(rh^2\frac{\partial\chi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\xi^2} \tag{3.12}$$

is the helical Laplacian operator. The dynamic equation for ν and ζ are

$$\begin{aligned}\frac{\partial\nu}{\partial t} &= \mathbf{h}\cdot(\nabla\chi\times\nabla\nu) \qquad (3.13)\\ h^2\frac{\partial\zeta}{\partial t} &= -2\kappa h^4\mathbf{h}\cdot(\nabla\chi\times\nabla\nu) + \mathbf{h}\cdot(\nabla(h^2\nu)\times\nabla\nu) - \mathbf{h}\cdot(\nabla(h^2\zeta)\times\nabla\chi)\end{aligned}$$

We will say that a helically symmetric flow is "swirl-free" if the velocity is perpendicular to $\mathbf{h}$, i.e. if $\psi = \nu = 0$. Note that this condition is maintained by the equations and that the fluid equations for u and ζ then reduce to

$$\begin{aligned}\Delta^*\chi &= -h^2\zeta\\ h^2\frac{\partial\zeta}{\partial t} &= -\mathbf{h}\cdot(\nabla(h^2\zeta)\times\nabla\chi).\end{aligned} \tag{3.14}$$

A direct calculation shows that the term on the right of the ζ equation is

$$\begin{aligned}\mathbf{h}\cdot(\nabla(h^2\zeta)\times\nabla\chi) &= r^{-1}(\chi_\xi\partial_r - \chi_r\partial_\xi)(h^2\zeta)\\ &= \mathbf{u}\cdot\nabla(h^2\zeta)\end{aligned} \tag{3.15}$$

so that (3.14) is

$$\partial_t(h^2\zeta) + \mathbf{u}\cdot\nabla(h^2\zeta) = 0. \tag{3.16}$$

which contains no stretching term.

Since $\omega_\xi = -h\psi_r = -h\nu_r = -h\partial_r(u_h/h)$, then

$$u_h(r) = -h(r)\int_0^r h^{-1}(r')\omega_\xi(r')\ dr'. \tag{3.17}$$

This relation is analogous to Stokes law relating angular velocity to axial vorticity for an axisymmetric flow as derived below. So we refer to this as the circulation formula for flow with helical symmetry.

In the case of axial symmetry with $\kappa = 0$, the above definitions simplify to

$$\begin{aligned}\mathbf{e}_\xi &= \mathbf{e}_z, \quad \mathbf{h} = -\frac{1}{r}\mathbf{e}_\theta\\ \mathbf{u} &= -\frac{\nu}{r}\mathbf{e}_\theta - \frac{1}{r}\nabla\chi\times\mathbf{e}_\theta = \frac{1}{r}\chi_z\mathbf{e}_r - \frac{\nu}{r}\mathbf{e}_\theta - \frac{1}{r}\chi_r\mathbf{e}_z\\ \omega &= -\frac{\zeta}{r}\mathbf{e}_\theta - \frac{1}{r}\nabla\psi\times\mathbf{e}_\theta = \frac{1}{r}\psi_z\mathbf{e}_r - \frac{\zeta}{r}\mathbf{e}_\theta - \frac{1}{r}\psi_r\mathbf{e}_z.\end{aligned} \tag{3.18}$$

Thus the components of $\mathbf{u}$ and ω in cylindrical coordinates are

$$\begin{aligned}(u_r, u_\theta, u_z) &= \frac{1}{r}(\chi_z, -\nu, -\chi_r)\\ (\omega_r, \omega_\theta, \omega_z) &= \frac{1}{r}(\psi_z, -\zeta, -\psi_r) \qquad (3.19)\end{aligned}$$

(3.20)

The fluid equations are

$$
\begin{aligned}
\psi &= \nu \\
\frac{\partial^2 \chi}{\partial z^2} &+ r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \chi}{\partial r} = -\zeta \\
\frac{\partial}{\partial t}\nu &+ (u_r\partial_r + u_z\partial_z)\nu = 0 \\
\frac{\partial}{\partial t}(\zeta/r^2) &+ (u_r\partial_r + u_z\partial_z)(\zeta/r^2) = \frac{1}{r^4}\partial_z(\nu^2)
\end{aligned} \tag{3.21}
$$

For the case of axial flow without swirl these simplify to

$$
\begin{aligned}
\psi &= \nu = 0 \\
\frac{\partial^2 \chi}{\partial z^2} &+ r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \chi}{\partial r} = -\zeta \\
\frac{\partial}{\partial t}(\frac{\zeta}{r^2}) &+ (u_r\partial_r + u_z\partial_z)(\frac{\zeta}{r^2}) = 0.
\end{aligned} \tag{3.22}
$$

