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The sound of a singularity

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The sound generated by the singularity of a rolled-up vortex sheet is computed. A strong pressure wave is generated, whose Fourier transform is a power of frequency, with the exponent a characteristic of the singularity.

The generation of sound by nonmechanical means has been the subject of much recent research.[1] In particular, the possibility of using sound as a diagnostic and probe to explore the properties of complicated flows has been raised.[2] One of those properties that has long been studied is the development of singularities in different situations,[3] such as the rolling up of a semi-infinite vortex sheet.[4] In this Letter we report the sound that is generated by this configuration.

A vortex sheet is a two-dimensional flow characterized by a vorticity

$$\overline{\omega}(\mathbf{y},t) = \frac{d\Gamma}{ds}\,\delta_{\text{sheet}}\,, \tag{1}$$

in which s is the arclength along the sheet, Γ is a circulation variable, and δ_{sheet} is a delta function with support on the sheet. The edge of a semi-infinite, initially flat, sheet rolls up into a tightly wound spiral at whose center there is a strong singularity.[4] Since aeroacoustic sound is generated by unsteady vorticity distributions and is linearly related to it,[1] we expect that the major role for sound generation will be played by the center, whose shape is described by

$$y_1 + iy_2 = r_0\Gamma^{1/p}\exp(i\theta_0 t\Gamma^{(p-2)/p})\,, \tag{2}$$

where p $(0<p<2)$ is an exponent characteristic of the initial circulation distribution, t is time, and r_0 and θ_0 are constants proportional to $a^{-1/p}$ and $a^{2/p}$, respectively, where a is the only dimensional quantity in this problem. It has dimensions of (velocity)(distance)$^{(1-p)}$ and it is fixed by the initial circulation distribution of $\Gamma = 2a|x|^p$ $(-\infty < x \leqslant 0)$. The relevant nondimensional variable λ, whose limit $\lambda\to 0$ describes the center of the spiral, is

$$\lambda = \Gamma/(a^{2/(2-p)}t^{p/(2-p)})$$

so that we are limited to times $t\Gamma^{(p-2)/2} \gg a^{-2/p}$, enough for the spiral to develop. Although the size of the spiral grows with time, its shape remains constant and in the region of interest is very nearly circular, so that the flow resembles that of a circular vortex and the velocity of the sheet may be written as

$$v_1 - iv_2 = (1/2\pi i)[\Gamma/(y_1+iy_2)]\,. \tag{3}$$

The far-field acoustic pressure generated by a low Mach number flow of unperturbed density ρ_0, velocity $\mathbf{v}$, and vorticity $\boldsymbol{\omega}$, is[5]

$$P = -\rho_0\int dt'\,d^3x'(\mathbf{v}\wedge\boldsymbol{\omega})\cdot\nabla G\,, \tag{4}$$

a convolution of the source $(\mathbf{v}\wedge\boldsymbol{\omega})$ with the Green's function G of the wave equation. In two dimensions, and in the frequency domain (with frequency variable ω), this Green's function is a Hankel function with outgoing wave boundary conditions:

$$G(\mathbf{x},\omega) = (i/8\pi)H_0^+(\omega|\mathbf{x}|/c)\,, \tag{5}$$

with asymptotic behavior

$$H_0^+(z) \underset{z\to\infty}{\to} \sqrt{(2/\pi z)}\,e^{-i[z-(\pi/4)]}$$

so that the radiation at a distance large compared with the dimension of the source and many wavelengths away from it is

$$\frac{P(\mathbf{x},t)}{\rho_0} = -\Re\left(\frac{\sqrt{2}e^{3\pi i/4}}{4\pi^{3/2}}\int_0^\infty d\omega\sqrt{\frac{\omega}{c|\mathbf{x}|}}\times e^{-i\omega[t-(|\mathbf{x}|/c)]}S(\omega)\right), \tag{6}$$

where $S(\omega)$ is given by

$$S(\omega) = \int_{-\infty}^{\infty} dt'\,e^{i\omega t'}\widetilde{S}(t')\,,$$

where

$$\widetilde{S}(t') = \int d^2y\,\overline{\omega}(\mathbf{y},t')\epsilon_{ab}\gamma_a v_b(\mathbf{y},t')e^{-i(\omega/c)\gamma\cdot\mathbf{y}}\,,$$

