LETTERS

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Variance in the sedimentation speed of a suspension

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The variance in the sedimentation speed for a homogeneous suspension of solid spheres in a Stokes fluid is calculated for a particular choice of the distribution function of the spheres. In the infinite particle number limit, the variance is found to be infinite.

The average sedimentation speed of a suspension was evaluated by Batchelor[1] using a renormalization method for a small volume fraction β of particles. This method was reformulated by Hinch[2] and Feuillebois,[3] the latter by treating a suspension which occupies a container of finite linear dimension. Batchelor's result is obtained in the continuum limit $N\to\infty$ and $a/R\to 0$ with $\beta = N(a/R)^3$ small but constant, in which N is the number of particles and a is their radius.

The variance in the sedimentation speed is of interest as a mathematical quantity and as a prediction of the variance in experimental observations. More importantly it would directly enter into a calculation of the rate of diffusion induced by particle interactions.[4]

In this letter we calculate the variance for a special, simple choice of the particle distribution function, which was first introduced by Batchelor.[1] We show that in the limit of infinite particle number N, the variance is infinite. Our calculations follow the method of Feuilleois[3] with some modifications.

As a model calculation, first consider the Stokes flow $\mathbf{u}$ caused by equal point forces located on the configuration $C^N = \{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ and each of magnitude $\mathbf{F} = 6\pi\mu a \mathbf{v}_{ST}$. The flow is in a container Ω_R with $\mathrm{vol}(\Omega_R) = 4\pi R^3/3$, and $\mathbf{v}_{ST}$ is the Stokes velocity with direction $\hat{e}$. For any point $\mathbf{x}\in\Omega_R$ we can write $\mathbf{u}$ as a superposition, $\mathbf{u}(\mathbf{x}) = (|\mathbf{F}|/\mu)\Sigma_1^N \mathbf{z}(\mathbf{x},\mathbf{x}_K,R)$. The flow $\mathbf{z}(\mathbf{x},\mathbf{y},R)$ is defined for any $\mathbf{x}\in\Omega_R$, $\mathbf{y}\in\Omega_R$, and solves

$$\nabla^2\mathbf{z} - \nabla\pi = -\hat{e}\delta(\mathbf{x}-\mathbf{y}), \quad \nabla\cdot\mathbf{z} = 0 \quad \text{in } \Omega_R, \tag{1}$$

$$\mathbf{z} = 0 \quad \text{on } \partial\Omega_R, \tag{2}$$

with π the corresponding pressure. One might refer to $\mathbf{z}$ as the Stokeslet in a box. Since the δ function scales like $|\mathbf{x}|^{-3}$ and ∇^2 like $|\mathbf{x}|^{-2}$ it follows that $\mathbf{z}(\mathbf{x},\mathbf{y},R) = R^{-1}\mathbf{z}(\mathbf{x}/R,\mathbf{y}/R,1)$. As shown by Feuillebois[3] the average of $\mathbf{z}$ is $E\mathbf{z} = 0$ if $\mathbf{y}$ is randomly distributed (uniformly) over Ω_R.

Suppose that the points $\mathbf{x}_K$ are randomly, independently distributed over Ω_R; then

$$\begin{aligned} E[u(\mathbf{x})^2] &= N(F/\mu)^2 E[z(\mathbf{x},\mathbf{y},R)^2] \\ &= N(6\pi a/R)^2 v_{ST}^2 E[z(\mathbf{x}/R,\mathbf{y}/R,1)^2] \\ &\approx N^{1/3}\beta^{2/3} v_{ST}^2\sigma, \end{aligned} \tag{3}$$

in which $\sigma = 36\pi^2 E[z(\mathbf{x}/R,\mathbf{y}',1)^2]$ is the variance of $\mathbf{z}$ over $\mathbf{y}' = \mathbf{y}/R$ randomly distributed in Ω_1. For all $\mathbf{x}$ away from $\partial\Omega_R$, σ is finite and nonzero. Therefore $E(u^2)\mapsto\infty$ in the limit $N\to\infty$ with β held constant. Furthermore, it can be shown that the distribution of $N^{-1/6}\mathbf{u}$ approaches a Gaussian, so that $\mathbf{u}$ has no distribution in the limit.

In the remainder of the letter we evaluate the effects of particle interactions and finite particle size. Suppose that the particle centers are distributed randomly and independently, except for the restriction that particles may not overlap. The resulting probability distribution for N particles is denoted $P(C^N)$, or $P(C^N|\mathbf{x}_0)$ if there is an additional particle known to be at $\mathbf{x}_0$.

The normalization of P is given by $\int P(C^N)dC^N = N!$. Since the particles are uniformly distributed over the box and are independent except for volume exclusion, then

$$P(\mathbf{x}_1|\mathbf{x}_0) \leqslant cNR^{-3}$$
$$\text{and } P(\mathbf{x}_1|\mathbf{x}_0) = P(\mathbf{x}_1) \quad \text{for } |\mathbf{x}_0 - \mathbf{x}_1| > 2a \tag{4}$$

in which c is a constant independent of N and R. We shall also use the approximation

$$(N!)^{-1}\int f(\mathbf{x},C^N)P(C^N)dC^N = \int f(\mathbf{x},\mathbf{x}_1)P(\mathbf{x}_1)d\mathbf{x}_1\cdot[1+O(\beta)], \tag{5}$$

which is valid if the right-hand side of (5) is finite. This approximation, introduced by Batchelor,[1] approximates the N-particle effect by an accumulation of 1-particle effects.