In the case of axial symmetry the relation (3.17) becomes the familiar circulation equation

$$u_\theta = r^{-1}\int_0^r \omega_z(r')r'dr'. \tag{3.23}$$

Finally we state the relations between the quantities used here and the corresponding quantities defined by Landman [11]. They are

$$
\begin{aligned}
&\kappa = \alpha^{-1} \quad \xi = \kappa\phi_L \quad \mathbf{h} = \kappa^{-1}\mathbf{h}_L \quad h^2 = \kappa^{-2}h_L^2 \\
&(\chi, \nu, \zeta, \psi) = \kappa(u_L, v_L, \zeta_L, \psi_L) \\
&\Delta^* = \kappa^{-2}\Delta_L^*
\end{aligned} \tag{3.24}
$$

in which the subscript L denotes the quantity used in [11].

4 Vortex Sheets with Helical Symmetry

First we parameterize the sheet by setting α to be constant on the symmetry curve, i.e. in the direction h, and β to be linear in angle. Write the point $\mathbf{X}(\alpha, \beta, t)$ in cylindrical co-ordinates as

$$\mathbf{X} = (z, r, \theta). \tag{4.1}$$

Then

$$\xi = \xi(\alpha, t) \qquad \theta(\alpha, \beta, t) = \psi(\alpha, t) + \beta. \tag{4.2}$$

The function ψ, which represents the angular coordinate of the curve $\beta =$ *constant*, is unrelated to the potential function of the previous section. Since $\xi = z + \kappa\theta$ and since r is constant along a symmetry curve, then

$$\begin{aligned} z &= -\kappa\beta + Z(\alpha, t) \\ r &= r(\alpha, t). \end{aligned} \tag{4.3}$$

It also follows that $\mathbf{X}_\beta$ is parallel to $\mathbf{h}$.

Similarly the vorticity density $-\mathbf{n}\times\nabla_\tau\Phi$ must be constant along a symmetry curve, i.e. it must be independent of β. Since $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$ do not depend on β, this implies Φ_α and Φ_β must be independent of β, i.e. that $\Phi = \phi_1(\alpha) + \phi_2\beta$. The parameterization must also be chosen so that $\mathbf{X}_\beta,\ \mathbf{X}_\alpha,\ \mathbf{n}$ form a right hand system. For example if the vortex sheet is a perturbation of a circular cylinder then $\mathbf{X}_\beta \approx \mathbf{e}_\theta,\ \mathbf{X}_\alpha \approx \mathbf{e}_z,\ \mathbf{n} \approx \mathbf{e}_r$ which is a right hand system.

In the case of helical symmetry with $\kappa \neq 0$, the symmetry lines are the helices $r =$ *constant*, $z + \kappa\theta =$ *constant* which are unbounded, so that the domain for β is $-\infty < \beta < \infty$. A curve $\beta =$ *constant* starts on a helix, say $\alpha = \alpha_1$, and continues until it returns to the same helix at $\alpha = \alpha_2$. The interval $(\alpha_1,\ \alpha_2)$ is most likely finite, in which case $r,\ \psi,$ and Z are required to be periodic on $(\alpha_1,\ \alpha_2)$.

The form of Φ may be simplified in two cases.

(1) *Axial or Helical Symmetry without swirl.* In this case the vortex lines coincide with the symmetry lines. Then the vorticity density σ, which is parallel to $\Phi_\beta\mathbf{X}_\alpha - \Phi_\alpha\mathbf{X}_\beta$, is also parallel to $\mathbf{h}$. Since $\mathbf{h}$ is parallel to $\mathbf{X}_\beta$, $\Phi_\beta = 0$, i.e.

$$\Phi = \phi(\alpha) \tag{4.4}$$

(2) *Axial or Helical Symmetry with Swirl.* Assume that the vortex lines are never tangent to the symmetry lines. Choose the parameter β to be constant along a vortex line; so that σ is parallel to $\mathbf{X}_\alpha$ Then $\Phi_\alpha = 0$, and

$$\Phi = \phi_2\beta = \Gamma\beta/2\pi, \tag{4.5}$$

in which Γ is analogous to the circulation around the sheet and is independent of (α, β, t).