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with $\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1$, and γ is the unit vector in the observer's direction. Using (1) we have

$$\widetilde{S}(t') = \int d\Gamma(\gamma_1 v_2 - \gamma_2 v_1) e^{-i(\omega/c)(\gamma_1 y_1 + \gamma_2 y_1)} .$$

Since the source of sound is radially symmetric, the radiation will be isotropic and we can take $(\gamma_1,\gamma_2) = (1,0)$. Expanding the exponential for a compact (small by comparison to the relevant acoustic wavelength) sound source and keeping the first nonvanishing contribution, we obtain, using (2),

$$\widetilde{S} = -\frac{i\omega}{2\pi c}\int_0 \Gamma\, d\Gamma \cos^2(\theta_0 t' \Gamma^{[(p-2)/p]}) , \qquad (7)$$

where the integration is limited to $\Gamma \sim 0$.

If we change variables to

$$\psi = \theta_0 t' \Gamma^{(p-2)/p} ,$$

which is basically $\lambda^{(p-2)/p}$, and remember $p < 2$, we obtain

$$\widetilde{S} = \frac{i\omega p}{(p-2)2\pi c}(\theta_0 t')^{2p/(2-p)} \times \int^{\infty} \psi^{(p+2)/(p-2)} \cos^2 \psi\, d\psi . \qquad (8)$$

Since $0 < p < 2$, the integral is convergent, and we have, up to a nondimensional multiplicative constant,

$$\widetilde{S} \cong (\omega/c) a^{4/(2-p)} t'^{2p/(2-p)}$$

and, in the frequency domain,

$$S \cong (1/c) a^{4/(2-p)} \omega^{2p/(p-2)} . \qquad (9)$$

From (6), we see that the singularity generates a pressure wave, whose Fourier transform is a power of frequency with an exponent that is characteristic of the initial circulation distribution and, if measurable, could be used as a diagnostic for the character of the singularity. This holds for frequencies such that the corresponding acoustic wavelength is small compared with distances over which large-scale variations occur, but large compared with the dimensions of the inner portion of the spiral.

If U is a typical velocity of the source and d a typical length, one has

$$a \sim U d^{(1-p)}, \quad \omega \sim U d^{-1} .$$

Using (6), it is straightforward to check that

$$P \sim \rho_0 U^2 (U/c)^{3/2} (d/x)^{1/2}$$

as it should for a two-dimensional flow.

Additional insight into this problem can be gained by looking at the pressure in the time domain. Substitution of (9) into (6) leads to

$$P(\mathbf{x},t) = P_0 |\mathbf{x}|^{-1/2} (t - |\mathbf{x}|/c)^{(7p-6)/(4-2p)} ,$$

with $P_0 \cong a^{4/(2-p)} c^{-3/2}$. For $p < \frac{6}{7}$, this is a strong pressure pulse propagating outward. The pressure gradient

$$\partial_{|\mathbf{x}|} P \cong \widetilde{P}_0 (t - |\mathbf{x}|/c)^{(9p-10)/(4-2p)}$$

is singular for $p < \frac{10}{9}$. As expected, a singularity in a flow represents violent variations that generate sound that is also singular.

The choice of the parameter p, which characterizes the type of vortex sheet spiral, depends on the physical situation. Kaden[6] analyzed the motion of an initially plane semi-infinite vortex sheet with a parabolic distribution of circulation and found a spiral with $p = \frac{1}{2}$. Kaden's solution also describes the inner portion of the rolled-up vortex sheet shed from the wing tips of an elliptically loaded wing at small incidence, as well as the starting vortex shed by the impulsive motion of a flat plate. For a starting vortex shed by the impulsive motion of a wedge of angle β or the diffraction of a shock by such a wedge, the resulting rollup will have $p = \pi/(2\pi - \beta)$.[7] For the Kelvin–Helmholtz problem in which a curvature singularity on the sheet immediately precedes the rollup, an argument based on matching the vortex strength at the singularity time suggests that $p = \frac{2}{3}$. However, no such spiral similarity solutions have yet been constructed for $p > 1$.[8]

In the small length scale of the rollup, the boundaries that generate the vortex sheet are far away. Thus they will not significantly affect the generation of sound by the vortex sheet.

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