For a given particle $\mathbf{x}_0$ with a configuration C^N of other particles, the velocity $\mathbf{v}_P$ of the particle can be decomposed following Faxen's law[1] as

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$$\mathbf{v}_P = \mathbf{v}_{ST} + (1 + \tfrac{1}{6}a^2\nabla^2)\mathbf{v}(\mathbf{x}_0, C^N) + \mathbf{w}(\mathbf{x}_0, C^N), \tag{6}$$

in which $\mathbf{v}(\mathbf{x}, C^N)$ is the N-particle Stokeslet in Ω_R, solving

$$\mu\nabla^2\mathbf{v} - \nabla P = \mathbf{F}\sum_{k=1}^{N}\left(1 + \frac{1}{6}a^2\nabla^2\right)\delta(\mathbf{x} - \mathbf{x}_K),$$

$$\nabla\cdot\mathbf{v} = 0, \tag{7}$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega_R. \tag{8}$$

The velocity $\mathbf{w}$ is a correction to Faxen's law caused by the boundary conditions on the N other particles, and it is defined by (6). If the $\mathbf{x}_K$ are distributed randomly and independently except for volume exclusion, then just as in (3),

$$E\{[(1 + \tfrac{1}{6}a^2\nabla^2)\mathbf{v}]^2\} \approx N^{1/3}\beta^{2/3}v_{ST}^2\sigma. \tag{9}$$

On the other hand, if we restrict to $N = 1$, then $\mathbf{v}$ and $\mathbf{w}$ have the following properties[1,3]:

$$\int |\mathbf{w}(\mathbf{x}_0,\mathbf{x}_1)|d\mathbf{x}_1 < ca^3|\mathbf{v}_{ST}|,$$

$$|\mathbf{w}(\mathbf{x}_0,\mathbf{x}_1)| + |(1 + \tfrac{1}{6}a^2\nabla^2)\mathbf{v}(\mathbf{x}_0,\mathbf{x}_1)|$$

$$< c|\mathbf{v}_{ST}| \quad \text{for} \quad |\mathbf{x}_0 - \mathbf{x}_1| > a. \tag{10}$$

Now we are prepared to evaluate the variance in $\mathbf{v}_P$. Since we know $E\mathbf{v}_P$ to be finite, it suffices to calculate $E[(\mathbf{v}_P - \mathbf{v}_{ST})^2]$, which can be approximated using (5) as

$$\begin{aligned} E[(\mathbf{v}_P - \mathbf{v}_{ST})^2] &\approx \int |\mathbf{w}(\mathbf{x}_0,\mathbf{x}_1)|^2 P(\mathbf{x}_1|\mathbf{x}_0)d\mathbf{x}_1 \\ &+ 2\int \mathbf{w}\cdot\left(1 + \frac{1}{6}a^2\nabla^2\right)\mathbf{v}(x_0,x_1)P(\mathbf{x}_1|\mathbf{x}_0)d\mathbf{x}_1 \\ &+ \int_{|\mathbf{x}_0 - \mathbf{x}_1| > a}\left[\left(1 + \frac{1}{6}a^2\nabla^2\right)\mathbf{v}(\mathbf{x}_0,\mathbf{x}_1)\right]^2 \\ &\cdot [P(\mathbf{x}_1|\mathbf{x}_0) - P(\mathbf{x}_1)]d\mathbf{x}_1 \\ &+ \int_{\substack{|\mathbf{x}_0 - \mathbf{x}_K| > a \\ \forall K}}\left[\left(1 + \frac{1}{6}a^2\nabla^2\right)\mathbf{v}(\mathbf{x}_0, C^N)\right]^2 P(C^N)dC^N. \end{aligned} \tag{11}$$

Because of (4) and (10) the first three integrals are bounded by $c\beta|\mathbf{v}_{ST}|^2$.

In the fourth integral the restriction ($|\mathbf{x}_0 - \mathbf{x}_K| > a\ \forall K$) is of no consequence and the effect of volume exclusion can be shown to be relatively small, so that the value of the integral is asymptotic to $N^{1/3}\beta^{2/3}v_{ST}^2\sigma$ just as in (3) or (9). This shows that the variance of $\mathbf{v}_P$ is infinite in the limit $N\to\infty$, with β constant.

The significance of this result is not yet clear to us, since an infinite variance is certainly unphysical. For very small particle concentrations, Brownian motion effects should dominate hydrodynamic interactions and the particle distribution should be random and independent. Our result suggests that this is valid only if $N^{1/3}\beta^{2/3}$ is small. If $N^{1/3}\beta^{2/3}$ is large, as in the usual continuum limit, it seems that hydrodynamic forces result in a strong dispersion of particles. Thus hydrodynamic forces dominate Brownian diffusion so that the particle positions will be no longer independent. Some correlation between particle positions may develop which would change the calculation of the velocity variance enough to make it finite. However, it may be that these correlations have a small effect on the average velocity, and that the model of random independent particles yields the correct sedimentation speed but is inadequate for calculating the variance.

Finally note that the infinite variance calculation above is equivalent to the calculation of infinite kinetic energy in Stokes fluid around a sedimenting sphere. If the Reynolds number were small but nonzero, this kinetic energy would be finite but large, and analogously we expect that the variance would be finite but large. This large variance still seems unphysical so that the remarks of the preceding paragraph are still relevant.

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[1] G. K. Batchelor, J. Fluid Mech. **52**, 245 (1972).
[2] E. J. Hinch, J. Fluid Mech. **83**, 695 (1977).
[3] F. Feuillebois, J. Fluid Mech. **139**, 145 (1984).
[4] This diffusive mechanism has been proposed by Batchelor (private communication).