Now calculate the quantities in the Birkhoff-Rott integral (2.22). A point $\mathbf{x}$ has the representation in cylindrical coordinates $\mathbf{x} = (z, r, \theta) = z\mathbf{e}_z + r\mathbf{e}_r(\theta)$ in which the basis vector $\mathbf{e}_r$ depends on θ and has derivative $\partial_\theta\mathbf{e}_r = r\mathbf{e}_\theta$. On the surface S,

$$\mathbf{X} = (-\kappa\beta + Z(\alpha, t),\ r(\alpha, t),\ \beta + \psi(\alpha, t)) \tag{4.6}$$

and thus its derivatives are given by

$$\begin{aligned} \mathbf{X}_\alpha &= Z_\alpha\hat{\mathbf{z}} + r_\alpha\hat{\mathbf{r}} + r\psi_\alpha\hat{\theta} \\ \mathbf{X}_\beta &= -\kappa\hat{\mathbf{z}} + r\hat{\theta} \\ &= -h^{-2}\mathbf{h} \end{aligned} \tag{4.7}$$

The vorticity density vector is then

$$\sigma = -|\mathbf{X}_\alpha \times \mathbf{X}_\beta|^{-1}(\Phi_\beta \mathbf{X}_\alpha - \Phi_\alpha \mathbf{X}_\beta) \tag{4.8}$$

in which

$$|\mathbf{X}_\alpha \times \mathbf{X}_\beta| = r(r_\alpha^2(1+\kappa^2/r^2) + (Z_\alpha + \kappa\psi_\alpha)^2)^{\frac{1}{2}} \tag{4.9}$$

and

$$\Phi_\beta \mathbf{X}_\alpha - \Phi_\alpha \mathbf{X}_\beta = ((\phi_2 Z_\alpha + \kappa\phi_{1\alpha})\hat{\mathbf{z}} + \phi_2 r_\alpha \hat{\mathbf{r}} + r(\phi_2\psi_\alpha - \phi_{1\alpha})\hat{\theta}) \tag{4.10}$$

In terms of cylindrical basis vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ evaluated at the point (α, β), the basis vectors $\hat{\mathbf{r}'}$, $\hat{\theta'}$ are given by

$$\begin{aligned} \hat{\mathbf{r}'} &= \cos(\theta' - \theta)\hat{\mathbf{r}} + \sin(\theta' - \theta)\hat{\theta} \\ \hat{\theta'} &= -\sin(\theta' - \theta)\hat{\mathbf{r}} + \cos(\theta' - \theta)\hat{\theta} \end{aligned} \tag{4.11}$$

Thus the vector $\Phi'_\beta \mathbf{X}'_\alpha - \Phi'_\alpha \mathbf{X}'_\beta$ evaluated at (α', β') is

$$\begin{aligned} \Phi'_\beta \mathbf{X}'_\alpha - \Phi'_\alpha \mathbf{X}'_\beta &= (\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})\hat{\mathbf{z}} \qquad (4.12) \\ &+ (\phi_2 r'_\alpha \cos(\theta' - \theta) - r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})\sin(\theta' - \theta))\hat{\mathbf{r}} \\ &+ (\phi_2 r'_\alpha \sin(\theta' - \theta) + r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})\cos(\theta' - \theta))\hat{\theta} \end{aligned}$$

Similarly

$$\begin{aligned} \mathbf{X} - \mathbf{X}' &= (z - z')\hat{\mathbf{z}} + (r - r'\cos(\theta' - \theta))\hat{\mathbf{r}} - r'\sin(\theta' - \theta)\hat{\theta} \qquad (4.13) \\ |\mathbf{X} - \mathbf{X}'|^2 &= (z - z')^2 + r^2 + r'^2 - 2rr'\cos(\theta - \theta') \qquad (4.14) \\ &= (-\kappa(\theta - \theta' - \psi + \psi') + Z - Z')^2 + r^2 + r'^2 - 2rr'\cos(\theta' - \theta) \end{aligned}$$

Now compute the cross product in the integral at $\theta = 0$; the value for $\theta \neq 0$ can be found by replacing θ' by $\theta' - \theta$. The cross product is

$$\begin{aligned} (\Phi'_\beta \mathbf{X}'_\alpha &- \Phi'_\alpha \mathbf{X}'_\beta) \times (\mathbf{X} - \mathbf{X}') \\ &= (-\phi_2 r r'_\alpha \sin\theta' + (r'^2 - rr'\cos\theta')(\phi_2\psi'_\alpha - \phi'_{1\alpha}))\hat{\mathbf{z}} \qquad (4.15) \\ &+ ((\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})r'\sin\theta' + (z - z')(\phi_2 r'_\alpha \sin\theta' + r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})\cos\theta'))\hat{\mathbf{r}} \\ &+ ((\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})(r - r'\cos\theta') - (z - z')(\phi_2 r'_\alpha \cos\theta' - r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})\sin\theta'))\hat{\theta} \end{aligned}$$

since $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\mathbf{z}}$ form a right hand system. In this expression $z - z' = \kappa(\theta' + \psi - \psi') + Z - Z'$ since $\theta = 0$.

Define the following integrals

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \cos\theta' |\mathbf{X}-\mathbf{X}'|^{-3} d\theta' \\
I_2 &= \int_{-\infty}^{\infty} |\mathbf{X}-\mathbf{X}'|^{-3} d\theta' \\
I_3 &= \int_{-\infty}^{\infty} \sin\theta' |\mathbf{X}-\mathbf{X}'|^{-3} d\theta' \\
I_4 &= \int_{-\infty}^{\infty} \theta'\cos\theta' |\mathbf{X}-\mathbf{X}'|^{-3} d\theta' \\
I_5 &= \int_{-\infty}^{\infty} \theta'\sin\theta' |\mathbf{X}-\mathbf{X}'|^{-3} d\theta'
\end{aligned}
\tag{4.16}
$$

in which

$$
|\mathbf{X}-\mathbf{X}'|^2 = (\kappa(\theta'+\psi-\psi') + Z - Z')^2 + r^2 + r'^2 - 2rr'\cos\theta'. \tag{4.17}
$$

First perform the β' integration in the Birkhoff-Rott integral and denote this integral as B. Since $d\beta' = d\theta'$, then

$$
\begin{aligned}
B &= \{-\phi_2 r r'_\alpha I_3 + (r'^2 I_2 - rr' I_1)(\phi_2\psi'_\alpha - \phi'_{1\alpha})\}\hat{\mathbf{z}} \\
&+ \{(\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})r' I_3 + (Z - Z' + \kappa(\psi-\psi'))(\phi_2 r'_\alpha I_3 \\
&+ r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_1) + \kappa(\phi_2 r'_\alpha I_5 + r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_4)\}\hat{\mathbf{r}} \\
&+ \{(\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})(rI_2 - r'I_1) - (Z - Z' + \kappa(\psi-\psi'))(\phi_2 r'_\alpha I_1 \\
&- r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_3) - \kappa(\phi_2 r'_\alpha I_4 - r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_5)\}\hat{\theta}
\end{aligned}
\tag{4.18}
$$

In the Birkhoff-Rott equation the velocity is

$$
\mathbf{X}_t = Z_t\hat{\mathbf{z}} + r_t\hat{\mathbf{r}} + r\psi_t\hat{\theta} \tag{4.19}
$$

then the equations for Z and r are

$$
\begin{aligned}
Z_t &= -\frac{1}{4\pi} PV\int - \phi_2 r r'_\alpha I_3 + (r'^2 I_2 - rr' I_1)(\phi_2\psi'_\alpha - \phi'_{1\alpha}) d\alpha' \\
r_t &= -\frac{1}{4\pi} PV\int (\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})r' I_3 + (Z - Z' + \kappa(\psi-\psi'))(\phi_2 r'_\alpha I_3 + r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_1) \\
&\quad + \kappa(\phi_2 r'_\alpha I_5 + r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_4) d\alpha'
\end{aligned}
\tag{4.20}
$$

while the equation for ψ is

$$
\begin{aligned}
\psi_t = -\frac{1}{4\pi r} \quad & PV\int (\phi_2 Z'_\alpha + \kappa\phi'_{1\alpha})(rI_2 - r'I_1) - (Z - Z' + \kappa(\psi-\psi'))(\phi_2 r'_\alpha I_1 \\
& - r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_3) - \kappa(\phi_2 r'_\alpha I_4 - r'(\phi_2\psi'_\alpha - \phi'_{1\alpha})I_5) d\alpha'
\end{aligned}
\tag{4.21}
$$

In the swirl-free case,i.e. $\phi_2 = 0$, these equations were derived earlier in [8], but without the simplification derived next.

Using the circulation formula (3.17) for helical flow the equation for ψ_t can be considerably simplified. Since $\omega_\xi = 0$ except on the sheet, u_h/h is piecewise constant as a function of r, i.e. constant on the line segments with z and θ fixed between successive crossings of the sheet. By symmetry $u_h = 0$ on the axis $r = 0$; its value everywhere else can be found by determining the jump $u_{h+} - u_{h-}$ across the sheet. Following (3.9), (4.7), (2.11) and (2.20),

$$\begin{aligned} h^{-1}(u_{h+} - u_{h-}) &= h^{-2}\mathbf{h} \cdot (\mathbf{u}_+ - \mathbf{u}_-) \\ &= \mathbf{X}_\beta \cdot \nabla_\tau \Phi \\ &= \Phi_\beta \end{aligned} \tag{4.22}$$

which is a constant. Recall that u_{h+} is the value in Ω_+ and u_{h-} is the value in Ω_- and that the axis $r = 0$ is contained in Ω_-. Then (4.22) implies that

$$h^{-1}u_h = \begin{cases} 0 & \text{in } \Omega_- \\ \Phi_\beta & \text{in } \Omega_+ \end{cases} \tag{4.23}$$

Now $h^{-1}u_h = \kappa u_z - r u_\theta$ and the vortex sheet velocity is the average of the velocities on either side. Thus

$$\begin{aligned} \kappa Z_t - r^2\psi_t &= \frac{1}{2}h^{-1}(u_{h+} + u_{h-}) \\ &= \frac{1}{2}h^{-1}u_{h+} \\ &= \frac{1}{2}\Phi_\beta. \end{aligned} \tag{4.24}$$

The resulting simple equation for the evolution of ψ is

$$\psi_t = \frac{\kappa}{r^2}z_t - \frac{1}{2r^2}\Phi_\beta. \tag{4.25}$$

A direct derivation of this equation from (4.21) is difficult and not illuminating.

The vortex sheet equations (4.20) and (4.25) can be simplified in each of the two cases above as follows:

Case 1. Helical flow without swirl: $\phi_1 = \phi(\alpha)$, $\phi_2 = 0$, i.e. $\Phi = \phi(\alpha)$. Then

$$\begin{aligned} Z_t &= \frac{1}{4\pi}PV\int \phi'_\alpha(r'^2 I_2 - rr'I_1)d\alpha' \\ r_t &= -\frac{1}{4\pi}PV\int \phi'_\alpha r'(\kappa I_3 - (Z - Z' + \kappa(\psi - \psi'))I_1 - \kappa I_4)d\alpha' \\ \psi_t &= \frac{\kappa}{r^2}Z_t \end{aligned} \tag{4.26}$$

The last equation for ψ does not influence the equations for Z and r; it only maintains the condition that $u_h = 0$ for swirl-free flow.

Case 2. Helical flow with swirl: $\phi_1 = 0,\ \phi_2 = \Gamma$ i.e. $\Phi = \Gamma\beta/2\pi$. Then

$$\begin{aligned}
Z_t &= -\frac{\Gamma}{8\pi^2} PV \int -rr'_\alpha I_3 + (r'^2 I_2 - rr' I_1)\psi'_\alpha d\alpha' \\
r_t &= -\frac{\Gamma}{8\pi^2} PV \int Z'_\alpha r' I_3 + (Z - Z' + \kappa(\psi - \psi'))(r'_\alpha I_3 + r'\psi'_\alpha I_1) + \kappa(r'_\alpha I_5 + r'\psi'_\alpha I_4) d\alpha' \\
\psi_t &= \frac{\kappa}{r^2} Z_t + \frac{\Gamma}{4\pi r^2}. \qquad (4.27)
\end{aligned}$$

A helically symmetric vortex sheet with swirl may be represented in the form of Case 2, if the vortex lines on the sheet are never tangent to the symmetry direction $\mathbf{h}$.

The equations above provide the velocity for a helical vortex sheet with or without swirl. The fluid velocity $\mathbf{u}_+,\ \mathbf{u}_-$ for points off the sheet is computed from similar formulas: The axial and radial velocities u_z and u_r at a point $\mathbf{x}$ are exactly equal to the formulas above for Z_t and r_t in which the values of $z = Z + \kappa\psi$ and r in the integrals are the the axial and radial components of $\mathbf{x}$. The azimuthal velocity u_θ is then given from the formula (4.23).

In this presentation the main features of helical symmetry are that the evolutions equations for the vortex sheet may be simplified in the two cases above and that the angular velocity equation for ψ_t is easily expressed in terms of Z_t and Γ.

5 Axi-Symmetric Sheets

For an axi-symmetric sheets the symmetry lines are the circles $r = constant,\ z = constant$, so that the parameter β lies in the interval $0 < \beta < 2\pi$. The interval for α is infinite if the vortex sheet forms an infinite cylinder, or finite with periodicity if the sheet forms a ring.

Equations for an axi-symmetric sheet can be derived as in the previous section. The formula (4.15) for the integrand in (2.22) is still valid, but now $\kappa = 0$ and $z = Z(\alpha, t) = z(\alpha, t)$.

The integrals corresponding to I_4 and I_5 are not needed since $\kappa = 0$. In addition the term $|\mathbf{X} - \mathbf{X}'|$ is periodic and even in θ', so that the integral with a factor $sin\theta'$ corresponding to I_3 vanishes. The remaining two integrals are

$$J_1 = \int_0^{2\pi} cos\theta' \left((z - z')^2 + r^2 + r'^2 - 2rr' cos\theta'\right)^{-3/2} d\theta'$$

$$J_2 = \int_0^{2\pi} \left((z - z')^2 + r^2 + r'^2 - 2rr' cos\theta'\right)^{-3/2} d\theta'. \qquad (5.1)$$

They can be written in terms of elliptic integrals (see [1] p. 590 or [9] pp.

904-905)

$$\begin{aligned} K(m) &= \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2} d\theta \\ E(m) &= \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2} d\theta \end{aligned} \tag{5.2}$$

of the first and second kind respectively. These also have the properties (see [9] p. 907)

$$\begin{aligned} \frac{dK}{dm} &= \frac{1}{2m(1-m)}(E - (1-m)K) = \frac{1}{2}\int_0^{\pi/2} \sin^2\theta(1 - m\sin^2\theta)^{-3/2} d\theta \\ \frac{dE}{dm} &= \frac{1}{2m}(E - K) \end{aligned} \tag{5.3}$$

After some manipulation, it follows that

$$\begin{aligned} J_1 &= 4\left(\frac{m}{4rr'}\right)^{3/2} \left\{\frac{2}{m}(E(m) - K(m)) + \frac{1}{1-m}E(m)\right\} \\ J_2 &= 4\left(\frac{m}{4rr'}\right)^{3/2} \frac{1}{1-m}E \end{aligned} \tag{5.4}$$

in which

$$m = \frac{4rr'}{(z-z')^2 + (r+r')^2} \tag{5.5}$$

These expressions will be used to simplify the integrals for z_t and r_t.

As before the potential difference Φ across the sheet can be expressed as $\Phi = \phi_1(\alpha) + \phi_2\beta$. The general equations for the evolution of an axi-symmetric vortex sheet are

$$\begin{aligned} z_t &= -\frac{1}{4\pi} PV \int (r'^2 J_2 - rr' J_1)(\phi_2\psi'_\alpha - \phi'_{1\alpha}) d\alpha' \\ &= -\frac{1}{\pi} PV \int r'\,(\phi_2\psi'_\alpha - \phi'_{1\alpha}) \left(\frac{m}{4rr'}\right)^{3/2} \left\{\frac{r'-r}{1-m}E - \frac{2r}{m}(E-K)\right\} d\alpha' \end{aligned} \tag{5.6}$$

$$r_t = -\frac{1}{4\pi} PV \int (z-z') r' (\phi_2\psi'_\alpha - \phi'_{1\alpha}) J_1 d\alpha' \tag{5.7}$$

$$= -\frac{1}{\pi} PV \int r'\,(\phi_2\psi'_\alpha - \phi'_{1\alpha})\,(z-z') \left(\frac{m}{4rr'}\right)^{3/2} \left\{\frac{2}{m}(E-K) + \frac{1}{1-m}E\right\} d\alpha'$$

$$\begin{aligned} \psi_t &= -\frac{1}{4\pi r}\phi_2 PV \int z'_\alpha (rJ_2 - r'J_1) - (z-z') r'_\alpha J_1 d\alpha' \\ &= -\frac{\phi_2}{2} r^{-2} \end{aligned} \tag{5.8}$$

The equations for an axi-symmetric vortex sheet can then be written in the two possible cases as before:

(1) Axi-Symmetric Vortex Sheet without Swirl: $\phi_1 = \phi(\alpha),\ \phi_2 = 0$ and

$$
\begin{aligned}
z_t &= \frac{1}{\pi} PV \int \phi'_\alpha r' \left(\frac{m}{4rr'}\right)^{3/2} \left\{ \frac{r'-r}{1-m} E - \frac{2r}{m}(E-K) \right\} d\alpha' && (5.9)\\
r_t &= \frac{1}{\pi} PV \int \phi'_\alpha r' (z-z') \left(\frac{m}{4rr'}\right)^{3/2} \left\{ \frac{2}{m}(E-K) + \frac{1}{1-m} E \right\} d\alpha' && (5.10)
\end{aligned}
$$

with $\phi' = \phi(\alpha')$, etc. In the swirl free case θ is constant on each vortex line, so that no equation is needed for ψ.

(2) Axi-Symmetric Vortex Sheet with Swirl: $\phi_1 = 0,\ \phi_2 = -\Gamma/2\pi$ and

$$
\begin{aligned}
z_t &= -\frac{\Gamma}{2\pi^2} PV \int r' \psi'_\alpha \left(\frac{m}{4rr'}\right)^{3/2} \left\{ \frac{r'-r}{1-m} E - \frac{2r}{m}(E-K) \right\} d\alpha' \\
r_t &= -\frac{\Gamma}{2\pi^2} PV \int r' \psi'_\alpha (z-z') \left(\frac{m}{4rr'}\right)^{3/2} \left\{ \frac{2}{m}(E-K) + \frac{1}{1-m} E \right\} d\alpha' \\
\psi_t &= \frac{\Gamma}{4\pi} r^{-2} && (5.11)
\end{aligned}
$$

in which Γ is the circulation (which is constant) $r = r(\alpha, t),\ r' = r(\alpha', t)$, etc. As in the case of helical symmetry, an axi-symmetric vortex sheet with swirl may be written in the form of Case 2 if the vortex lines on the sheet are never tangent to the symmetry direction $\hat{\theta}$.

The velocity off the sheet may be obtained as in the case of helical symmetry. The velocities u_z and u_r at a point $\mathbf{x}$ are equal to the formulas for z_t and r_t in which z and r in the integrals are the coordinates of the point $\mathbf{x}$. The azimuthal velocity u_θ is given by

$$
u_\theta = \begin{cases} 0 & \text{inside the sheet} \\ \Gamma/2\pi r & \text{outside the sheet.} \end{cases} \qquad (5.12)
$$

As before, "inside" means the component of $R^3 - S$ containing the axis $r = 0$; the remaining component is "outside."

6 Two Simple Flows

For a purely rotating flow generated by a cylindrical vortex sheet of radius R and circulation Γ, the vorticity and velocity are

$$
\begin{aligned}
\omega &= (\Gamma/2\pi R)\delta(r-R)\hat{\mathbf{z}} \\
\mathbf{u} &= \begin{cases} 0 & r < R \\ (\Gamma/4\pi R)\hat{\theta} & r = R \\ (\Gamma/2\pi r)\hat{\theta} & r > R. \end{cases} && (6.1)
\end{aligned}
$$

The corresponding potential functions are $\phi_- = 0$, $\phi_+ = -(\Gamma/2\pi)\theta$ and $\Phi = -(\Gamma/2\pi)\theta$. The vortex sheet may be parameterized as

$$\begin{aligned} z(\alpha,t) &= \alpha \\ r(\alpha,t) &= R \\ \psi(\alpha,t) &= (\Gamma/4\pi R^2)t. \end{aligned} \tag{6.2}$$

This is directly seen to be a solution of the equations (5.11) for an axi-symmetric vortex sheet with swirl.

For a purely axial flow with velocity U inside a cylindrical vortex sheet of radius R, the vorticity and velocity are

$$\begin{aligned} \omega &= U\hat{\theta}\delta(r-R) \\ \mathbf{u} &= \begin{cases} U\hat{\mathbf{z}} & r < R \\ \frac{1}{2}U\hat{\mathbf{z}} & r = R \\ 0 & r > R \end{cases} \end{aligned} \tag{6.3}$$

The corresponding potential functions are $\phi_- = -Uz$, $\phi_+ = 0$ and $\Phi = Uz$. The sheet may be parameterized as

$$\begin{aligned} z(\alpha,t) &= \alpha + \frac{1}{2}Ut \\ r(\alpha,t) &= R \\ \psi(\alpha,t) &= 0 \end{aligned} \tag{6.4}$$

We now check that this solves the equations (5.9) for a non-rotating axi-symmetric flow. The first equation for z becomes

$$\begin{aligned} z_t &= -\frac{UR^2}{4\pi} PV \int_{-\infty}^{\infty} J_1 - J_2 d\alpha' \\ &= U\frac{R^2}{4\pi} PV \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{1-\cos\theta}{((\alpha-\alpha')^2 + 2R^2(1-\cos\theta))^{3/2}} d\theta d\alpha' \\ &= U\frac{R^2}{2\pi} \int_0^{2\pi} \frac{1-\cos\theta}{2R^2(1-\cos\theta)} d\theta \\ &= U/2 \end{aligned} \tag{6.5}$$

using integral #3.252(7) from [9] (p. 296) for the α' integral. The equation for r is $r_t = 0$ since $(z-z') = \alpha - \alpha'$ is odd in $(\alpha - \alpha')$, while all of the other terms in the integral are even. These two equations for r_t and z_t agree with (5.9) above.

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A Plemejl Formulas in Three Dimensions

In this appendix we derive the Plemejl formulas (2.9) and (2.10) above. Consider the integral

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int_S \sigma(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} ds(\mathbf{x}') \tag{A.1}$$

and its limits as $\mathbf{x}$ approaches a point $\mathbf{x}_0$ on S from either side. Break $\mathbf{u}$ into two parts

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}_\varepsilon(\mathbf{x}) + \mathbf{w}_\epsilon(x) \tag{A.2}$$

in which $\mathbf{v}_\varepsilon$ is the integral over $S_\varepsilon = S \cap \{|\mathbf{x}' - \mathbf{x}_0| < \varepsilon\}$ and $\mathbf{w}_\varepsilon$ is the integral over $S - S_\varepsilon$. The second integral is continuous at $\mathbf{x}_0$ and

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbf{w}_\varepsilon(\mathbf{x}_0) &= \lim_{\varepsilon\to 0} \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{w}_\varepsilon(\mathbf{x}) \\ &= \frac{1}{4\pi} PV \int_S \sigma(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} ds(\mathbf{x}') \end{aligned} \tag{A.3}$$

In the first integral $\mathbf{v}_\varepsilon$, first consider $\mathbf{x}$ approaching $\mathbf{x}_0$ on a normal $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$ pointing into Ω_+, the region "above" S. For ε small, the surface S_ε can be approximated by a flat disc of radius ε, centered at $\mathbf{x}_0$ and normal to $\mathbf{n}$. Also we may replace $\sigma(\mathbf{x}')$ by $\sigma_0 = \sigma(\mathbf{x}_0)$. The integral is then approximated, after substituting $\mathbf{x}_0 + \mathbf{x}'$ for $\mathbf{x}'$, as

$$\mathbf{v}_\varepsilon(\mathbf{x}) \cong \frac{1}{4\pi} \int_{|\mathbf{x}'|\le\varepsilon,\ \mathbf{x}'\perp\mathbf{n}_0} \sigma_0 \times \frac{\mathbf{x}-(\mathbf{x}_0+\mathbf{x}')}{|\mathbf{x}-(\mathbf{x}_0+\mathbf{x}')|^3} ds(\mathbf{x}') \tag{A.4}$$

Let $\mathbf{x}'$ have radial coordinates (r,θ) in the disc. Since $\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{n}_0$, with $\delta > 0$ in Ω_+ and $\delta < 0$ in Ω_-, the term $\mathbf{x}'$ in the numerator integrates to 0 by oddness, while $\mathbf{x} - \mathbf{x}_0 = \delta\mathbf{n}_0$, $|\mathbf{x} - (\mathbf{x}_0 + \mathbf{x}')|^3 = (\delta^2 + r^2)^{3/2}$ and $ds(\mathbf{x}') = rd\theta dr$. Thus

$$\begin{aligned} \mathbf{v}_\varepsilon(\mathbf{x}) &\cong \frac{1}{4\pi}\sigma_0 \times \mathbf{n}_0 \int_0^\varepsilon 2\pi \frac{\delta r}{(\delta^2 + r^2)^{3/2}} dr \\ &= \frac{1}{2}\left(sgn(\delta) - \frac{\delta}{(\delta^2+\varepsilon^2)^{1/2}} \right) \sigma_0 \times \mathbf{n}_0 \end{aligned} \tag{A.5}$$

Finally for $\mathbf{x} \to \mathbf{x}_0$ in Ω_+,

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{v}_\varepsilon(\mathbf{x}) &= \lim_{\delta\to 0} \mathbf{v}_\varepsilon \\ &= \frac{1}{2}\sigma_0 \times \mathbf{n}_0 \end{aligned} \tag{A.6}$$

and similarly for $\mathbf{x} \to \mathbf{x}_0$ in Ω_-

$$\lim_{\varepsilon\to 0} \lim_{\mathbf{x}\to\mathbf{x}_0} v_\varepsilon(\mathbf{x}) = -\frac{1}{2}\sigma_0 \times \mathbf{n}_0. \tag{A.7}$$

Finally we show that the restrictions in the above calculation can be removed. The integral over the surface S_ε can also be written as an integral over the planar disk D_ε. If this integral and the integral in (A.4) are subtracted, the singularity of size $|\mathbf{x} - \mathbf{x}'|^{-2}$ is canceled leaving an integrand of size $|\mathbf{x} - \mathbf{x}'|^{-1}$. Since this is absolutely integrable, the resulting integral, which is the difference between the integrals over S_ε and D_ε, goes to 0 as $\varepsilon \to 0$.

We have established the limit as $\mathbf{x} \to \mathbf{x}_0$ along a normal direction. Since the limit is approached uniformly in x_0 and since $\sigma(\mathbf{x}_0)$ and $\mathbf{n}(\mathbf{x}_0)$ are continuous in $\mathbf{x}_0$, then the same limit is achieved as $\mathbf{x} \to \mathbf{x}_0$ along any direction (in Ω_+ and Ω_- separately).

The result is then

$$\lim_{\tilde{\mathbf{x}} \to \mathbf{x}_+} \mathbf{u}(\tilde{\mathbf{x}}) = \frac{1}{4\pi} PV \int \sigma(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} ds(\mathbf{x}') \pm \frac{1}{2} \sigma(\mathbf{x}) \times \mathbf{n} \tag{A.8}$$

as $\tilde{\mathbf{x}}$ approaches $\mathbf{x}$ in $\Omega_\pm$.